A Linear Programming Approach to Max-sum Problem: A Review

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Max-Sum Problem

$$\max_{\mathbf{x} \in X^T} \left[\sum_{t \in T} g_t(x_t) + \sum_{\{t,t'\} \in E} g_{tt'}(x_t, x_{t'}) \right]$$

e.g. the MAP problem on MRFs

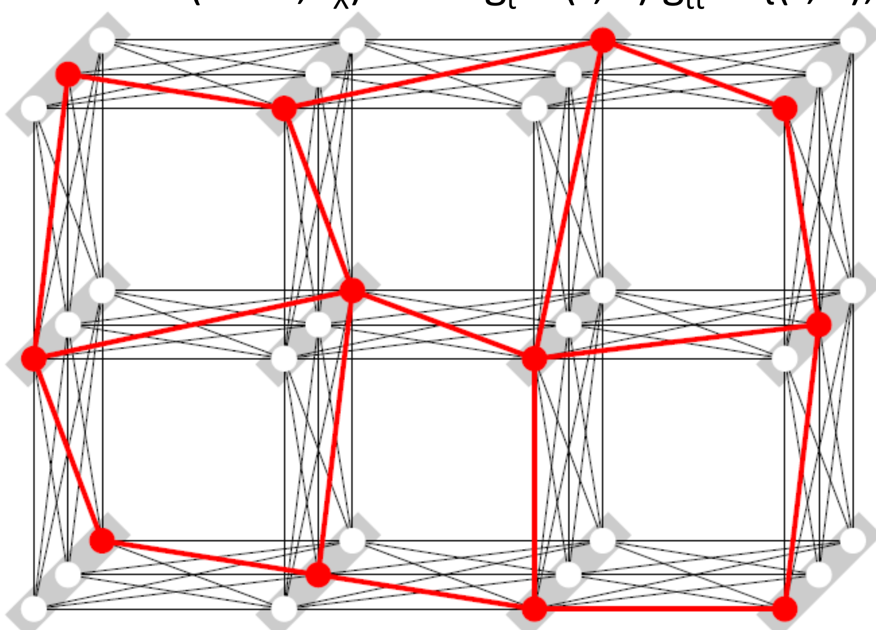
Formulation of the Problem

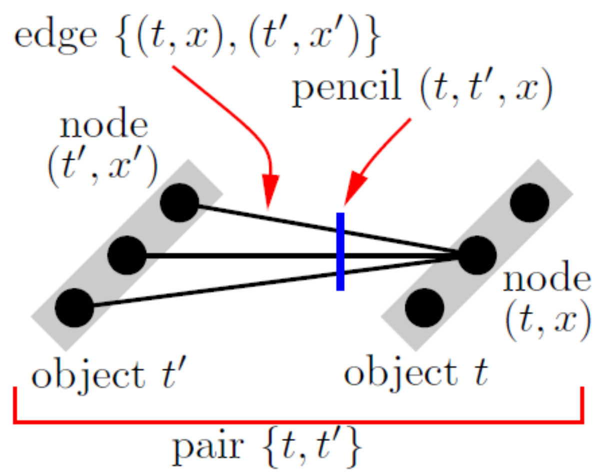
G = (T, E)

T is a set of objects, $x_t \in X$ is a labeling on t

$E \subseteq \binom{T}{2}$

G' = (T × X, E_X) g_t = (t, x) $g_{tt'}$ = {(t, x), (t', x'}



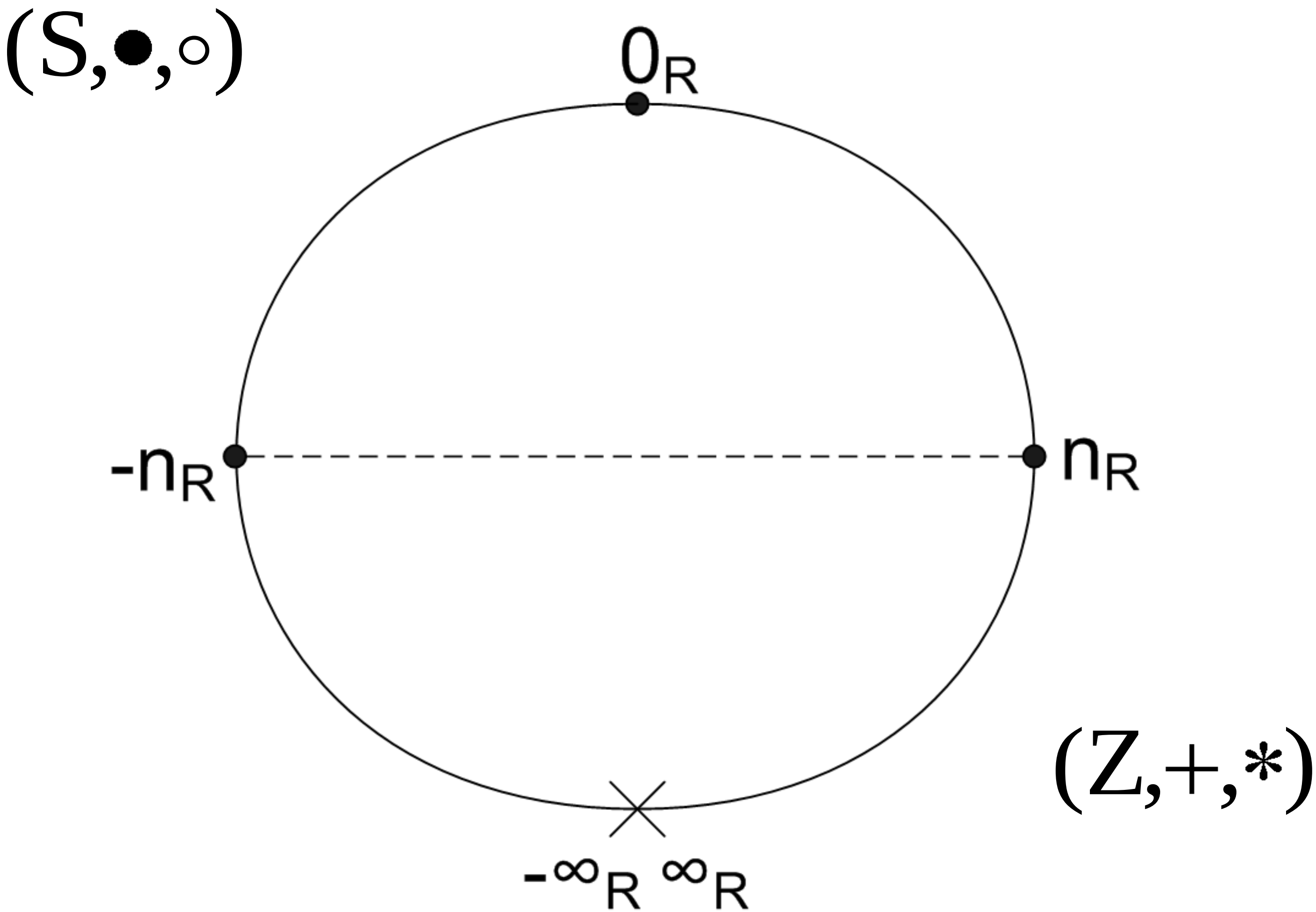


Commutative Semirings

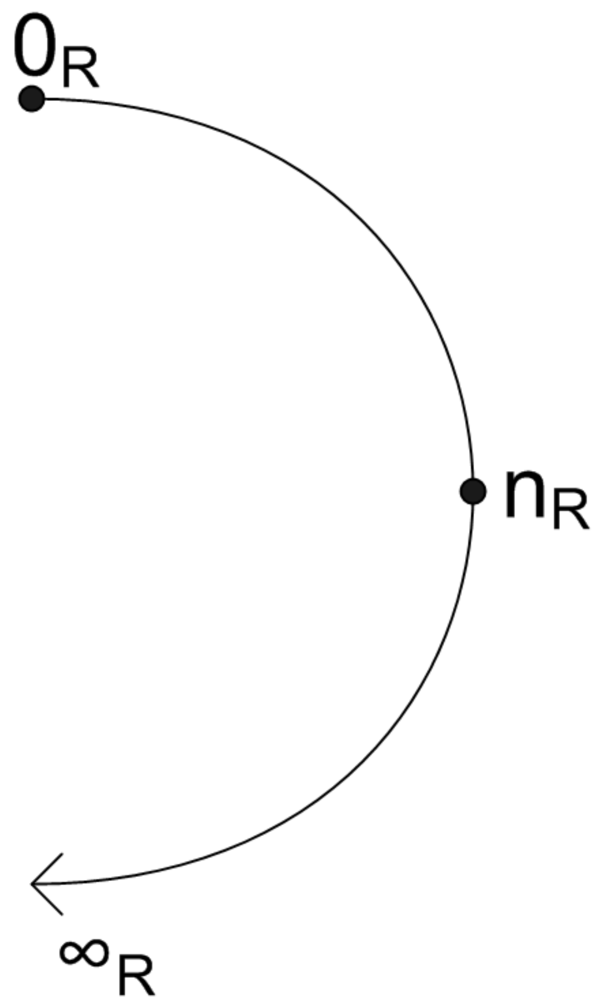
$$\bigoplus_{\mathbf{x}\in X^{|T|}} \left[\bigotimes_{t} g_t(x_t) \otimes \bigotimes_{\{t,t'\}} g_{tt'}(x_t, x'_t) \right]$$

$(S, \oplus, \otimes)$	task
$(\{0,1\}, \vee, \wedge)$	or-and problem, CSP
$([-\infty, \infty), \min, \max)$	min-max problem
$([-\infty, \infty), \max, +)$	max-sum problem
$([0, \infty), +, *)$	sum-product problem

Rings



Semirings



Semirings
CSP

Denote a problem by $(G,X,\bar{g})$ – Graph, Domain, Constraints
Let $\bar{g}_t(x)$, $\bar{g}_{tt'}(x,x') = \{0,1\}$ say if an assignment is allowed or forbidden

$$\bar{L}_{G,X}(\bar{\mathbf{g}}) = \left\{ \mathbf{x} \in X^T \;\middle|\; \bigwedge_{t} \bar{g}_t(x_t) \wedge \bigwedge_{\{t,t'\}} \bar{g}_{tt'}(x_t, x_{t'}) = 1 \right\}$$

Arc consistency in CSP

$$\bar{g}_{tt'}(x, x') \in \{0, 1\}$$

$$\bigvee_{x'} \bar{g}_{tt'}(x, x') = \bar{g}_t(x), \quad \{t, t'\} \in E,\ x \in X$$

The **kernel** can be obtained by iteratively applying the following relations until no more 0 assignments are made (arc consistency algorithm)

$$\bar{g}_t(x) := \bar{g}_t(x) \wedge \bigvee_{x'} \bar{g}_{tt'}(x, x'),$$

$$\bar{g}_{tt'}(x, x') := \bar{g}_{tt'}(x, x') \wedge \bar{g}_t(x) \wedge \bar{g}_{t'}(x')$$

Semirings
Max-sum

Denote a problem by (G,X,g) – Graph, Assignments, Weights

$$F(\mathbf{x} \mid \mathbf{g}) = \sum_{t \in T} g_t(x_t) + \sum_{\{t,t'\} \in E} g_{tt'}(x_t, x_{t'})$$

$$L_{G,X}(\mathbf{g}) = \operatorname*{argmax}_{\mathbf{x} \in X^T} F(\mathbf{x} \mid \mathbf{g})$$

Equivalent Transformations

Also known as ERs (Wainwright)

A problem is called equivalent if (G,X,g) and (G,X,g') produce the same problem, denoted as g~g'

The simplest such transformation adds a number $\phi_{tt'}(x)$ to $g_t(x)$ while removing from $g_{tt'}(x,x')$

This formulation corresponds to *potentials* or *messages* from message passing

$$g_t^{\varphi}(x) = g_t(x) + \sum_{t' \in N_t} \varphi_{tt'}(x),$$

$$g_{tt'}^{\varphi}(x, x') = g_{tt'}(x, x') - \varphi_{tt'}(x) - \varphi_{t't}(x')$$

Schlesinger's Upper Bound

$$u_t = \max_x g_t(x), \quad u_{tt'} = \max_{x,x'} g_{tt'}(x, x')$$

$$U(\mathbf{g}) = \sum_t u_t + \sum_{\{t,t'\}} u_{tt'}$$

$$U^*(\mathbf{g}) = \min_{\varphi \in \mathbb{R}^P} \left[\sum_t \max_x g_t^\varphi(x) + \sum_{\{t,t'\}} \max_{x,x'} g_{tt'}^\varphi(x, x') \right]$$

Triviality

(t,x) is a maximal node if $g_t(x) = u_t$
{(t,x), (t',x')} is a maximal edge if $g_{tt'}(x,x') = u_{tt'}$

$\bar{g}_t(x) = [[g_t(x) = u_t]]$ $\bar{g}_{tt'}(x) = [[g_{tt'}(x,x') = u_{tt'}]]$

A max-sum problem is **trivial** if a labeling can be formed of a subset of its maximal nodes and edges

Theorem 4. *Let C be a class of equivalent max-sum problems. Let C contain a trivial problem. Then, any problem in C is trivial if and only if its height is minimal in C.*

Triviality

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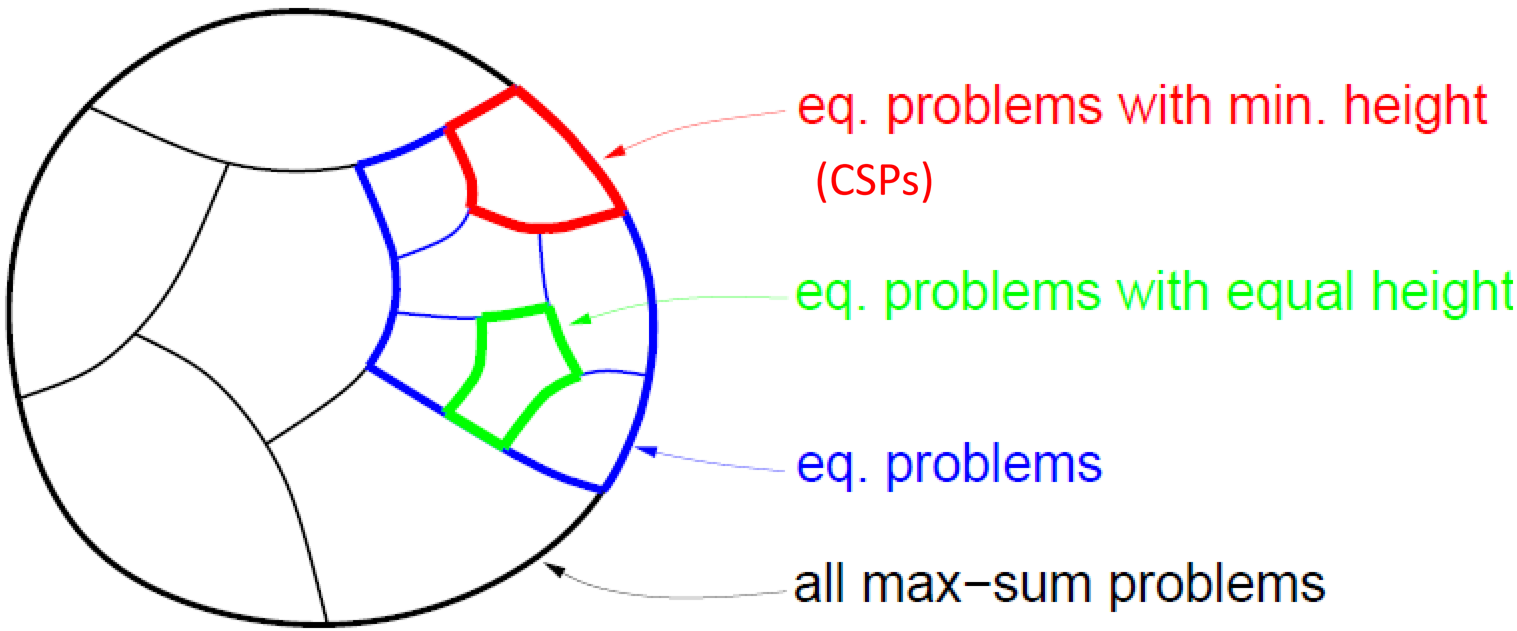
1. minimize the problem height by equivalent transformations and
2. test the resulting problem for triviality.

Testing for triviality of a max-sum problem is correspondent to solving the CSP generated by its maximal nodes and edges

A CSP is a tight solution to all max-sum problems it can be equivalently transformed into

Equivalent Transformations



Linear Programming Relaxation

$$\sum_{x'} \alpha_{tt'}(x, x') = \alpha_t(x), \quad \{t, t'\} \in E, \; x \in X,$$

$$\sum_{x} \alpha_t(x) = 1, \qquad t \in T,$$

$$\boldsymbol{\alpha} \geq \mathbf{0},$$

This gives the polytope $\Lambda_{G,X}$
which has a set of
optimal vertices given by $\Lambda_{G,X}(\mathbf{g}) = \underset{\alpha \in \Lambda_{G,X}}{\operatorname{argmax}} \langle \mathbf{g}, \boldsymbol{\alpha} \rangle$

$$\langle \mathbf{g}, \boldsymbol{\alpha} \rangle = \sum_{t} \sum_{x} \alpha_t(x)\, g_t(x) + \sum_{\{t,t'\}} \sum_{x,x'} \alpha_{tt'}(x, x')\, g_{tt'}(x, x')$$

Duality of the Relaxations

$$\langle \mathbf{g}, \boldsymbol{\alpha} \rangle \to \max_{\boldsymbol{\alpha}} \qquad \sum_{t\in T} u_t + \sum_{\{t,t'\}\in E} u_{tt'} \to \min_{\boldsymbol{\varphi},\mathbf{u}} \tag{11a}$$

$$\sum_{x'\in X} \alpha_{tt'}(x,x') = \alpha_t(x) \qquad \varphi_{tt'}(x) \in \mathbb{R}, \qquad \{t,t'\}\in E,\ x\in X \tag{11b}$$

$$\sum_{x\in X} \alpha_t(x) = 1 \qquad u_t \in \mathbb{R}, \qquad t\in T \tag{11c}$$

$$\sum_{x,x'\in X} \alpha_{tt'}(x,x') = 1 \qquad u_{tt'} \in \mathbb{R}, \qquad \{t,t'\}\in E \tag{11d}$$

$$\alpha_t(x) \geq 0 \qquad u_t - \sum_{t'\in N_t} \varphi_{tt'}(x) \geq g_t(x), \qquad t\in T,\ x\in X \tag{11e}$$

$$\alpha_{tt'}(x,x') \geq 0 \qquad u_{tt'} + \varphi_{tt'}(x) + \varphi_{t't}(x') \geq g_{tt'}(x,x'), \qquad \{t,t'\}\in E,\ x,x'\in X \tag{11f}$$

Theorem 5. *The height of $(G, X, \mathbf{g})$ is minimal of all its equivalents if and only if $(G, X, \bar{\mathbf{g}})$ is relaxed-satisfiable. If it is so, then $\Lambda_{G,X}(\mathbf{g}) = \bar{\Lambda}_{G,X}(\bar{\mathbf{g}})$.*

More theorems fall out

Theorem 6. *Let* $(G, X, \bar{\mathbf{g}}^*)$ *be the kernel of a CSP* $(G, X, \bar{\mathbf{g}})$. *Then,* $\bar{\Lambda}_{G,X}(\bar{\mathbf{g}}) = \bar{\Lambda}_{G,X}(\bar{\mathbf{g}}^*)$.

Theorem 7. *A nonempty kernel of* $(G, X, \bar{\mathbf{g}})$ *is necessary for its relaxed satisfiability and, hence, for minimal height of* $(G, X, \mathbf{g})$.

Finding the kernel does not guarantee finding a solution for the minimal upper bound

Obvious by approach from CSPs

For problems of boolean variables |X| = 2 finding the kernel is necessary and sufficient for finding the upper bound

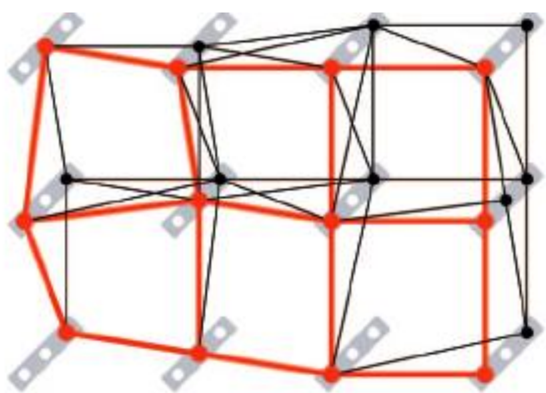
$$\begin{array}{c} \bar{\mathbf{g}} \text{ satisfiable} \\ \mathbf{g} \text{ trivial} \end{array} \Rightarrow \begin{array}{c} \bar{\mathbf{g}} \text{ relaxed–satisfiable} \\ \text{height of } \mathbf{g} \text{ minimal} \end{array} \Rightarrow \text{kernel of } \bar{\mathbf{g}} \text{ nonempty}$$

(Super) Submodularity

Known that the (super) submodularity property produces max-sum problems with tractable solutions by conversion to max-flow/min-cut problems

Has been suggested that supermodularity is the discrete counterpart of convexity. Lots of work shows that the LP relaxation for a supermodular max-sum problem is tight

Supermodular max-sum problems will always form a **lattice CSP** with a tractable solution



An application (not just theory!)