Exact Inference Algorithms Bucket-elimination

COMPSCI 276, Spring 2013

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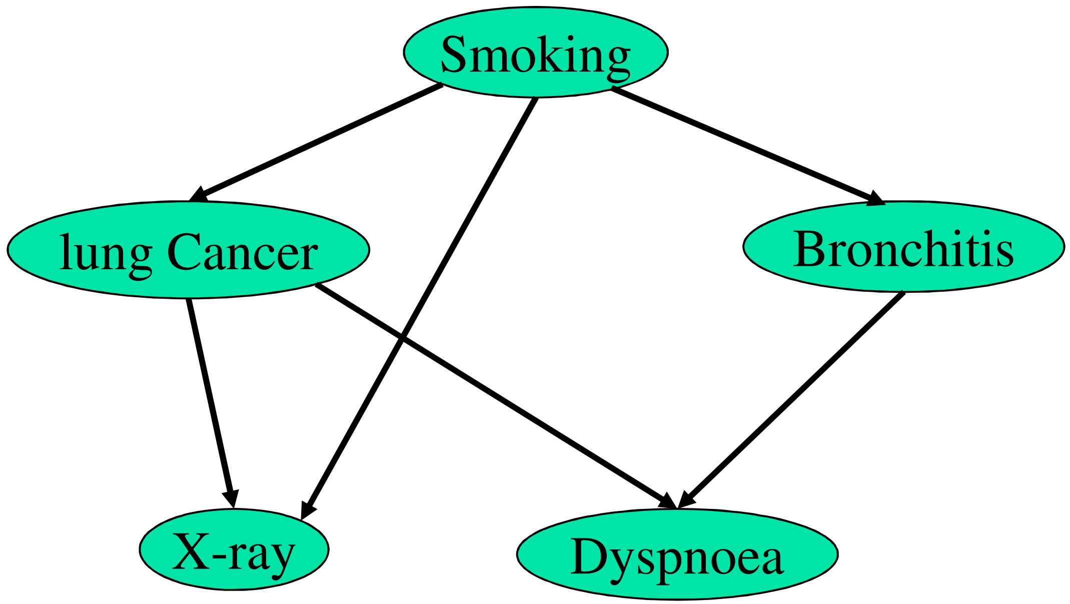
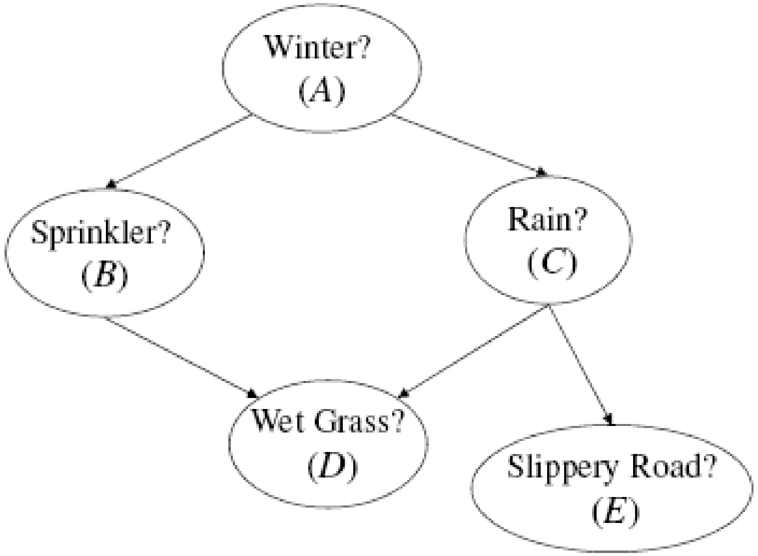
(Reading: class notes chapter 4 , Darwiche chapter 6)

Belief Updating



P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

A Bayesian Network



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Probabilistic Inference Tasks

- Belief updating: E is a subset {X1,…,Xn}, Y subset X-E, P(Y=y|E=e)
- P(e)? $\mathbf{BEL(X_i) = P(X_i = x_i \mid evidence)}$

Finding most probable explanation (MPE)

$$\bar{x}^* = \arg\max_{\bar{x}} P(\bar{x}, e)$$

- Finding maximum a-posteriory hypothesis

$$(a_1^*,\ldots,a_k^*) = \arg\max_{\bar{a}} \sum_{X/A} P(\bar{x}, e)$$

$A \subseteq X$: **hypothesis variables**

- Finding maximum-expected-utility (MEU) decision

$$(d_1^*,\ldots,d_k^*) = \arg\max_{\bar{d}} \sum_{X/D} P(\bar{x}, e) U(\bar{x})$$

$D \subseteq X$: **decision variables**

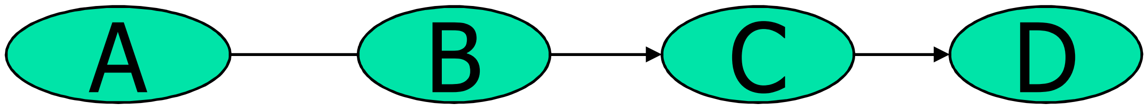
$U(\bar{\mathbf{x}})$: **utility function**

Belief updating is NP-hard

- Each sat formula can be mapped to a Bayesian network query.
- Example: (u,~v,w) and (~u,~w,y) sat?

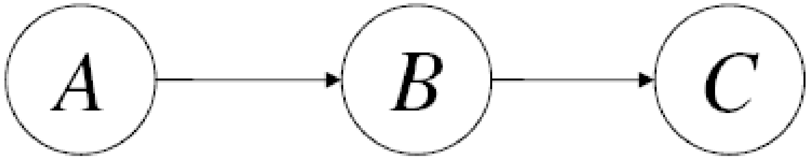
A simple network

Given:



- How can we compute P(D)?, P(D|A=0)? P(A|D=0)?
- Brute force O(k^4)
- Maybe O(4k^2)

Elimination as a Basis for Inference



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.9
true	false	.1
false	true	.2
false	false	.8

B	C	$\Theta_{C\|B}$
true	true	.3
true	false	.7
false	true	.5
false	false	.5

To compute the prior marginal on variable C, $\Pr(C)$

we first eliminate variable A and then variable B

Elimination as a Basis for Inference

- There are two factors that mention variable A, Θ_A and $\Theta_{B|A}$
- We multiply these factors first and then sum out variable A from the resulting factor.
- Multiplying Θ_A and $\Theta_{B|A}$:

A	B	$\Theta_A \Theta_{B|A}$
true	true	.54
true	false	.06
false	true	.08
false	false	.32

- Summing out variable A:

B	$\sum_A \Theta_A \Theta_{B|A}$
true	$.62 = .54 + .08$
false	$.38 = .06 + .32$

Elimination as a Basis for Inference

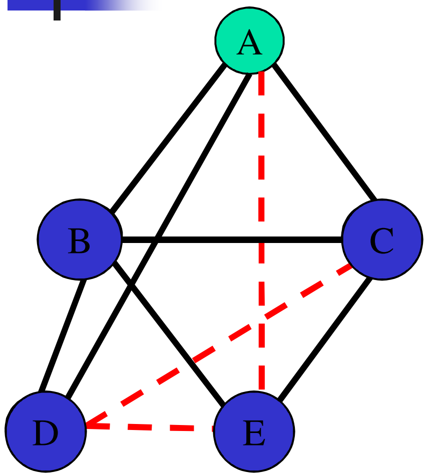
- We now have two factors, $\sum_A \Theta_A \Theta_{B|A}$ and $\Theta_{C|B}$, and we want to eliminate variable B
- Since B appears in both factors, we must multiply them first and then sum out B from the result.
- Multiplying:

B	C	$\Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	true	.186
true	false	.434
false	true	.190
false	false	.190

- Summing out:

C	$\sum_B \Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	.376
false	.624

Belief updating: P(X|evidence)=?



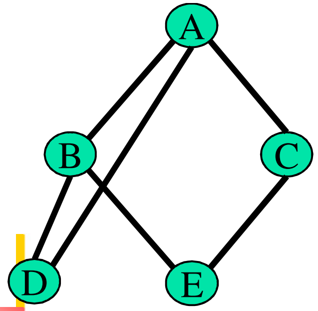
"Moral" graph

$$P(a|e=0) \propto P(a,e=0)=$$

$$\sum_{e=0,d,c,b} P(a)\underbrace{P(b|a)}P(c|a)\underbrace{P(d|b,a)P(e|b,c)}=$$

$$P(a)\sum_{e=0}\sum_{d}\sum_{c}P(c|a)\sum_{b}\underbrace{P(b|a)P(d|b,a)P(e|b,c)}_{h^B(a,d,c,e)}$$

Variable Elimination

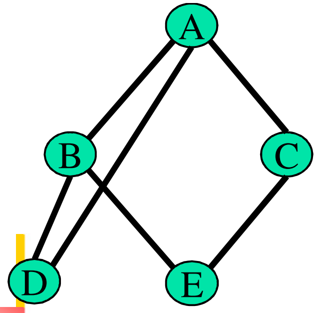


Backwards Computation,

Ordering: a, e, d, c, b

$$P(a, e=0) = P(a) \sum_{e=0} \sum_d \sum_c P(c|a) \sum_b P(b|a) P(d|a,b) P(e|b,c)$$

$$P(a) \sum_{e=0} \sum_d \sum_c P(c|a) \lambda_B(a,d,c,e)$$
$$P(a) \sum_{e=0} \sum_d \lambda_C(a,d,e)$$
$$P(a) \sum_{e=0} \lambda_D(a,e)$$
$$P(a) \lambda_D(a, e=0)$$



Backwards Computation,

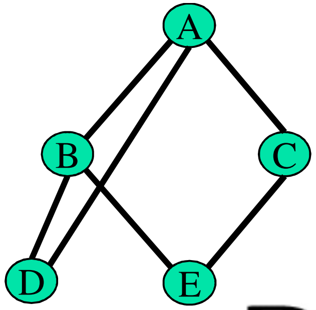
Ordering: a, e, d, c, b

$P(a, e=0) = P(a) \sum_{e=0} \sum_d \sum_c P(c|a) \sum_b P(b|a)$
$P(d|a,b) P(e|b,c)$

$P(a) \sum_{e=0} \sum_d \sum_c P(c|a) \lambda_B(a,d,c,e)$
$P(a) \sum_{e=0} \sum_d \lambda_C(a,d,e)$
$P(a) \sum_{e=0} \lambda_D(a,e)$
$P(a) \lambda_D(a, e=0)$

The bucket elimination Process:

$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a)$
$bucket(C) = \quad P(c|a) \quad || \quad \lambda_B(a,d,c,e)$
$bucket(D) = \quad || \quad \lambda_C(a,d,e)$
$bucket(E) = \quad e=0 \quad || \quad \lambda_D(a,e)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_D(a, e=0)$



Backwards Computation = Elimination

Using a different

Ordering: a, b, c, d, e

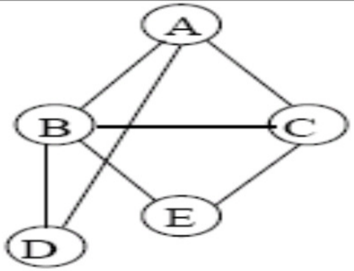
$P(a)\sum_b P(b|a)\sum_c P(c|a)\sum_d P(d|b,a)\sum_{e=0} P(e|b,c)$

$= P(a)\sum_b P(b|a)\sum_c P(c|a)P(e=0|b,c)\sum_d P(d|b,a)$
$= P(a)\sum_b P(b|a)\lambda_D(a,b)\sum_c P(c|a)P(e=0|b,c)$
$= P(a)\sum_b P(b|a)\lambda_D(a,b)\lambda_C(a,b)$
$= P(a)\lambda_B(a)$

The Bucket elimination process:

$bucket(E) =$	$P(e|b,c),\ e=0$
$bucket(D) =$	$P(d|a,b)$
$bucket(C) =$	$P(c|a)$
$bucket(B) =$	$P(b|a)$
$bucket(A) =$	$P(a)$

Bucket Elimination and Induced Width



Ordering: a, b, c, d, e

$bucket(E) = \quad P(e|b,c),\ e = 0$
$bucket(D) = \quad P(d|a,b)$
$bucket(C) = \quad P(c|a) \ \ || \ P(e = 0|b,c)$
$bucket(B) = \quad P(b|a) \ \ || \ \ \lambda_D(a,b), \lambda_C(b,c)$
$bucket(A) = \quad P(a) \ \ || \ \ \lambda_B(a)$

Ordering: a, e, d, c, b

$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a)$
$bucket(C) = \quad P(c|a) \ \ || \ \ \lambda_B(a,c,d,e)$
$bucket(D) = \quad || \ \lambda_C(a,d,e)$
$bucket(E) = \quad e = 0 \ \ || \ \lambda_D(a,c)$
$bucket(A) = \quad P(a) \ \ || \ \ \lambda_E(a)$

Factors: Sum-Out Operation

The result of summing out variable X from factor $f(\mathbf{X})$

is another factor over variables $\mathbf{Y} = \mathbf{X} \setminus \{X\}$:

$$\left(\sum_X f\right)(\mathbf{y}) \stackrel{def}{=} \sum_x f(x, \mathbf{y})$$

B	C	D	f_1
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

B	C	$\sum_D f_1$
true	true	1
true	false	1
false	true	1
false	false	1

	$\sum_B \sum_C \sum_D f_1$
$\top$	4

Factors: Sum-Out Operation

The sum-out operation is commutative

$$\sum_{Y}\sum_{X} f = \sum_{X}\sum_{Y} f$$

No need to specify the order in which variables are summed out.

If a factor f is defined over disjoint variables $\mathbf{X}$ and $\mathbf{Y}$

then $\sum_{\mathbf{X}} f$ is said to marginalize variables $\mathbf{X}$

If a factor f is defined over disjoint variables $\mathbf{X}$ and $\mathbf{Y}$

then $\sum_{\mathbf{X}} f$ is called the result of projecting f on variables $\mathbf{Y}$

Factors: Multiplication Operation

B	C	D	f_1
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

D	E	f_2
true	true	0.448
true	false	0.192
false	true	0.112
false	false	0.248

The result of multiplying the above factors:

B	C	D	E	$f_1(B, C, D)f_2(D, E)$
true	true	true	true	$0.4256 = (.95)(.448)$
true	true	true	false	$0.1824 = (.95)(.192)$
true	true	false	true	$0.0056 = (.05)(.112)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
false	false	false	false	$0.2480 = (1)(.248)$

Factors: Multiplication Operation

The result of multiplying factors $f_1(\mathbf{X})$ and $f_2(\mathbf{Y})$

is another factor over variables $\mathbf{Z} = \mathbf{X} \cup \mathbf{Y}$:

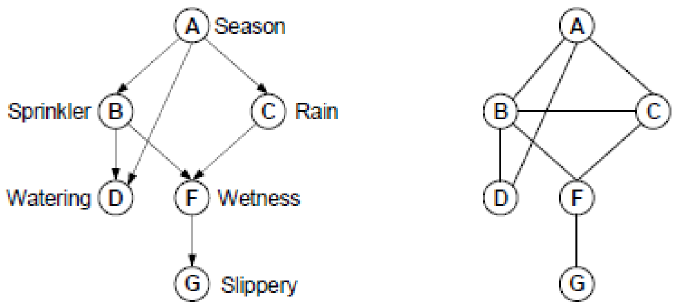
$$(f_1 f_2)(\mathbf{z}) \stackrel{def}{=} f_1(\mathbf{x}) f_2(\mathbf{y}),$$

where $\mathbf{x}$ and $\mathbf{y}$ are compatible with $\mathbf{z}$; that is, $\mathbf{x} \sim \mathbf{z}$ and $\mathbf{y} \sim \mathbf{z}$

Factor multiplication is commutative and associative

It is meaningful to talk about multiplying a number of factors without specifying the order of this multiplication process.

A Bayesian network
ordering: C,B,E,D,G



(a) Directed acyclic graph (b) Moral graph

$$P(a, g=1) = \sum_{c,b,e,d,g=1} P(a,b,c,d,e,g) = \sum_{c,b,f,d,g=1} P(g|f)P(f|b,c)P(d|a,b)P(c|a)P(b|a)P(a).$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\sum_f P(f|b,c)\sum_d P(d|b,a)\sum_{g=1} P(g|f). \quad (4.1)$$

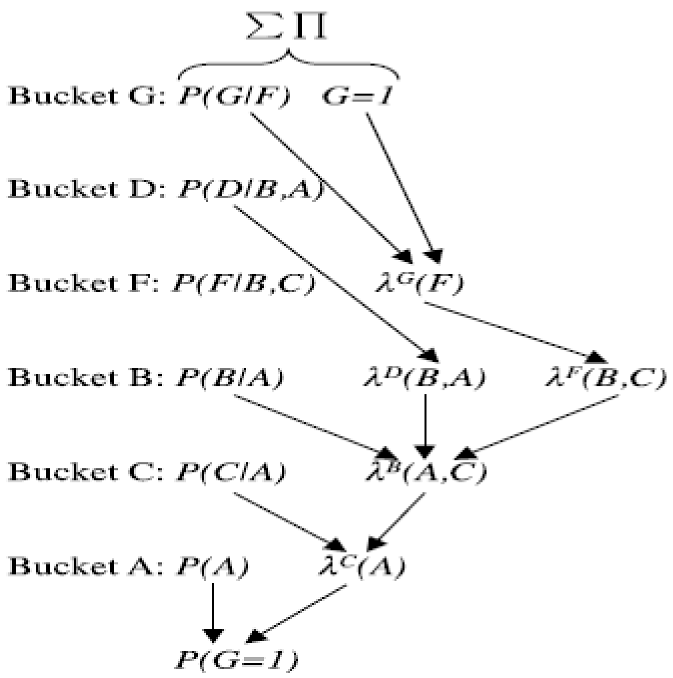
$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\sum_f P(f|b,c)\lambda_G(f)\sum_d P(d|b,a). \quad (4.2)$$

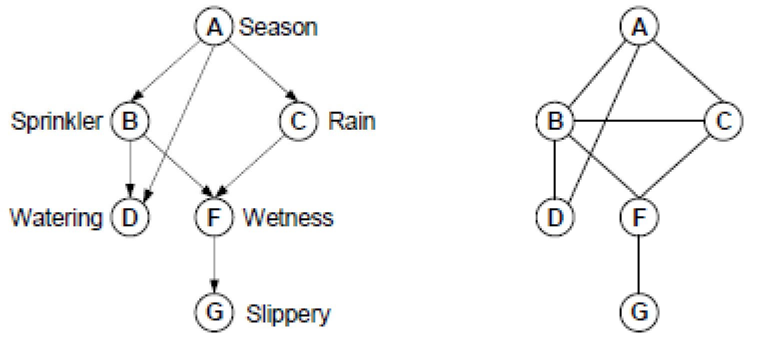
$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\lambda_D(a,b)\sum_f P(f|b,c)\lambda_G(f) \quad (4.3)$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\lambda_D(a,b)\lambda_F(b,c) \quad (4.4)$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\lambda_B(a,c) \quad (4.5)$$

A Bayesian network
ordering: C,B,E,D,G

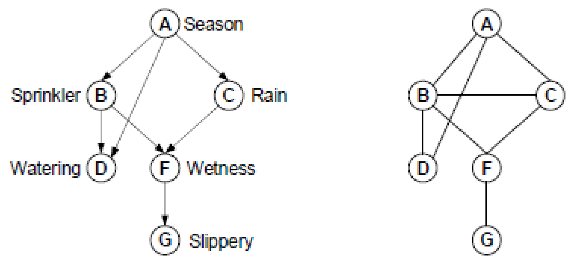




(a) Directed acyclic graph (b) Moral graph

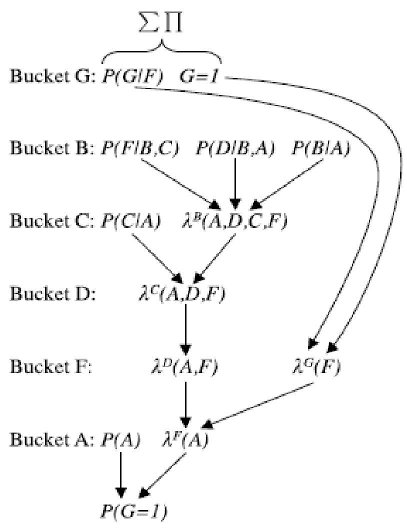
Figure 4.2: Bucket elimination along ordering $d_1 = A, C, B, F, D, G$.

A different ordering



(a) Directed acyclic graph (b) Moral graph

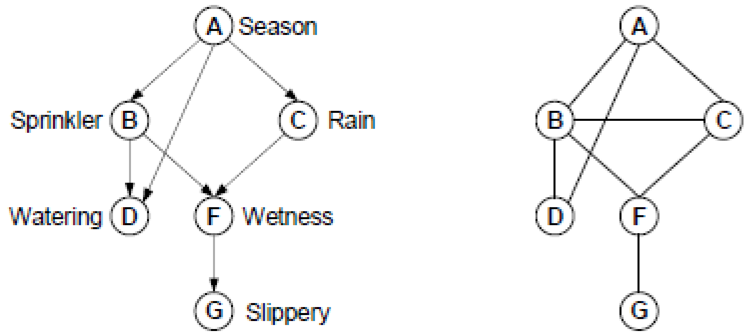
$$P(a, g=1) = P(a) \sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b) P(f|b,c) \sum_{g=1} P(g|f)$$
$$= P(a) \sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b) P(f|b,c)$$
$$= P(a) \sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \lambda_B(a,d,c,f)$$
$$= P(a) \sum_f \lambda_g(f) \sum_d \lambda_C(a,d,f)$$
$$= P(a) \sum_f \lambda_G(f) \lambda_D(a,f)$$
$$= P(a) \lambda_F(a)$$



(a)

Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

A Bayesian network processed along 2 orderings



(a) Directed acyclic graph (b) Moral graph

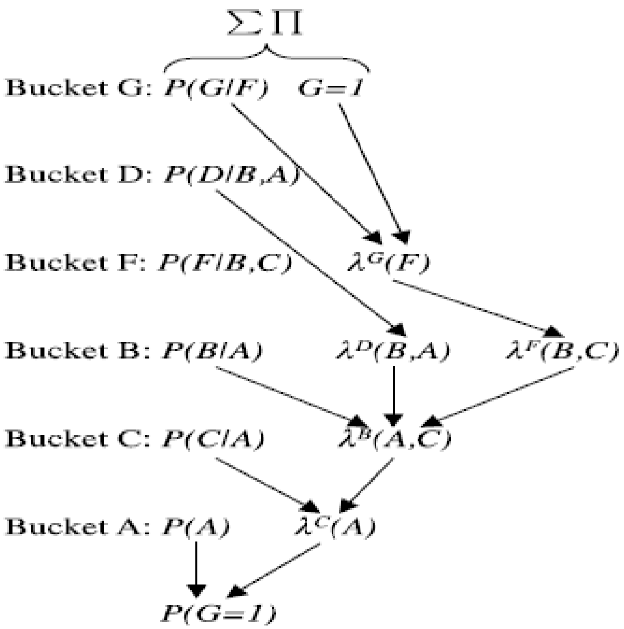
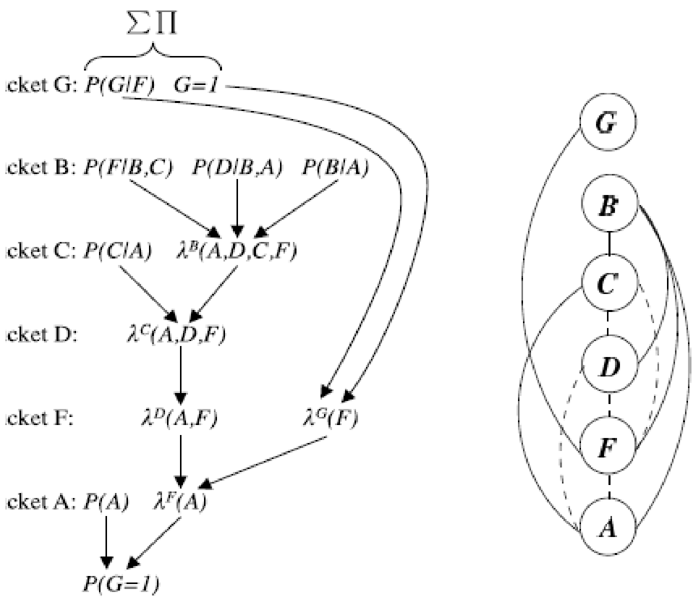


Figure 4.2: Bucket elimination along ordering $d_1 = A, C, B, F, D, G$.

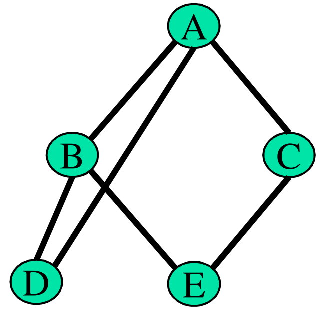


(a) (b)

Γhe bucket's output when processing along $d_2 = A, F, D, C, B, G$

Bucket elimination

Algorithm *BE-bel* (Dechter 1996)

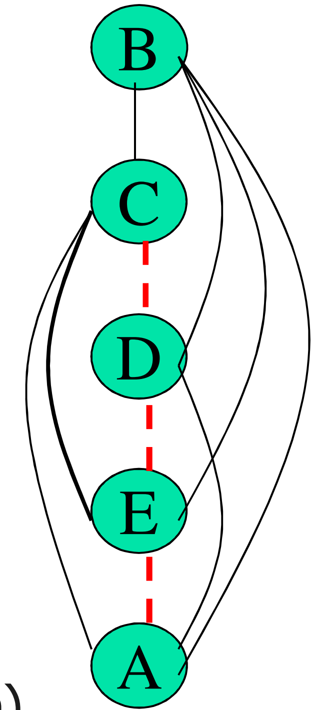


$\sum_b \prod$ ← Elimination operator

bucket B:	P(b|a) P(d|b,a) P(e|b,c)	
bucket C:	P(c|a)	$h^B(a,d,c,e)$
bucket D:		$h^C(a,d,e)$
bucket E:	e=0	$h^D(a,e)$
bucket A:	P(a)	$h^E(a)$

P(a|e=0)

W*=4
"induced width"
(max clique size)



BE-BEL

Input: A belief network $\{P_1, ..., P_n\}$, d,e.
Output: belief of X_1 given e.

1. **Initialize:**
2. **Process buckets** from $p = n$ to 1
   for matrices $\lambda_1, \lambda_2, ..., \lambda_j$ in $bucket_p$ do
   - **If** (observed variable) $X_p = x_p$ assign $X_p = x_p$ to each λ_i.
   - **Else**, (multiply and sum) $\lambda_p = \sum_{X_p} \Pi_{i=1}^{j} \lambda_i$. Add λ_p to its bucket.
3. **Return** $Bel(x_1) = \alpha P(x_1) \cdot \Pi_i \lambda_i(x_1)$

ALGORITHM BE-BEL

Input: A belief network $\mathcal{B} = < \mathcal{X}, \mathcal{D}, \mathcal{G}, \mathcal{P} >$, an ordering $d = (x_1, \ldots, x_n)$; evidence e

output: The belief $P(x_1|e)$ and probability of evidence $P(e)$

1. Partition the input functions (CPTs) into $bucket_1$, ..., $bucket_n$ as follows: **for** $i \leftarrow n$ **downto** 1, put in $bucket_i$ all unplaced functions mentioning x_i. Put each observed variable in its bucket. Denote by ψ_i the product of input functions in $bucket_i$.
2. **backward: for** $p \leftarrow n$ **downto** 1 **do**
3. **for** all the functions $\psi_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_j}$ in $bucket_p$ **do**

   If (observed variable) $X_p = x_p$ appears in $bucket_p$,

   assign $X_p = x_p$ to each function in $bucket_p$ and then

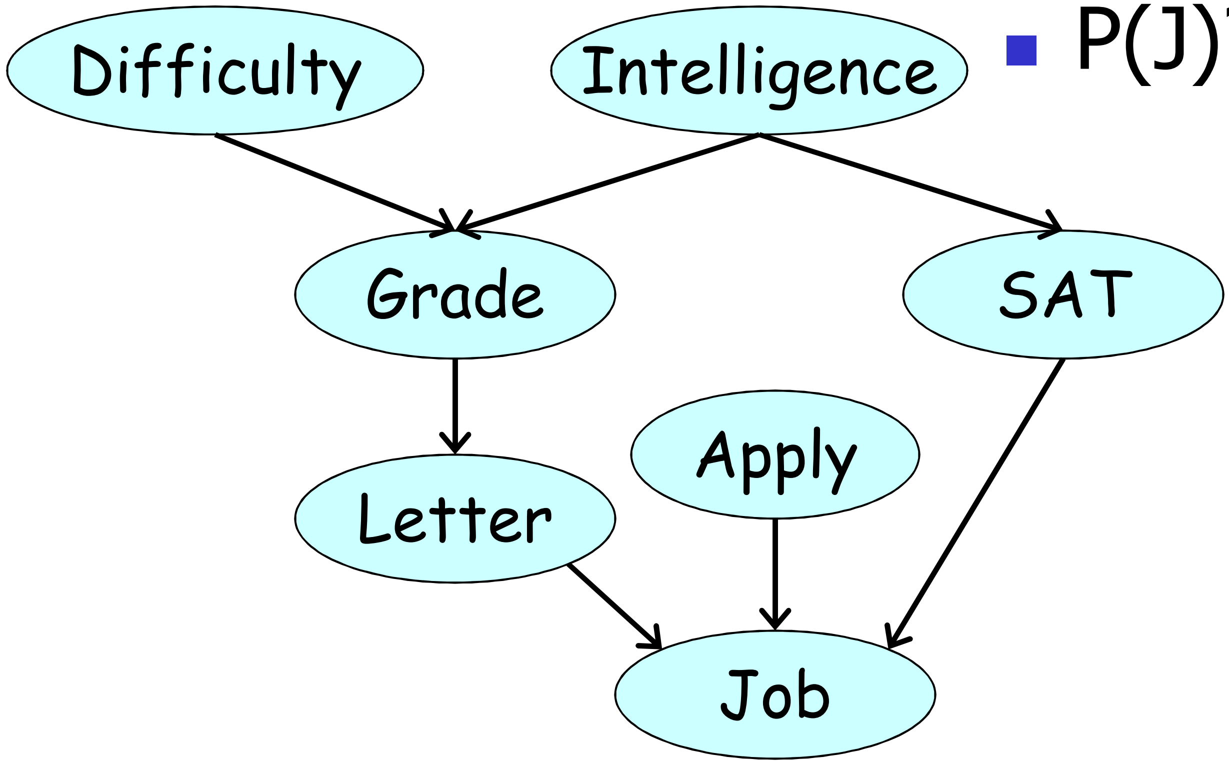
   put each resulting function in the bucket of the *closest* variable in its scope.

   else,
4. $S_p \leftarrow scope(\psi_p) \cup \bigcup_{i=0}^{j} scope(\lambda_i) - \{X_p\}$
5. $\lambda_p \leftarrow \sum_{X_p} \psi_p \cdot \prod_{i=1}^{j} \lambda_{S_i}$
6. add λ_p to the bucket of the latest variable in S_p,
8. **return** $P(e) = \alpha = \sum_{X_1} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$

   return: $P(x_1|e) = \frac{1}{\alpha}\psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$

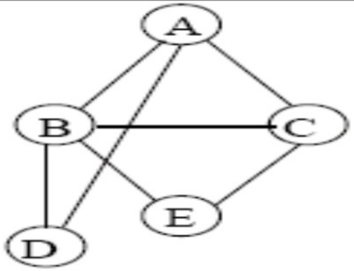
Figure 4.4: BE-bel: a sum-product bucket-elimination algorithm

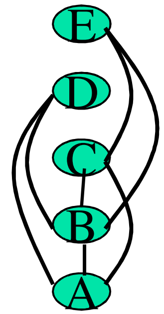
Student Network example



- P(J)?

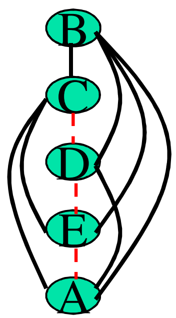
Bucket Elimination and Induced Width





Ordering: a, b, c, d, e

$bucket(E) = \quad P(e|b,c),\ e = 0$
$bucket(D) = \quad P(d|a,b)$
$bucket(C) = \quad P(c|a) \quad || \ P(e = 0|b,c)$
$bucket(B) = \quad P(b|a) \quad || \quad \lambda_D(a,b), \lambda_C(b,c)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_B(a)$



Ordering: a, e, d, c, b

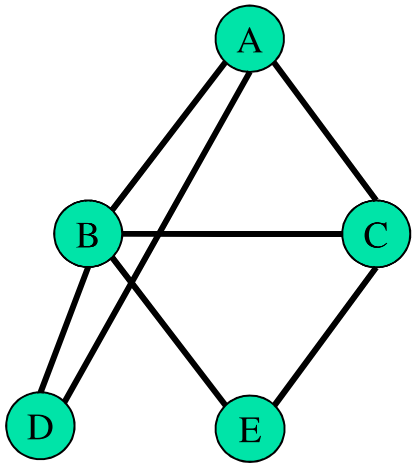
$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a)$
$bucket(C) = \quad P(c|a) \quad || \quad \lambda_B(a,c,d,e)$
$bucket(D) = \quad || \ \lambda_C(a,d,e)$
$bucket(E) = \quad e = 0 \quad || \ \lambda_D(a,c)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_E(a)$

Complexity of elimination

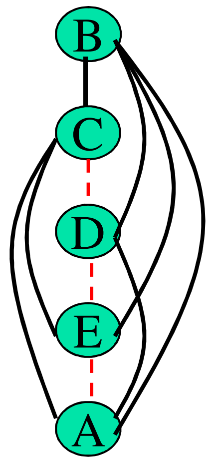
$$O(n\exp(w^*(d))$$

$w^*(d)$ – the induced width of moral graph along ordering d

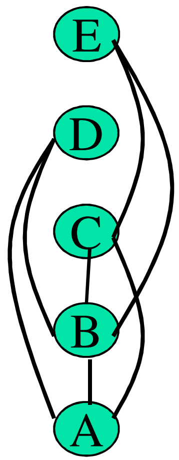
The effect of the ordering:



"Moral" graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

Complexity of bucket elimination

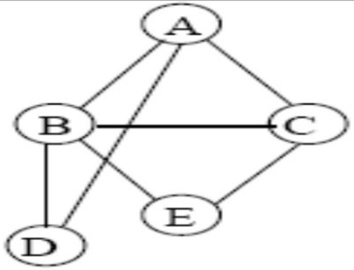
Theorem
Given a belief network having n variables, observations e, the complexity of BE-BEL along d, is time and space

$$O(n \cdot exp(w*(d))$$

where $w*(d)$ is the induced width of the moral graph whose edges connecting evidence to earlier nodes, were deleted.

More accurately: O(r exp(w*(d)) where r is the number of cpts.
For Bayesian networks r=n. For Markov networks?

Handling Observations



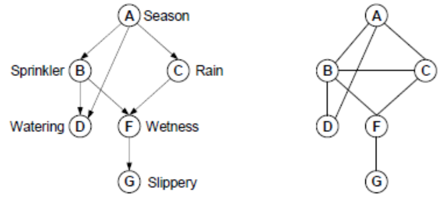
Observing $b = 1$

Ordering: a, e, d, c, b

$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a), b = 1$
$bucket(C) = \quad P(c|a), \quad || \quad P(e|b=1,c)$
$bucket(D) = \quad || \quad P(d|a,b=1)$
$bucket(E) = \quad e = 0 \quad || \quad \lambda_C(e,a)$
$bucket(A) = \quad P(a), \quad || \quad P(b=1|a) \ \lambda_D(a), \lambda_E(e,a)$

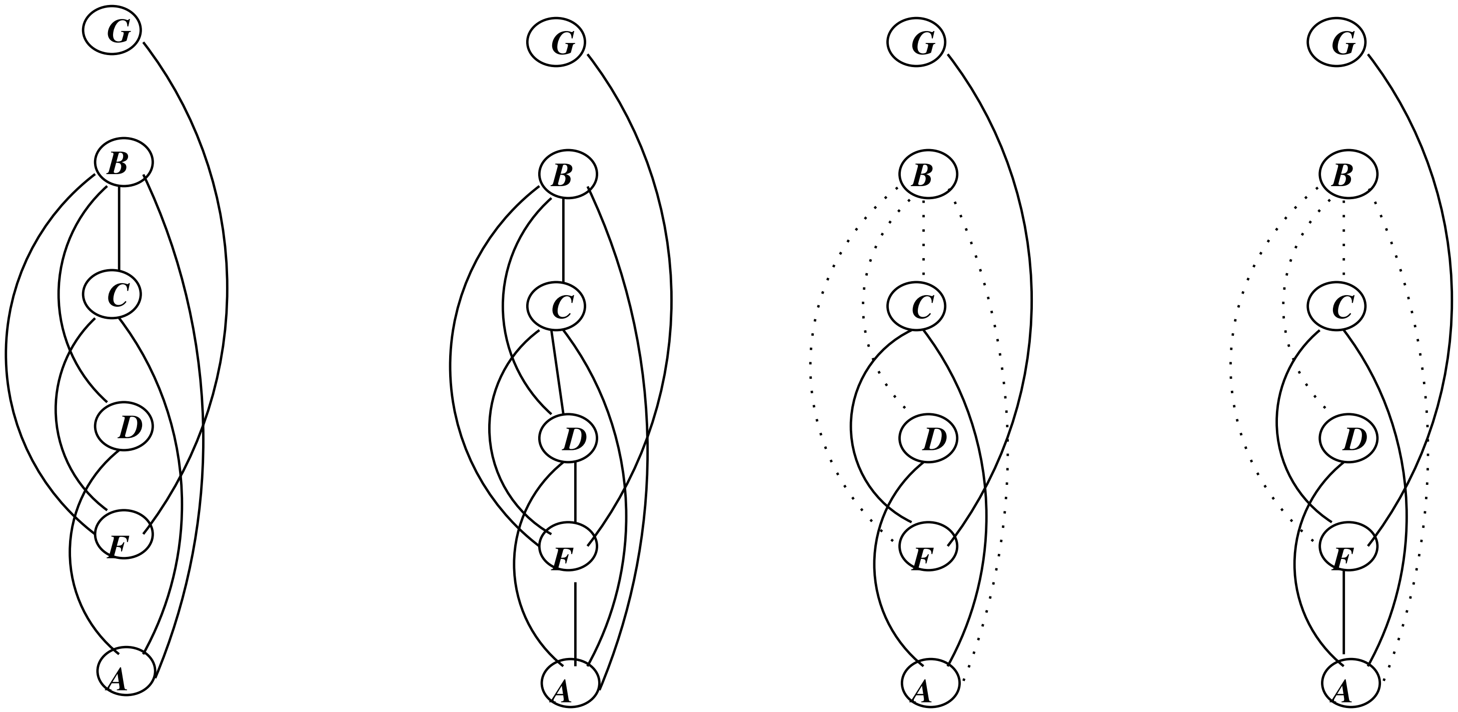
Ordering: a, b, c, d, e

$bucket(E) = \quad P(e|b,c), \ e = 0$
$bucket(D) = \quad P(d|a,b)$
$bucket(C) = \quad P(c|a) \quad || \quad \lambda_E(b,c)$
$bucket(B) = \quad P(b|a), b = 1 \quad || \quad \lambda_D(a,b), \lambda_C(a,b)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_B(a)$



(a) Directed acyclic graph (b) Moral graph

The impact of observations

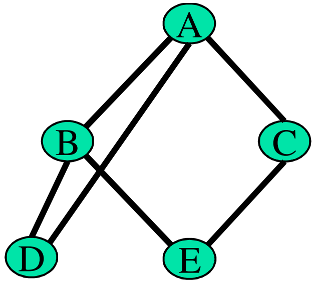


(a) (b) (c) (d)

Ordered graph

Induced graph

Ordered conditioned graph



"Moral" graph

Irrelevant buckets for

BE-BEL

Buckets that sum to 1 are **irrelevant**.
Identification: no evidence, no new functions.

Recursive recognition : ($bel(a|e)$)

$bucket(E) =$ $P(e|b,c),\ \ e = 0$
$bucket(D) =$ $P(d|a,b)$,...skipable bucket
$bucket(C) =$ $P(c|a)$
$bucket(B) =$ $P(b|a)$
$bucket(A) =$ $P(a)$

Complexity: Use induced width in moral graph without irrelevant nodes, then update for evidence arcs.

Use the ancestral graph only

Pruning Nodes

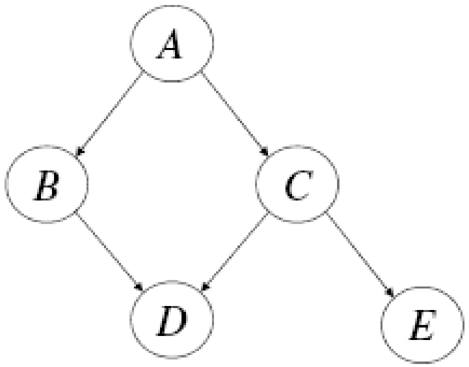
Given a Bayesian network $\mathcal{N}$ and query $(\mathbf{Q}, \mathbf{e})$

one can remove any leaf node (with its CPT) from the network as long as it does not belong to variables $\mathbf{Q} \cup \mathbf{E}$, yet not affect the ability of the network to answer the query correctly.

If $\mathcal{N}' = \mathrm{pruneNodes}(\mathcal{N}, \mathbf{Q} \cup \mathbf{E})$

then $\mathrm{Pr}(\mathbf{Q}, \mathbf{e}) = \mathrm{Pr}'(\mathbf{Q}, \mathbf{e})$, where Pr and Pr' are the probability distributions induced by networks $\mathcal{N}$ and $\mathcal{N}'$, respectively.

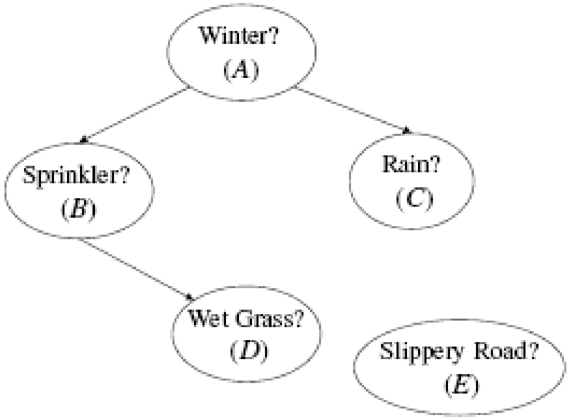
Pruning Nodes: Example



network structure   joint on B, E joint on B

Pruning Edges: Example

A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25



A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

A	Θ_A
true	.6
false	.4

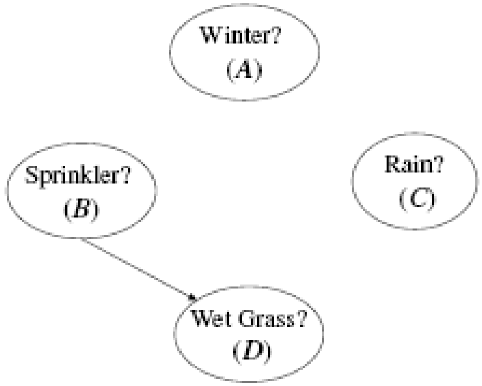
B	D	$\sum_C \Theta_{D\|BC}^{C=\text{false}}$
true	true	.9
true	false	.1
false	true	0
false	false	1

E	$\sum_C \Theta_{E\|C}^{C=\text{false}}$
true	0
false	1

Evidence $\mathbf{e}$: $C = \text{false}$

Pruning Nodes and Edges: Example

B	$\Theta'_B = \sum_A \Theta_{B\|A}^{A=\text{true}}$
true	.2
false	.8



C	$\Theta'_C = \sum_A \Theta_{C\|A}^{A=\text{true}}$
true	.8
false	.2

A	Θ_A
true	.6
false	.4

B	D	$\Theta'_{D\|B} = \sum_C \Theta_{D\|BC}^{C=\text{false}}$
true	true	.9
true	false	.1
false	true	0
false	false	1

Query $\mathbf{Q} = \{D\}$ and $\mathbf{e} : A = \text{true}, C = \text{false}$

Probabilistic Inference Tasks

- Belief updating:

$$\mathrm{BEL}(X_i) = P(X_i = x_i \mid \mathrm{evidence})$$

- Finding most probable explanation (MPE)

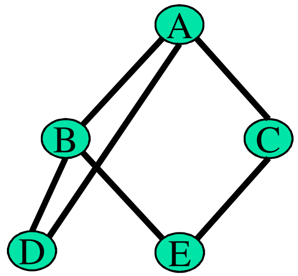
$$\bar{x}^* = \arg\max_{\bar{x}} P(\bar{x}, e)$$

- Finding maximum a-posteriory hypothesis

$$(a_1^*, \ldots, a_k^*) = \arg\max_{\bar{a}} \sum_{X/A} P(\bar{x}, e)$$

$A \subseteq X$:
hypothesis variables

Finding $MPE = \max_{\bar{x}} P(\bar{x})$

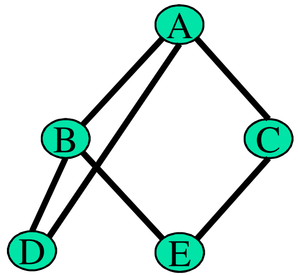


Algorithm *BE-mpe*

$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)P(d|a,b)P(e|b,c)$$

Finding $MPE = \max_{\bar{x}} P(\bar{x})$



Algorithm *elim-mpe* (Dechter 1996)

$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c \mid a)P(b \mid a)P(d \mid a,b)P(e \mid b,c)$$

$\max_b \prod$ ← Elimination operator

bucket B: $\underbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$

bucket C: $P(c|a)$ $\quad h^B(a, \underbrace{d, c, e})$

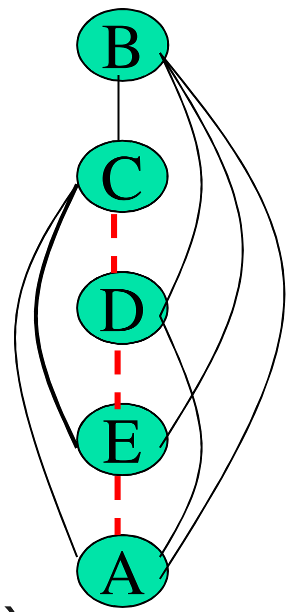
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $\quad h^D(a,e)$

bucket A: $P(a)$ $\quad h^E(a)$

MPE

W*=4
"induced width"
(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_c P(c \mid a') \times h^B(a', d', c, e')$

3. $d' = \arg\max_d h^C(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_a P(a) \cdot h^E(a)$

B: P(b|a) P(d|b,a) P(e|b,c)

C: P(c|a) $h^B(a,d,c,e)$

D: $h^C(a,d,e)$

E: e=0 $h^D(a,e)$

A: P(a) $h^E(a)$

Return (a', b', c', d', e')

Algorithm BE-mpe

Input: A belief network $\mathcal{B} =< X, D, G, \mathcal{P} >$, where $\mathcal{P} = \{P_1, ..., P_n\}$; an ordering of the variables, $d = X_1, ..., X_n$; observations e.

Output: The most probable assignment given the evidence.

1. **Initialize:** Generate an ordered partition of the conditional probability function, $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is X_i. Put each observed variable in its bucket. Let ψ_i be the input function in a bucket and let h_i be the messages in the bucket.

2. **Backward:** For $p \leftarrow n$ downto 1, do
for all the functions $h_1, h_2, ..., h_j$ in $bucket_p$, do

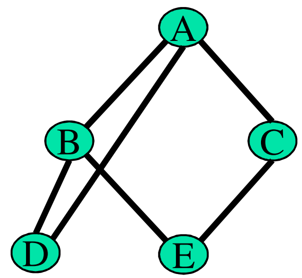
- **If** (observed variable) $bucket_p$ contains $X_p = x_p$, assign $X_p = x_p$ to each function and put each in appropriate bucket.
- else, $S_p \leftarrow \bigcup_{i=1}^{j} scope(h_i) \cup scope(\psi_p) - \{X_p\}$. Generate functions $h_p \Leftarrow \max_{X_p} \psi_p \cdot \Pi_{i=1}^{j} h_i$ Add h_p to the bucket of the largest-index variable in S_p.

3. **Forward:**

- Generate the mpe cost by maximizing over X_1, the product in $bucket_1$.
- (generate an mpe tuple)
  For $i = 1$ to n along d do: Given $\overline{x}_{i-1} = (x_1, ..., x_{i-1})$ Choose $x_i = argmax_{X_i} \psi_i \cdot \Pi_{\{h_j \in\ bucket_i\}} h_j(\overline{x}_{i-1})$

Finding MAP

Algorithm *BE-map*

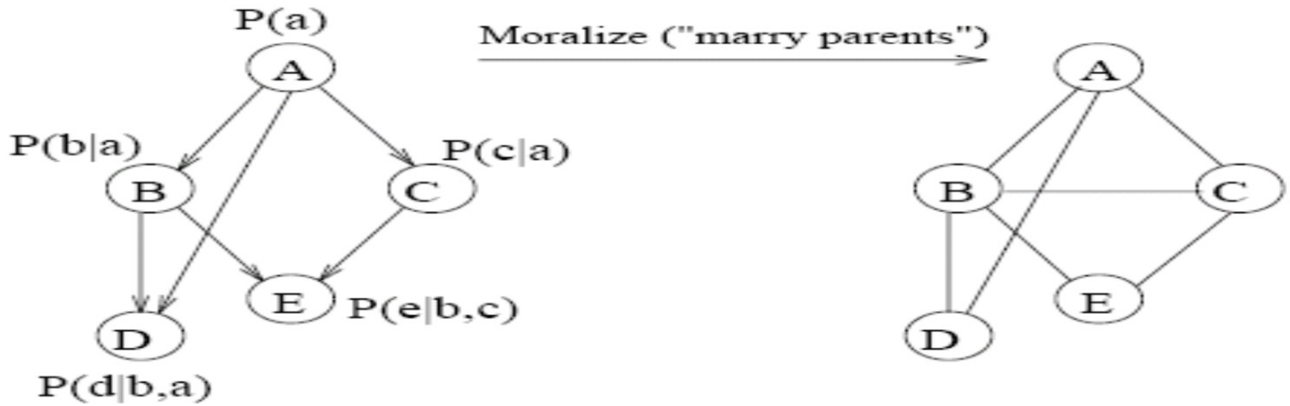


$$\sum \textbf{\textit{ and max :}}$$

$$MPE = \max_{a,c} P(a)P(c \mid a)\sum_{e,d,b} P(b \mid a)P(d \mid a,b)P(e \mid b,c)$$

Finding the MAP

(An optimization task)



Variables A and B are the hypothesis variables.

Ordering: $a, \ b, \ c, \ d, \ e$

$max_{a,b} P(a, b, e = 0) = \max_{a,b} \sum_{c,d,e=0} P(a, b, c, d, e)$
$= \max_a P(a) \max_b P(b|a) \sum_c P(c|a) \sum_d P(d|b, a)$
$\sum_{e=0} P(e|b, c)$

Ordering: $a, \ e, \ d, \ c, \ b$ .... illegal ordering
$\max_{a,b} P(a, e, e = 0) = \max_{a,b} \sum P(a, b, c, d, e)$
$\max_{a,b} P(a, b, e = 0) = \max_a P(a) \max_b P(b|a) \sum_d \cdot$
$\max_c P(c|a) P(d|a, b) P(e = 0|b, c)$

Algorithm BE-MAP

Algorithm BE-map

Input: A Bayesian network $\mathcal{B} = < X, D, G, \mathcal{P} > P = \{P_1, ..., P_n\}$; a subset of hypothesis variables $A = \{A_1, ..., A_k\}$; an ordering of the variables, d, in which the A's are first in the ordering; observations e. ψ_i is the input function in the bucket of X_i.

Output: A most probable assignment $A = a$.

1. **Initialize:** Generate an ordered partition of the conditional probability functions, $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is X_i.

2. **Backwards** For $p \leftarrow n$ downto 1, do

for all the message functions $\beta_1, \beta_2, ..., \beta_j$ in $bucket_p$ and for ψ_p do

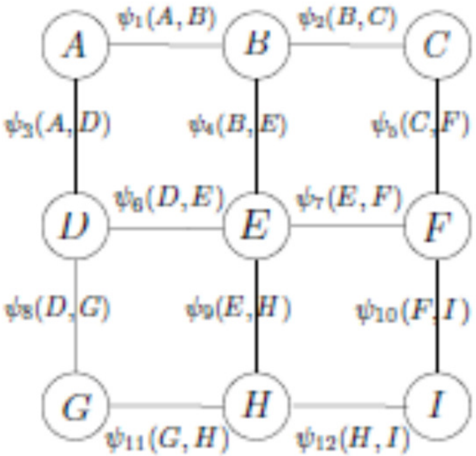
- **If** (observed variable) $bucket_p$ contains the observation $X_p = x_p$, assign $X_p = x_p$ to each β_i and ψ_p and put each in appropriate bucket.
- else, $S_p \leftarrow scope(\psi_p) \cup \bigcup_{i=1}^{j} scope(\beta_i) - \{X_p\}$. If X_p is not in A, then $\beta_p \Leftarrow \sum_{X_p} \psi_p \cdot \Pi_{i=1}^{j} \beta_i$;
  else, $(X_p \in A)$, $\beta_p \Leftarrow \max_{X_p} \psi_p \cdot \prod_{i=1}^{j} \beta_i$
  Place β_p in the bucket of the largest-index variable in S_p.

3. **Forward:** Assign values, in the ordering $d = A_1, ..., A_k$, using the information recorded in each bucket in a similar way to the forward pass in BE-mpe.

Variable ordering:
Restricted: Max buckets should
Be processed after sum buckets

Theorem 4.2.3 *Algorithm BE-map is complete for the map task for orderings started by the hypothesis variables. Its time and space complexity are are $O(r \cdot k^{w_d^*(E)+1})$ and $O(n \cdot k^{w_d^*(E)})$, respectively, where n is the number of variables in graph, k bounds the domain size and $w_d^*(E)$ is the conditioned induced width of its moral graph along d. (prove as an exercise.)* □

BE for Markov networks queries



(a)

D	E	$\psi_6(D,E)$
0	0	20.2
0	1	12
1	0	23.4
1	1	11.7

(b)

Complexity of bucket elimination

Theorem

Given a belief network having n variables, observations e, the complexity of elim-mpe, elim-bel, elim-map along d, is time and space

O(nexp(w*+1)) and O(n exp(w*)), respectively

where $w*(d)$ is the induced width of the moral graph whose edges connecting evidence to earlier nodes, were deleted.

More accurately: O(r exp(w*(d)) where r is the number of cpts.
For Bayesian networks r=n. For Markov networks?

Finding small induced-width

- NP-complete
- A tree has induced-width of ?
- Greedy algorithms:
  - Min width
  - Min induced-width
  - Max-cardinality
  - Fill-in (thought as the best)
  - See anytime min-width (Gogate and Dechter)

Type of graphs

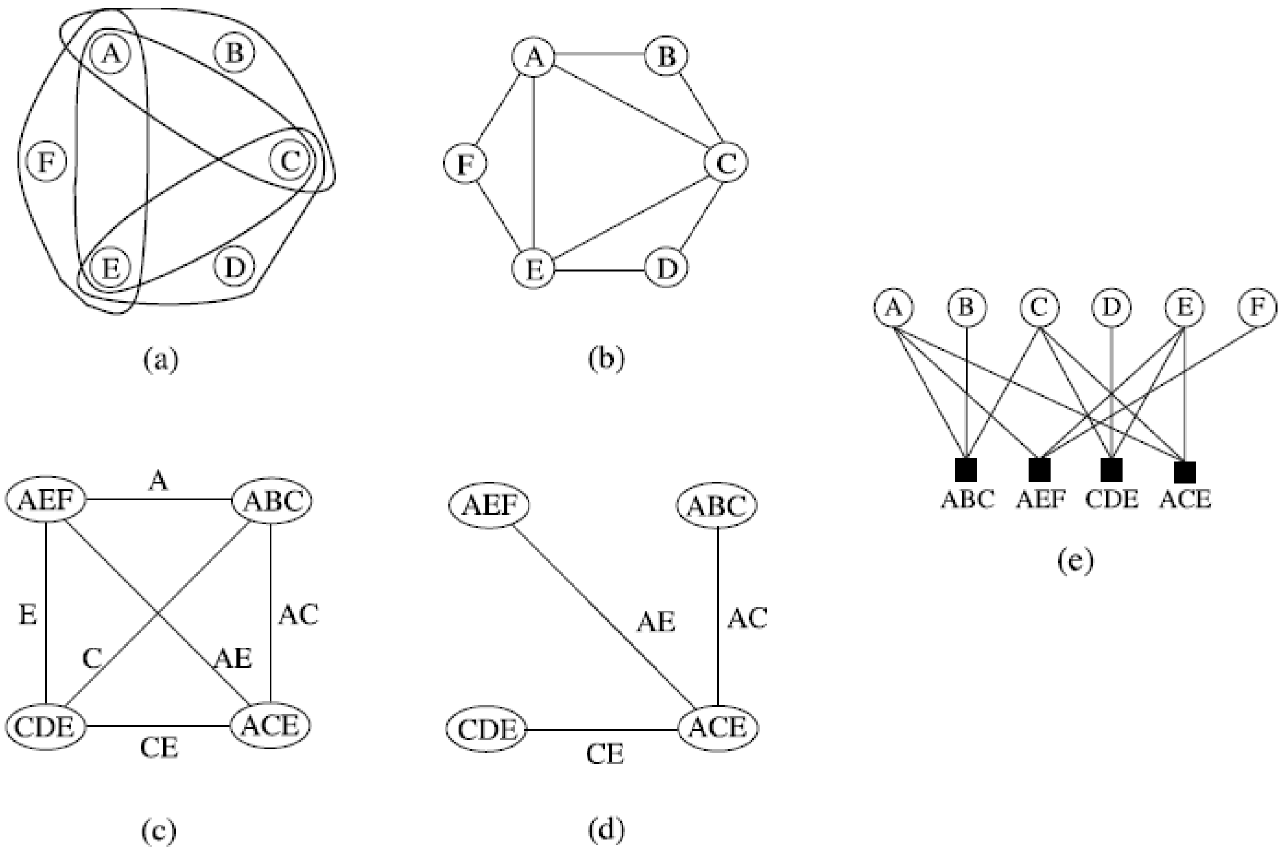


Figure 5.1: (a)Hyper, (b)Primal, (c)Dual and (d)Join-tree of a graphical model having scopes ABC, AEF, CDE and ACE. (e) the factor graph

The induced width

Definition 5.2.1 (width) *Given an undirected graph $G = (V, E)$, an* ordered graph *is a pair (G, d), where $V = \{v_1, ..., v_n\}$ is the set of nodes, E is a set of arcs over V, and $d = (v_1, ..., v_n)$ is an ordering of the nodes. The nodes adjacent to v that precede it in the ordering are called its* parents*. The* width of a node *in an ordered graph is its number of parents. The* width of an ordering *d of G, denoted $w_d(G)$ (or w_d for short) is the maximum width over all nodes. The* width of a graph *is the minimum width over all the orderings of the graph.*

Definition 5.2.3 (induced width) *The* induced width of an ordered graph *(G, d), denoted $w^*{}_d$), is the width of the* induced ordered graph *along d obtained as follows: nodes are processed from last to first; when node v is processed, all its parents are connected. The* **induced width** *of a graph, denoted by w^*, is the minimal induced width over all its orderings. Formally*

$$w^*(G) = \min_{d \in orderings} w^*{}_d(G)$$

Min-width ordering

MIN-WIDTH (MW)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$

output: A min-width ordering of the nodes $d = (v_1, ..., v_n)$.

1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in G with smallest degree.
3. put r in position j and $G \leftarrow G - r$.
   (Delete from V node r and from E all its adjacent edges)
4. **endfor**

Proposition: algorithm min-width finds a min-width ordering of a graph

Greedy orderings heuristics

MIN-INDUCED-WIDTH (MIW)

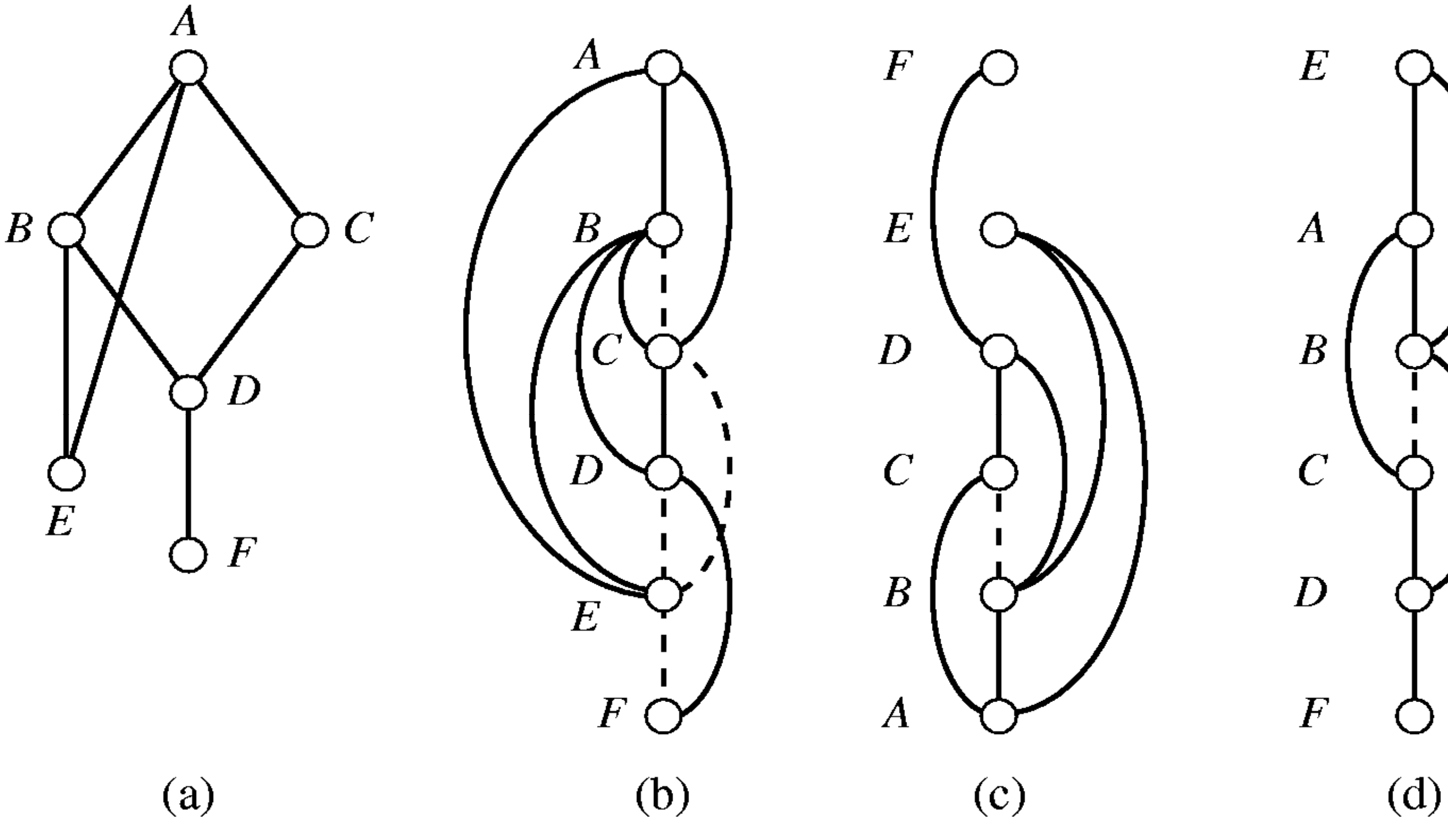
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in V with smallest degree.
3. put r in position j.
4. connect r's neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5. remove r from the resulting graph: $V \leftarrow V - \{r\}$.

Theorem: A graph is a tree iff it has both width and induced-width of 1.

MIN-FILL (MIN-FILL)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in V with smallest fill edges for his parents.
3. put r in position j.
4. connect r's neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5. remove r from the resulting graph: $V \leftarrow V - \{r\}$.

Different Induced-graphs



Induced-width for chordal graphs

- **Definition:** A graph is chordal if every cycle of length at least 4 has a chord
- Finding w* over chordal graph is easy using the **max-cardinality ordering:** order vertices from 1 to n, always assigning the next number to the node connected to a largest set of previously numbered nodes. Lets d be such an ordering
- A graph along max-cardinality order has no fill-in edges iff it is chordal.
- On chordal graphs width=induced-width.

Max-cardinality ordering

MAX-CARDINALITY (MC)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. Place an arbitrary node in position 0.
2. **for** $j = 1$ to n do
3. $r \leftarrow$ a node in G that is connected to a largest subset of nodes in positions 1 to $j - 1$, breaking ties arbitrarily.
4. **endfor**

Proposition 5.3.3 *[56] Given a graph $G = (V, E)$ the complexity of max-cardinality search is $O(n + m)$ when $|V| = n$ and $|E| = m$.*

What is the complexity of min-fill? Min-induced-width? $O(n^3)$

K-trees

Definition 5.3.4 (k-trees) *A subclass of chordal graphs are k-trees. A k-tree is a chordal graph whose maximal cliques are of size $k+1$, and it can be defined recursively as follows: (1) A complete graph with k vertices is a k-tree. (2) A k-tree with r vertices can be extended to $r+1$ vertices by connecting the new vertex to all the vertices in any clique of size k. A partial k-tree is a k-tree having some of its arcs removed. Namely it will clique of size smaller than k.*

Which greedy algorithm is best?

- MinFill, prefers a node who add the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)
- Complexity of greedy orderings?
  - MW is $O(n^2)$…maybe $O(nlogn + m\ \ )$?
  - MIW: O($O(n^3)$,
  - MF ($O(n^3)$,
  - MC is O(m+n), m edges.

Recent work in my group

- **Vibhav Gogate and Rina Dechter.** "A Complete Anytime Algorithm for Treewidth". *In UAI 2004.*
- **Andrew E. Gelfand, Kalev Kask, and Rina Dechter.** "Stopping Rules for Randomized Greedy Triangulation Schemes" in *Proceedings of AAAI 2011.*
- Potential project