# On Minimal Tree-Inducing Cycle-Cutsets and Their Use in a Cutset-Driven Local Search

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**Abstract**

We prove that in grids of any size there exists a minimal cycle-cutset that its complement induces a single connected tree. More generally, any cycle-cutset in a grid can be transformed to a tree-inducing cycle-cutset, no bigger than the original one. We use this result to improve the known lower bounds on the size of a minimal cycle-cutset in some cases of grids, thus equating the lower bound to the known upper bound. In addition, we present a cycle-cutset driven stochastic local search algorithm in order to approximate the minimal energy of a sum of unary and binary potentials. We show that this method is on-par and even surpasses the state-of-the-art on some grid problems, when both are initialized by elementary means.

## 1 Introduction

A *cycle-cutset* in a graph $G = (V, E)$ is a subset $C$ of the vertices of $G$, such that the graph induced on $V' = V \backslash C$ is acyclic, i.e. a forest. *The minimal cycle-cutset problem* is finding a cycle-cutset of minimal cardinality. The problem is of interest to a wide variety of applications, including distributed computing and artificial intelligence in the context of Bayesian inference and constraint satisfaction. For example, the run-time of the method of conditioning for inference in Bayesian networks depends exponentially on the size of the cutset on which it is applies. Due to the importance of the problem, it has been extensively studied, although the problem was proven to be NP-complete for general graphs. Some of the findings include polynomial algorithms solving the problem for some specific graph classes and lower and upper-bounds on the size of the minimal cycle-cutset. In particular, Luccio [1] has previously presented lower and upper bounds on the size of the minimal cycle-cutset of grids. It was shown that the minimal cycle-cutset of the grid $M_{m,n}$ is of size at least

$$\frac{(m-1)(n-1)+1}{3}$$

and at most

$$\frac{mn}{3} + \frac{m+n}{6} + o(m,n)$$

Luccio's upper bound was later significantly improved by Madelaine and Stewart [2], who have shown an upper bound matching Luccio's lower bound in many cases and differing from Luccio's lower bound by at most 2, in other cases apart from when $m = 5$ and $n \geq 5$.

The rest of the paper is constructed as follows. In Section 2, we present the main definitions used throughout the paper. In Section 3, we show that the forest induced by a minimal cycle-cutset in grids can always be made into a single connected component, i.e a single tree. We then use this result in Section 4 in order to improve Luccio's lower bound in some cases. Thus matching the lower bound with upper bound of Madelaine and Stewart in those cases.

## 2 Preliminaries

**Definition 1** (Cycle cutset, Partition to trees)**.** Let $G = (V, E)$ be an undirected graph. A subset $C \subseteq V$ is a *cycle-cutset* in $G$ iff the graph $F$ induced by $V' = V \backslash C$ on $G$ is a forest. We define *the partition $T$ of $F$ to trees* as

$$T = \{t = (V_t, E_t) : \text{t is a connected component of F}\}$$

i.e. $T$ is a set of (connected) trees, and $V'$ can be written as $V' = \biguplus_{t \in T} V_t$.

**Definition 2** (Tree-degree )**.** Let $c \in C$ be a vertex in cycle-cutset $C$ of $G$ and $t \in T$ a tree induced by $C$, and denote by $N(c)$ the neighbors of $c$ in the graph induced by $V' \cup \{c\}$ on $G$, we define *the tree-degree of $c$ in $t$ over $C$* to be d, if $|N(c) \cap V_t| = d$ and for every other tree $t \neq t' \in T$ it holds that $|N(c) \cap V_{t'}| \leq 1$. In general, we say that the tree-degree of c is d if the condition above holds for some t, and that the tree-degree is undefined otherwise. If $d \geq 2$ we call $N(c) \cap V_t$ the *in-tree neighbors of c.*

Given a cutset $C$ inducing a forest with more than one connected component, we would like to replace some of the vertices of $C$ with other vertices, or remove some vertices altogether, so to receive a new cutset $C'$ that induces a single tree.

It is easy to see that every cutset vertex of tree-degree equal or less than 1, can be removed from the cutset while maintaining its validity.

let $c \in C$ be a cutset vertex of tree-degree 2 in $t \in T$, and let $u, v \in N(c) \cap V_t$ be its two in-tree neighbors, then it easy to see that there exists a (unique) path from u to v in t, and that by removing c from the cutset, a single cycle is formed in the cutset-induced graph. Thus, it is obvious that adding any vertex in that cycle to the cutset thereafter will produce a valid cutset again. In conclusion, any cutset vertex of tree-degree 2 can be replaced by any vertex along the (unique) path between its two in-tree neighbors while maintaining the cutset's validity.

Let $c \in C$ be a cutset vertex of tree-degree 3 in $t \in T$, and let $u, v, w \in N(c) \cap V_t$ be its three in-tree neighbors, then there exists a (unique) path in t between each pair of in-tree neighbors of c. The intersection of all three paths corresponding to possible pairings is a single vertex $c' \in T$, and therefore it is the only vertex (in F) that c can be replaced with while maintaining the cutset's validity.

We note that for cutset vertices of tree-degree greater than 3, the intersection of all paths between the in-tree neighbors may be empty, and therefore these cutset vertices would be irreplaceable. Additionally, cutset vertices of undefined tree-degree are necessarily irreplaceable.

**Definition 3** (Equivalent cutset vertices)**.** Let $c \in C$ be a cutset vertex and let $c' \in V$ be a vertex of the graph $G$. We say that $c'$ is *equivalent to c (under C),* if $C \backslash \{c\}$ is not a cutset, while $(C \backslash \{c\}) \cup \{c'\}$ is a cutset.

Let c be a cutset vertex of tree-degree d, and let $c'$ be an equivalent vertex, i.e. $c$ can be replaced by $c'$ while maintaining the cutset's validity. It should be noted that the tree-degree of $c'$ is d as well (in the forest induced by $V' \cup \{c', c\}$). Moreover, if the degree of c in the graph $H$ induced by $V' \cup \{c\}$ is p (note that $p \geq d$), then the number of connected components in $H$ is smaller by $p - d$ from the number of connected components in the forest F induced by $V'$. Consequently, it can be shown that a forest F is connected iff for each vertex $c$ in the inducing cutset C the tree-degree of c is defined and is equal to its degree in the graph induced by $V' \cup \{c\}$. As a result, given a minimal cutset C, if there exists a sequence of replacement moves, such that at each step a cutset vertex is replaced with an equivalent vertex, whose induced-degree is equal to its tree-degree, a valid cutset may be produced, which is minimal as well as induces a connected tree.

**Definition 4** (Induced degree)**.** Let $c \in C$ be a cutset vertex. The *induced degree of $c$ under $C$* is the degree of $c$ in the graph induced by $V' \cup \{c\}$.

We summarize our previous observations in the following lemma

**Lemma 1.** *(a) If c is cutset vertex of tree-degree 2, then every vertex along the path its two in-tree neighbors are of equivalent to c.*

*(b) If c is a cutset vertex of tree-degree 3, then there exists a unique vertex $c'$ which is equivalent to c.*

*(c) A cutset induces a single tree iff the tree-degree of every cutset vertex is equal to its induced degree.*

**Definition 5** (Boundary of a tree)**.** Let $t \in T$ be a tree, we define *the boundary of t* to be all cutset vertices touching t and some other tree, and denote it B(t), i.e

$$B(t) = \{c \in C : N(c) \cap V_t \neq \emptyset, \dots \\ \dots \exists t \neq t' \in T, \, s.t. \, N(c) \cap V_t \neq \emptyset\}$$

Note that for any tree t and vertex $c \in B(t)$ the tree-degree of c is either undefined or strictly smaller than its induced-degree.

**Definition 6** (The $n \times m$ grid graph)**.** Let $m, n \geq 2$. T*he* $n \times m$ *grid graph* $M_{n,m}$ is an undirected graph whose vertex set is $V(M_{n,m}) = \{v_{i,j} : 0 \leq i < n, 0 \leq j < m\}$ and the edge set $E(M_{n,m})$ is defined by

$$E(M_{n,m}) = \{(v_{i,j}, v_{i+1,j}) : 0 \leq i < n-1, 0 \leq j < m\} \\ \cup \{(v_{i,j}, v_{i,j}) : 0 \leq i < n, 0 \leq j < m-1\}$$

# 3 Connectivity of the Induced Graphs for Grids

**Theorem 1.** *Let G be a grid graph and let C be a cutset such that the induced forest F is disconnected, i.e. $|T| \geq 2$, then there exists a a series of replacement moves, such that the resulting cutset $C'$ has no more elements than C, and the forest induced by $C'$ contains a single tree, i.e. $|C'| \leq |C|$ and the graph induced by $V \backslash C'$ is connected.*

**Corollary 1.** *In particular, it follows from Theorem 1 that there exists a minimal cutset that induces a connected tree.*

In the following we will prove this proposition using two lemmas.

**Lemma 2.** *Let G be a grid graph and let C be a cutset of G, that the induces a disconnected forest F, i.e. $|T| \geq 2$. Let $t \in T$ be a connected component of F. If there exists a vertex $c \in B(t)$ in the boundary of t with an tree-degree of 2, then c can be replaced with a vertex $c'$, that has a degree of 2 in the graph induced by $V' \cup \{c, c'\}$, thus reducing the number of connected components in the induced forest.*

***Proof of Lemma 2.*** As observed before, $c$ can be replaced with any of the vertices along the path from its two in-tree neighbors. If there exists such a vertex $c'$ with induced-degree 2, then replacing $c$ with $c'$ would reduce the number of connected components in the induced graph. To be exact, if the induced-degree of $c$ is $p$, then the number of connected components would decrease by $p-2$.

We will show that there exists such a vertex $c'$ equivalent to $c$ of induced degree 2. Let $u$, $v$ be the two in-tree neighbors of $c$, and assume to contrary that the degrees of all vertices along the path from $u$ to $v$ in the graph induced by $V' \cup \{c\}$ are equal or greater than 3.

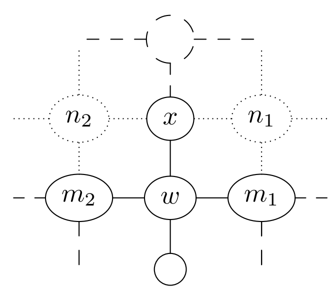


Figure 1: Neighborhood of a vertex $w$ of degree 4. Unbroken lines represent vertices and edges in the forest, dotted lines represent vertices and edges in the cutset, and dashed lines represent vertices and edges of unknown status

Assume that there exists a vertex $w$ along the path of degree 4. (refer to Figure 1, showing only the neighborhood of $w$, not necessarily including neither $u$ nor $v$). One promptly notes that it follows that the following vertex along the path, denoted by $x$ (or the previous one, in case $w = v$) must be of degree 2, in contradiction to the negated assumption. Otherwise, $x$ is of degree at least 3 and either $n_1$ or $n_2$ must be in $V'$. Assume w.l.o.g. that $n_1 \in V'$, then the graph induced by $V'$ contains the cycle $w$, $x$, $n_1$, $m_1$, in contradiction to the assumption that C is a cutset.

Therefore, assume that all vertices along the path have a degree of 3 in the graph induced by $V' \cup \{c\}$. One notes that the path must not bend, i.e. all the path's vertices must lie on a straight line (in contradiction to the assumption that they form a path from $u$ to $v$). Assume to contrary that the path bends at vertex $y$, namely assume w.l.o.g. that the previous vertex $x$ lies below $y$ and that the following vertex $z$ lies to right of $y$ (see Figure 2). It follows that $x$'s additional neighbor $r$ (not on the path) must lie to its left (or it will form a cycle $r, x, y, z$), and similarly $z$'s additional neighbor $s$ must lie above it. therefore, the introduction of $y$'s additional neighbor at each of the possible positions $p_1$ and $p_2$ would result in a cycle either with $x$ and $r$, or with $z$ and $s$ - a contradiction.

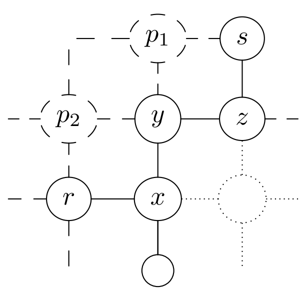


Figure 2: Neighborhood of a bend in a path

It follows that if the boundary of t contains a vertex c of tree-degree of 2, then c can be replaced with an equivalent vertex while reducing the number of connected components in the induced graph. ■

**Lemma 3.** *Let $G$ be a grid graph and let $C$ be a cutset of $G$, that induces a disconnected forest $F$, i.e. $|T| \geq 2$. if there does not exist a tree $t \in T$ and a vertex $c \in B(t)$, having tree-degree (defined and) is equal or less than 2, then:*

***a.*** *for every tree $t \in T$, there exist at least 2 vertices in the boundary of t of tree-degree 3.*

***b.*** *There exists a series of replacement moves, such that the forest induced by the final cutset $C'$ is composed of less connected components than the original forest.*

Note that the conditions of this lemma are the complementary of the conditions of lemma 1 (ignoring cutset vertices of tree-degree less than 2, which can be trivially removed).

To prove this lemma we will first show that a vertex in a grid can be equivalent to at most 2 cutset vertices, in which case the topology in the neighborhood of the vertex must be of a specific form. To do so, we first set to prove the following general claim.

**Lemma 4.** *Let $G = (V, E)$ is a planner graph, $v \in V'$ a vertex equivalent to cutset vertices $c_1, \ldots, c_k \in C$, which are of tree-degree 3, and let $d$ be the degree of $v$ in the graph induced by $V' \cup \{c_i\}_{i=1}^{k}$, then $d \geq k + 2$.*

***Proof of Lemma 4*** . Let $c \in C$ be a cutset vertex of tree-degree 3 and let $v \in V'$ be its (unique) equivalent vertex. Consider Figure 3 and note that the three paths $p_1$, $p_2$, $p_3$ from $c$ to $v$ in the graph induced by $V' \cup \{c\}$ partition the plane to three parts, denoted as $A$, $B$ and $C$ in Figure 3:

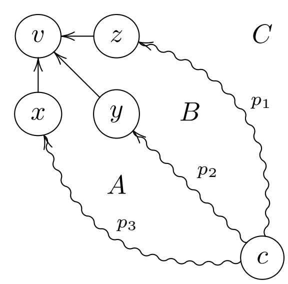


Figure 3: A vertex equivalent to a cutset vertex of tree-degree 3 in a planner graph. A solid line marks a single edge, and a squiggly line marks a path of arbitrary length

Assume that there exists another cutset vertex $c' \in C$ of tree-degree 3, such that its equivalent vertex is $v$ as well, and assume w.l.o.g that $c'$ lies in section A. In addition, assume to the contrary of the lemma's claim that $d < 4$. Then, $x$, $y$ and $z$ are the only neighbors of $v$ in the graph induced by $V' \cup \{c, c'\}$. Since $v$ is equivalent to $c'$, there must be three paths from $c'$ to $v$ in the graph $H$ induced by $V' \cup \{c'\}$. We note that intersection of either two of these three paths is only $c'$ and $v$, as otherwise, there is a path between two in-tree neighbors of $c'$ which does not pass through $v$, in contradiction to the assumption that $v$ is equivalent to $c'$. As $x$, $y$ and $z$ are the only neighbors of $v$ it follows that there must be a path from $c'$ to $v$ in $H$ passing through $z$, but since $G$ is a planner graph, every path from $c'$ to $z$ in $H$ not passing through $v$ first must intersect with either $p_2$ or $p_3$. Assume w.l.o.g that the path from $c'$ to $v$ through $z$ intersects with $p_2$ at vertex $t$, then there exist two paths from $t$ to $v$ - one passing through $z$ and the other passing through $y$, thus forming a cycle in the graph induced by $V'$, in contradiction to the assumption that it is a forest. Therefore, there is no path from $c'$ to $z$ in $H$ not passing through $v$. As there must be three paths from $c'$ to $v$, it follows that $v$ must have (at least) one other neighbor in H, through which a third path from $c'$ to $v$ must pass.

The general claim follows by induction. ■

In the context of grids, one may use the fact that the maximal degree of a vertex in a grid is 4 along with the previous lemma, and conclude that every vertex $v \in V'$ is equivalent to at most 2 cutset vertices of tree-degree 3. A closer inspection of the previous proof shows that in case a vertex $v \in V'$ is indeed equivalent to 2 cutset vertices $c$ and $c'$ of tree-degree 3, then the induced graph must be of the following general cycle topology (focused on the relevant vertices):

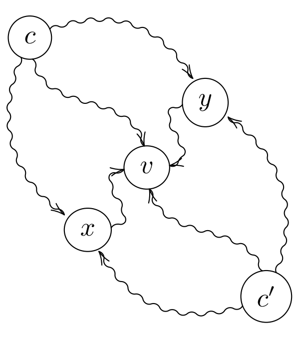


Figure 4: Topology of a vertex equivalent of 2 vertices of tree-degree of 3

Equipped with the previous observations we are now ready to prove lemma 3.

***Proof of Lemma 3.*** **a.** Let $t \in T$ be a tree and let $c \in B(t)$ be a cutset vertex on the boundary of $t$, such that there does not exist a vertex $c' \in B(t)$, which is higher than $c$ (i.e. with a higher y-coordinate value). Since $c$ is on the boundary of $t$ it touches at least 2 trees, and from the assumption on the tree-degree of the cutset vertices in the graph, it follows that it touches exactly 2 trees, as otherwise it would have tree-degree equal or less than 2. Let $s \in T$ be the second tree touching $c$. In addition, it should be noted that $c$ does not lie on the edges of the grid, as vertices there have a degree equal or less than 3, and therefore must have an tree-degree of 2 or less if they are boundary vertices. Denote by $N(c)$ the neighbors of c in $G$, then for a similar reason it holds that $C \cap N(c) = \emptyset$, i.e. no neighbor of $c$ is a cutset node. (refer to diagram 5).

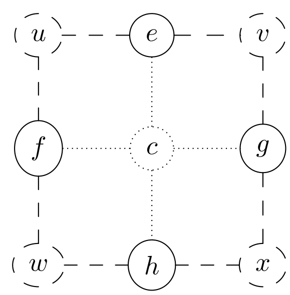


Figure 5: Neighborhood of a an extremal cutset vertex. Unbroken lines represent vertices and edges in the forest, dotted lines represent vertices and edges in the cutset, and dashed lines represent vertices and edges of unknown status.

Assume that vertex $f$ to the left of $c$ belongs to $t$, then $e$ must belong to $t$ too, since if $u$ is not a forest vertex then $e$ is connected to $f$. Otherwise, $u$ is a cutset vertex, and if $e \in s$ then $u$ is a cutset vertex in the boundary of $t$, which is higher than $c$, in contradiction to the assumption. It can be shown similarly that if $e \in t$ then $g \in t$, and that if $g \in t$ then $f \in t$. All in all we get that if either one of $e$, $f$ or $g$ belongs to $t$ than all three belong to $t$. This shows as well that if $h \in t$ then $e, f, g \notin t$, since as shown before, if either on of $e$, $f$ or $g$ belongs to $t$ all of them belong to $t$ and along with the assumption that $h \in t$ we get that $e, f, g, h \in t$, in contradiction to the assumption that $c$ is a boundary vertex. It follows that either $|N(c) \cap V_t| = 1$ and $|N(c) \cap V_s| = 3$, or the other way around. In both cases, it follows that $c$ has an tree-degree of 3 (either in $s$ in the former case or in $t$ in the latter). Similarly, it can be shown that any vertex $c_2 \in B(t)$, such that there does not exist a boundary vertex $c' \in B(t)$, which is lower than $c_2$, has an tree-degree of 3.

**b.** Let $c \in B(t)$ be a vertex touching trees $t$ and $s$ $(t, s \in T)$ with tree-degree 3 in $s$, then it can be replaced by an equivalent node $v \in V'$. if $v$ is of degree 3 in the graph induced by $V' \cup \{c\}$, then this replacement reduces (by 1) the number of connected components of $F$. Otherwise, the number of connected components after the replacement remains the same as before. Denote by $T'$ the partition of the forest induced by $(V' \cup \{c\}) \setminus \{v\}$ to trees. If there exist a tree $t' \in T'$ and a vertex $c' \in B(t')$ of tree-degree equal or less than 2, then we return to the previous situation. Otherwise, denote by $N(v)$ the neighbors of $v$ in the graph induced by $V' \cup \{c, v\}$, and let $t' \in T'$ be the tree such that $|N(v) \cap V_{t'}| = 1$, then from the preceding claims it follows that there must exist another cutset vertex $v \neq u \in B(t')$ with tree-degree 3 in the graph induced by $(V' \cup \{c\}) \setminus \{v\}$. As before, $u$ can be replaced with a unique vertex $w \in (V' \cup \{c\}) \setminus \{v\}$, and this process continues as long as there does not exist a boundary vertex of tree-degree less than 3. As the graph is finite and each vertex of tree-degree 3 can be replaced by a unique vertex, this process ought to stop or a vertex $x$ that was previously removed from the cutset will be added to it again. Since at each step the vertex removed is different from the one added in the previous step, the latter condition can only occur if $x$ is equivalent to 2 cutset vertices $q$ and $p$, and $x$ was added to the cutset instead of $q$, while on a previous step, $p$ was added instead of $x$. Let $C$ be the cutset before $x$ was added instead of $q$, then as can be seen from Figure 4, two of $q$, $p$ and $x$ must be cutset vertices, and in two of these configurations one of the cutset vertices is of tree-degree 2, we will denote this vertex by $y$. (In Figure 4, in each of the configurations where $v$ is a cutset vertex, the other cutset vertex has an tree-degree of 2). As seen before, since $y$ has an tree-degree of 2, it is guaranteed that it can be replaced with a vertex with induced degree of 2, thus reducing number of connected components in the induced forest. ■

***Proof of Theorem 1.*** Using lemmas 2 and 3 the main theorem follows immediately: Let $G$ be a grid graph and let $C$ be a cutset such that the induced forest $F$ is disconnected. Assume w.l.o.g. that $C$ does not contain vertices of tree-degree less than 2. If there exists a tree $t \in T$ and a boundary vertex $c \in B(t)$ of tree-degree 2, then lemma 2 shows that it can be replaced with another vertex while reducing the number of connected components. Otherwise, there does not exist a tree $t \in T$ and a boundary vertex $c \in B(t)$ of tree-degree 2, and therefore lemma 3 shows that there exists a finite series of replacement moves, such that the forest induced by the resulting cutset contains less connected components than the original forest. ■

All in all, we see that for every cutset that induces a forest with more than one connected component a series of replacement moves can be made, which reduces the number of connected components in the induced forest while not adding cutset nodes. As this can be done as long as the induced forest is disconnected, we see by induction that given a cutset $C$, there exists a cutset $C'$, such that $|C'| \leq |C|$ and the forest induced by $C'$ contains only one connected component, i.e. the induced forest is a single tree. In particular, if the initial cutset $C$ is minimal, then the resulting cutset $C'$ is minimal as well.

# 4 Improved Lower Bounds

We will use the results of the previous section in order to improve the known lower bound on the size of the minimal cutset of the grid $M_{n,m}$. In particular, we will show that our lower bound is equal to the upper bound in these seleceted cases. In the following we denote by

$olb_{n,m}$ the old lower bound of [1], i.e.

$$olb_{n,m} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$$

and by $nlb_{n,m}$ the new lower bound obtained by us.

**Lemma 5.** *Let $G := M_{n,m} = (V, E)$ be the $n \times m$ grid graph, and $C \subseteq V$ a cycle-cutset, such that the graph $T = (V_T, E_T)$ induced by $V' = V \backslash C$ is a single tree. Denote by $\alpha$ the number of cutset vertices which lie along the boundaries of the grid but not in its corners, i.e.*

$$\alpha = |C \cap (\{(0,j), (n-1,j) : 0 < j < m-1\} \cup \{(i,0), (i,m-1) : 0 < i < n-1\})|$$

*and by $\beta$ the number of cutset vertices which lie in the corner of the grid, i.e. $\beta = |C \cap \{(0,0), (0,m-1), (n-1,0), (n-1,m-1)\}|$. Denote by $n_C$ the cardinality of $C$, and by $p$ the number of connected components of the graph induced by $C$ (not $V \backslash C$). Then it holds that*

$$(n-1)(m-1) + \alpha + 2\beta \leq 2n_C + p \qquad (1)$$

***Proof of Lemma 5.*** Denote $n_T = |V_T|$. If every vertex of $T$ is of degree 4 in $G$, then it can be easily shown that the number of edges incident to $T$ from a vertex in $C$ is $2n_T + 2$. Since there are $2(n-2) + 2(m-2) - \alpha$ vertices of $T$ which lie along the boundaries of $G$, each of which reducing the number of incident edges by 1 (as each of these vertices is only of degree 3 in $G$), and $4 - \beta$ vertices of $T$ which lie in the corners of $G$, each of which reducing the number of incident edge by 2, we get that the number of edges incident to $T$ from $C$ is

$$\begin{aligned} r &:= 2n_T + 2 - [2(n-2) + 2(m-2) - \alpha] - 2(4-\beta) \\ &= 2n_T - 2n - 2m + \alpha + 2\beta + 2 \end{aligned}$$

It can be shown similarly, that if all the connected components of $C$ are trees, then the number of edges incident to $C$ from $T$ is

$$s := 2n_C + 2p - \alpha - 2\beta$$

and if not all the connected components of $C$ are trees, then the number of edges incident to $C$ from $T$ is bound from above by $s$.[1]

Using the facts that $n_T + n_C = n \cdot m$ and that $r \leq s$ one receives

$$2n \cdot m - 2n_C - 2n - 2m + \alpha + 2\beta + 2 \leq 2n_C + 2p - \alpha - 2\beta$$

which after reorganizing gives us the requested inequality:

$$(n-1)(m-1) + \alpha + 2\beta \leq 2n_C + p$$

Where the equality hold if all the connected components of $C$ are trees. ■

We note that using the trivial facts that the number $p$ of connected components of $C$ is smaller than the $|C| = n_C$, and that $\alpha + 2\beta \geq 1$, as there must be at least one cutset vertex along the boundaries of the grid one receives from lemma 5 that it holds that

$$(n-1)(m-1) + 1 \leq 3n_C$$

which is a restatement of Luccio's lower bound.

Assume that $n \equiv r \mod 3$ and that $m \equiv s \mod 3$ $(0 \leq r, s \leq 2)$, i.e. $n = 3q + r$ and $m = 3p + s$, and assume that w.l.o.g that $r \leq s$. Additionally, assume that $n_C = olb_{n,m}$, then

$$\begin{aligned} n_C = olb_{n,m} &= \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil \\ &= \left\lceil \frac{9pq + 3pr + 3qs + rs - 3p - s - 3q - r + 2}{3} \right\rceil \\ &= 3pq + p(r-1) + q(s-1) + \left\lceil \frac{(r-1)(s-1)+1}{3} \right\rceil \end{aligned}$$

referring to table 1 we see that the value of the fraction is either $\frac{1}{3}$ and $\frac{2}{3}$, unless $r = 0$ and $s = 2$, and therefore we get that

$$n_C = 3pq + p(r-1) + q(s-1) + \mathbb{1}[r \neq 0 \vee s \neq 2] \quad (2)$$

[1] It should be mentioned that using the results of the previous section, it can be shown that there always exists a cutset, whose connected components are indeed trees (along with all the other aformentioned desired properties).

| r | s | $(r-1)(s-1)+1$ | $(*)$ |
|---|---|---|---|
| 0 | 0 | 2 | 2 |
| 0 | 1 | 1 | 3 |
| 0 | 2 | 0 | 1 |
| 1 | 1 | 1 | 3 |
| 1 | 2 | 1 | 3 |
| 2 | 2 | 2 | 2 |

Table 1: Significant functions of r and s

Plugging equation 2 in inequality 1 we get

$$\begin{aligned} p &\geq (m-1)(n-1) - 2s + \alpha + 2\beta \\ &= 3pq + q(s-1) + p(r-1) + (r-1)(s-1) \\ &\quad - 2 \cdot \mathbb{1}[r \neq 0 \vee s \neq 2] + \alpha + 2\beta \\ &= n_C + (r-1)(s-1) - 3 \cdot \mathbb{1}[r \neq 0 \vee s \neq 2] + \alpha + 2\beta \end{aligned}$$

Rearranging the expression we get the following inequalities

$$0 \leq n_C - p \leq \underbrace{3 \cdot \mathbb{1}[r \neq 0 \vee s \neq 2] - (r-1)}_{(*)} - \alpha - 2\beta \qquad (3)$$

$$\alpha + 2\beta \leq \underbrace{3 \cdot \mathbb{1}[r \neq 0 \vee s \neq 2] - (r-1)}_{(*)} \qquad (4)$$

We will use these equation in proving the improvements to the lower bounds of [1].

**Theorem 2.** *Let $m, n \geq 4$, such that $n \equiv 0 \mod 3$ and $m = 2 \mod 3$, and assume that at least one of $n$ and $m$ is even, then the size of the minimal cutset of the $n \times m$ grid $M_{n,m}$ (or the $m \times n$ grid $M_{m.n}$) is at least of size $olb_{n,m} + 1$, i.e. $nlb_{n,m} = olb_{n,m} + 1$.*

*Proof of theorem 2 .* As stated by Inequality 4 and Table 1, we can see that $\alpha + 2\beta \leq 1$, which implies that $\alpha = 1$ and $\beta = 0$, i.e. there exists a single cutset vertex along the boundaries of the grid, not including the corners. Assume w.l.o.g that it is located along the right boundary of the grid. Focusing on the $2 \times 2$ grids containing each of the four corners of the grid, we note that since $m, n \geq 4$ all four grids are disjoint, and since there exists only a single cutset vertex along the boundaries, in at least three of the four $2 \times 2$ grids three of the vertices cannot be cutset vertices, and therefore the forth - inner - vertex must be a cutset vertex, in order to open the cycle formed by the $2 \times 2$ grid (refer to Figure 6, depicting the upper-left corner of the grid, using the same convention as before). Assume w.l.o.g that $v_{1,1}$, $v_{1,m-2}$, $v_{n-2,1}$ are cutset vertices, and assume that w.l.o.g that $n$ is even. Following from Inequality 3 we see that the number of connected components of the cutset is equal to the number of cutset vertices, that is $C$ is an independent set. Since $v_{1,1}$ is a cutset vertex, it follows that $v_{3,1}$ must be a cutset vertex too, in order to break the cycle formed by $v_{2,0}$, $v_{2,1}$, $v_{3,1}$ and $v_{3,0}$. For a similar reason, $v_{5,1}$ must be in the cutset, and so on and so forth: for every odd number $i$, $v_{i,1}$ must be a cutset vertex. Since $n$ is even by assumption, we get that $n-3$ is odd and therefore $v_{n-3,1}$ is a cutset vertex. Remembering that $v_{n-2,1}$ is a cutset vertex we get a contradiction to the fact that $C$ is an independent set.

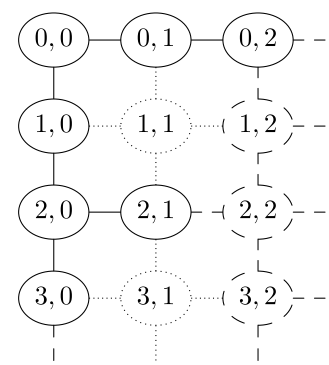


Figure 6: The upper-left corner of a grid

■

Let $m, n$ be two integers at least one of which is even. Assume w.l.o.g. that $m$ is even and that $m \geq 6$. If $n \geq 9$ and $n \equiv 0 \mod 3$, then [2] have shown (in case iii in [2]) that the upper bound on the size of the minimal cutset is $ub_{n,m} = olb_{n,m} + 1 = nlb_{n,m}$, i.e. the upper bound is equal to our new lower bound. If $n \geq 11$ and $n \equiv 2 \mod 3$, then they have shown (in case iv) that $ub_{n,m} = olb_{n,m} + 1 = nlb_{n,m}$. To conclude, in every case in which [2] have shown an upper bound applicable in the conditions of Theorem 2, the upper bound is equal to the lower bound.

**Definition 7.** Let $v_{i,j}$ be a vertex in the grid graph $M_{n,m}$, we say that it is an *even vertex* if $i+j$ is even, and that it is an *odd vertex* if $i+j$ is odd. Let $V_e$ be the set of all even vertices in $M_{n,m}$, and define $E_e = \{(v_{i,j}, v_{k,l}) \in V_e^2 : |i-k| = 1, |j-l| = 1\}$. We call the graph $G_e = (V_e, E_e)$ *the even semi-grid*, and call adjacent vertices in the semi-grid *semi-neighbors*. The *odd semi-grid* $G_o = (V_o, E_o)$ is defined similarly on the set of odd vertices $V_o$.

**Lemma 6.** *Let $G = (V, E)$ be a grid graph, and $A \subseteq V$ an independent set in $G$, and let $S$ be a connected component of the graph induced by $A \cap V_e$ ($A \cap V_o$) in the even (odd) semi-grid, then:*

***a.*** *if no vertex in $S$ lies on the boundaries of $G$, then there exists a cycle in the graph induced by $V \backslash A$ in $G$.*

***b.*** *if there exists a cycle in $S$ (with edges in $E_e$ ($E_o$)), then $S$ separates $G$ to (at least) two connected components, i.e. the graph induced by $V \backslash S$ in $G$ contains at least two connected components.*

***c.*** *if there exist two vertices in $S$ on the boundaries of $G$, then $S$ separates $G$ to (at least) two connected components.*

*Proof.* **a.** The proof of the lemma is by induction on the number of vertices in $S$.

**b.** The proof follows from the fact that a cycle in $G_o$ separates the plane to two (non-empty) regions.

**c.** The proof follows by connecting the two vertices on the boundaries of $S$ by a new edge in $G_o$ ($E_e$) and using previous claim. □

**Definition 8** (stem)**.** Let $G = (V, E)$ be a grid graph, and $A \subseteq V$ an independet set, and let $S$ be a connected component of the graph induced by $A \cap V_e$ ($A \cap V_o$) in the even (odd) semi-grid, we call the vertices of $S$ along the boundaries of the grid *the stems of $S$.*

**Definition 9** (even/odd semi-tree)**.** Let $G = (V, E)$ be a grid graph, and $A \subseteq V$ an independent set, and let $S$ be a connected component of the graph induced by $A \cap V_e$ ($A \cap V_o$) in the even (odd) semi-grid, that does not contain a cycle, then we call $S$ *an even (odd) semi-tree.*

Let $G$ be a grid, and $C$ an independent set, such that the graph induced by $V \backslash C$ on $G$ is a tree. Then, following from lemma 6 is that $C$ can be partitioned to a set of disjoint single-stemmed semi-trees

**Theorem 3.** *Let $m, n \geq 4$, such that both $n \equiv 0 \mod 3$ and $m = 0 \mod 3$ or both $n \equiv 2 \mod 3$ and $m = 2 \mod 3$, and assume that both $n$ and $m$ are even, then the size of the minimal cutset of the $n \times m$ grid $M_{n,m}$ (or the $m \times n$ grid $M_{m.n}$) is at least of size $olb_{n,m} + 1$, i.e. $nlb_{n,m} = olb_{n,m} + 1$.*

*Proof of Theorm 3.* We see from Inequality 4 and Table 1 that in both these cases $\alpha + 2\beta \leq 2$, which may result in several scenarios:

1. If $\alpha = 0$ and $\beta = 1$, then we get from Inequality 3 that the cutset is an independent set and therefore using a claim similar to that of Theorem 2 we receive a contradiction.

2. If $\alpha = 1$ and $\beta = 0$, then we get from Inequality 3 that all the connected components of the cutset are singletons apart from one which contains two vertices. Following the lines of the proof for Theorem 2, we note that at least in three of $2 \times 2$ grids located the corners of the $M_{n,m}$ the inner vertex must be a cutset vertex. Assume w.l.o.g that the cutset vertex lies along the bottom edge of the grid, and that in all $2 \times 2$ grids in the corners except maybe that in the lower-right one the inner vertex is a cutset vertex. As before, every second vertex along the second row and along the second column of the graph must be cutset vertices, giving us two pairs of adjacent cutset vertices - in the upper-right corner and in the lower-left corner, in contradiction to the fact that only one such pair should exist.

3. If $\alpha = 2$ and $\beta = 0$, then again we get that the cutset is an independent set. If at least one of the cutset vertices does not lie in a $2 \times 2$ grid in a corner of the graph, then there are at least three $2 \times 2$ grids in the corners, in which the inner vertex is a cutset vertex. As there are two pairs of corners along a single boundary of the graph such that their inner vertex is a cutset vertex and only a single vertex that is on a boundary

of the grid, we get that there exists a boundary such that in both its corners the inner vertex of the $2 \times 2$ grid is a cutset vertex, and there is no cutset vertex along it. Since it follows as before, that every second vertex along the edge is a cutset vertex, we get that there are two adjacent cutset vertices, in contradiction to the fact the the cutset is an independent set.

Therefore, assume that both cutset along the boundaries of the grid, lie in the $2 \times 2$ grid of adjacent corners $A$ and $B$, then in the $2 \times 2$ grid of the two remaining corners $C$ and $D$, the inner vertex is a cutset vertex. Since there is no cutset vertex along the boundary between $C$ and $D$, we get that there exist two adjacent cutset vertices, similarly to before - a contradiction.

Finally, assume that the cutset vertices along the boundaries of the grid lie in the $2 \times 2$ grid of opposite corners. Assume w.l.o.g that they lie in the $2 \times 2$ grids of the upper-left and lower-right corners (refer to Figure 7. Note that in this case both boundary vertices are necessarily odd, and as the only possible stems for the cutset, all cutset vertices in the graph are odd as well. Additionally, as before we get that $v_{i,1}, v_{1,i}, v_{j,m-2}, v_{n-2,j} \in C$ for every even $i$ and odd $j$. Note that the frame of the grid is separated to two parts be the cutset vertices. Since the the graph induced by $V \backslash C$ is a connected tree by assumption, we get that there must be a path from one part to the other though the inner part of the grid. Assume w.l.o.g that the path begins at $v_{0,j}$ for a (necessarily) odd $j$ ($3 \leq j \leq m-3$) and goes though $v_{1,j}$ to $v_{2,j}$. Since $v_{2,j-1}, v_{2,j+1}, v_{3,j} \notin C$ since they are even, we get that $v_{3,j-1}, v_{3,j+1} \in C$ (as otherwise two cycles would form). Since all four semi-neighbors of $v_{1,j}$ are cutset vertices, we get that the path must continue 2 steps at a time in the same direction. In general, assume the path reaches vertex $v_{k,l}$ such that $k \equiv 0 \mod 2$ and $l \equiv j \mod 2 \equiv 1 \mod 2$, then since $v_{k-1,l}$, $v_{k,l-1}$, $v_{k+1,l}$, $v_{k,l+1}$ are all even, and therefore not cutset vertices, we get that all four semi-neighbors of $v_{k,l}$ must be cutset vertices, and the path must continue 2 steps in the same direction. As a result, we get that the path must be connected to the other part of the grid's frame though a vertex of either the form $v_{k,1}$ with $k \equiv 0 \mod 2$, i.e. an even $k$, or the form $v_{n-2,l}$ with $l \equiv 1 \mod 2$, i.e. an odd $l$. However, as seen before, all vertices of such forms are necessarily cutset vertices, and therefore the path could not pass though them - a contradiction.

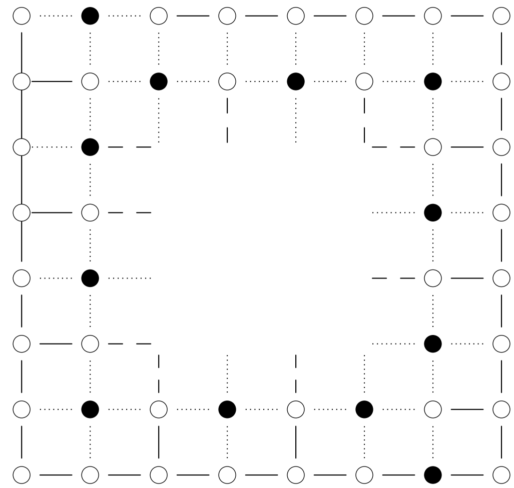

Figure 7: A possible frame of a $8 \times 8$ grid with a cutset of size $lb_{8,8}$. Cutset vertices are marked with a black circle and tree vertices are marked with white circle. Other notation as before.

■

Let $m, n$ be two even integers, and assume that $m \geq 6$. If $n \geq 9$ and $n \equiv 0 \mod 3$, then again case (iii) in [2] shows that the upper bound on the size of the minimal cutset is $ub_{n,m} = olb_{n,m} + 1 = nlb_{n,m}$. If $n \geq 11$ and $n \equiv 2 \mod 3$, in case (iv) in [2] shows that $ub_{n,m} = olb_{n,m} + 1 = nlb_{n,m}$. To conclude, in every case in which [2] have shown an upper bound applicable in the conditions of Theorem 3, the upper bound is equal to the lower bound.

# 5 Activate Cutset

We would like to use the notion of cycle-cutset in order to expand an optimal algorithm for energy minimization in trees presented in [3], thus receiving an approximate energy minimization algorithm that can handle arbitrary unary and binary potentials over domains of finite size.

## 5.1 Problem Definition

let $\bar{x} = x_1, \ldots, x_N$ be a set of variables over domain $\{1, \ldots, k\}$, for every $i \in \{1, \ldots, N\}$ let $\varphi_i : \{1, \ldots, k\} \to \mathbb{R}$ be a unary potential, and for every $1 \leq i < j \leq N$ let $\psi_{i,j} : \{1, \ldots, k\}^2 \to \mathbb{R}$ be a binary potential, then the problem of energy minimization is finding

$$\bar{x}^* = argmin_{\bar{x}} \sum_i \varphi_i(x_i) + \sum_{i<j} \psi_{i,j}(x_i, x_j)$$

Given an instance of the problem, a matching graph can be builts, where every variable $x_i$ is assigned a vertex, and two vertices $x_i$ and $x_j$ are connected iff the matching potential $\psi_{i,j}$ is not identically zero.

## 5.2 Our approach

Although the problem is generally NP-hard, it is tractable in some cases one of which is when the underling graph is cycle-free, i.e a forest. The algorithm for solving this case, presented in [3] as ACTIVATE, is essentially a form of belief propagation: After a root for every tree is found, every vertex $v$ sends its parent $pa_v$ a message of the optimal assignment to the tree beneath $v$ given the value of $pa_v$. Once all the message reach the root of the tree, the root decides on an optimal assignment for itself and passes massages to the children to set their values accordingly. We will refer to this algorithm as the TREE-ALGORITHM.

If the graph does contain cycles, the cycles can be opened by finding a cycle-cutset $C$, setting the values of the variables of $C$ to some values, and using the exact algorithm on the remaining (cycle-free) graph. This method is known as "conditioning", and by doing so one receives the optimal assignment given the assignment to the cutset variables. By conditioning the cutset variables to every possible assignment, one receives an exact algorithm for the problem with running time exponential in the size of the cutset.

In our algorithm, named ACTIVATE-CUTSET, after a cutset is chosen and the exact optimal solution given the values of the cutset variables is found, a new cutset is chosen and the process repeats with the cutset vertices retaining the values obtained by the exact TREE-ALGORITHM in previous iteration (in which they were not a part of the cutset). Since in every iteration an optimal assignment given the values of the cutset vertices is found, the enegry of the system can only decrease from iteration to iteration. Therefore, the resulting algorithm is a local search algorithm, in which in every update step the entire complement of the cutset is assigned an optimal solution given the cutset variables .

## 5.3 Algorithm Description

Given an instance of the energy minimization problem ACTIVATE-CUTSET operates as follows:

1. Either initializes the nodes randomly or sets their values to *undefined* (see 5.4)
2. Finds the set of nodes which are a part of a tree, i.e. node that are not a part of any (simple) cycle using the algorithm described in [3].
3. Selects a cycle-cutset C (i.e. a subset $C \subseteq V$ of the nodes such that the induced graph on $T = V \backslash C$ is a cycle free).
4. Updates the values of the nodes:
   - Given the values of of the cycle-cutset C, it uses the TREE-ALGORITHM to find the optimal assignment for T.
   - In an optional implementation, given the values of the tree nodes T, it uses some local search algorithm (i.e. HOPFIELD MODEL, $GLS^+$[5], etc.) in order to update the values C. If this feature is implemented these two update stages are performed alternately until convergence (at a local maxima) or until a set number of iterations has passed.
5. Repeats stages 3-4 until the specified time bound has passed.

## 5.4 Initialization

As noted in Algorithm Description above there are two variants in regard to the variable initialization - either randomly or to a special *undefined* value. Cycle-cutest nodes holding an *undefined* value are effectively

ignored until their value is set to valid one. The latter initialization scheme is used only when the cutset nodes values are not updated in the update stage (stage 4 in the "Algorithm Description"). In this case the nodes keep their *undefined* value until they are chosen to take part in a tree, and update their value as part of the update stage.
It should be mentioned that this initialization scheme has been explored, since some experiments suggested that the cutset nodes have a tendency to draw the algorithm to high local minima. It should be noted that as a consequence of the initialization to an *undefined* value, the graph energy is not defined in the first iterations of the algorithm until all nodes are assigned a valid value, and therefore the graph's energy may initially increases as more nodes are assigned values).

## 5.5 Cutset Selection

The cutset selection algorithm is based on the algorithm described in [4], where the cutset is built by randomly choosing a node who has a more than two neighbors not already in the cutset or a tree with probability proportional to the number of (non-cutset-non-tree) neighbors; once no node has more than 2 such neighbors, i.e. once the induced graph on the nodes not already a part of the cutset or a tree is a union of disjoint simple cycle, each cycle is opened by randomly picking a node from each with uniform probability. After each new cutset node is chosen the tree forming algorithm of [3], is run on its neighbors, in order to check if the new cutset node's removal allowed the forming of new trees.
Motivated by the ideas presented in [3] the probability a node is chosen in our implementation is governed not only by its degree, but by the number of iterations in which it has not changed its value and by the number of iterations it was not a part of the cycle-cutset. several linear combination of the aforementioned parameters were tested, but no major difference was noted.
In addition, in order to encourage highly connected forests, i.e. forests in which the number of trees is (relatively) small and are composed mainly from a single big tree, the first stage of [4] is split to 2 stages: At first, the program tries to add to the cutset nodes which do not already have a node pointing to them as a parent, and only when no such nodes remains the program begins to add nodes which are pointed as the parent of another node.
In order to reduce the cutset's size, after the forest is formed a message is passed from the roots down to the leaves, notifying the nodes what is the root of their tree, and essentially defining explicitly the partition of the forest into trees. Then, each cutset node of tree-degree less than 1 is removed from the cutset (and added to the forest nodes). Once all the redundant cutset nodes have been removed, the tree forming algorithm is run again (only on the forest nodes) in order to form a well directed forest.

## 5.6 Experiments

We have run ACTIVATE-CUTSET on the set of grid problems from problem set of the PASCAL2 Probabilistic Inference Challenge (PIC2011)[2]. The variables were initialized to the undefined value, and in every update step only trees bigger than the 10% of the grid size were updated. In addition, in order to avoid stagnation, if the assignment is not improved within 10 iteration, i.e. by 10 different cutsets, the algorithm is restarted by setting the cutset vertices to the undefined value. For comparison, we have run $GLS^+$ [5], considered to be the state-of-the-art in energy minimization on the problem set. Each algorithm was run on each problem 10 times a bounded amount of time. Since ACTIVATE-CUTSET was not implemented to employ a sophisticated initialization algorithm (such as the usage of mini-buckets by $GLS^+$), $GLS^+$ was set to a random initial assignment. Note that in the context of these problems an assignment obtaining **maximal** energy (sometimes refereed to "Goodness") is required, and therefore higher values are better. Following are the energies obtained by each of the algorithm compared to the global maxima. For each problem instance Figure 8 depicts the average, minimal and maximal goodness obtained by ACTIVATE, in the four left bars, and the corresponding values obtained by $GLS^+$ in the right bars.

[2] http://www.cs.huji.ac.il/project/PASCAL/index.php

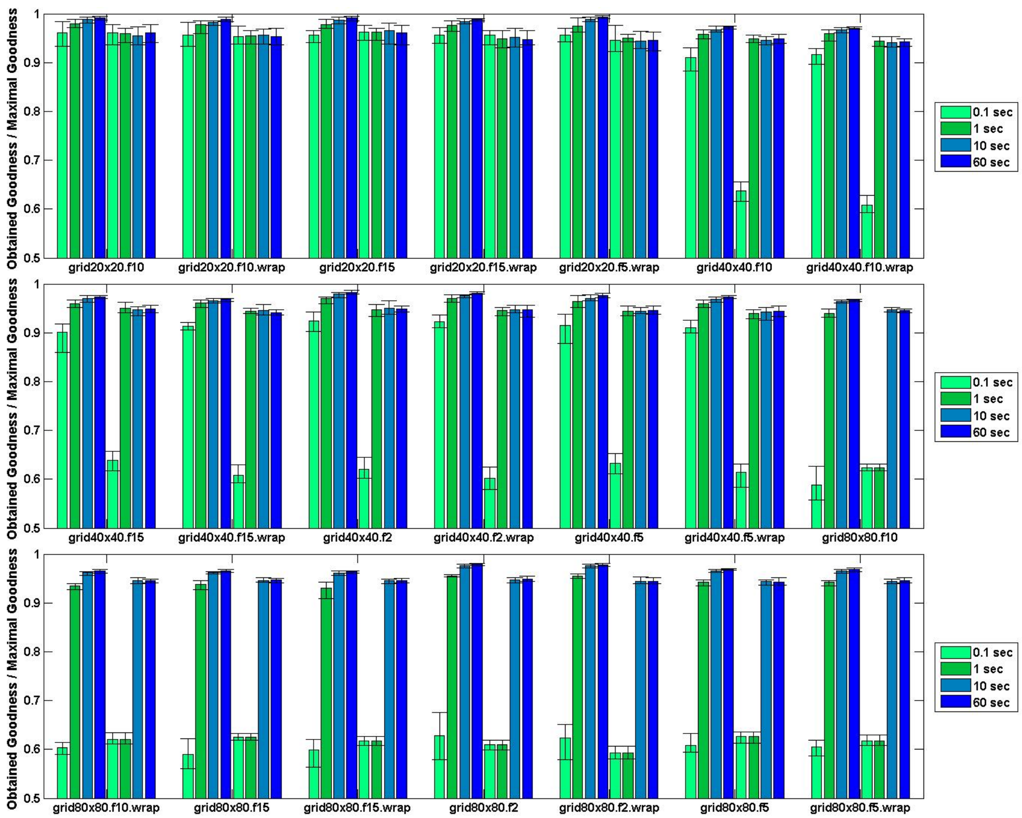


Figure 8: Ratio of obtained goodness to maximal goodness in grid problems of PIC2011 at several time bounds. In each set the 4 left bar correspond to ACTIVATE's results, and the 4 right bars correspond to the results of $GLS^+$.

It can be seen in Figure 8 that for most problems ACTIVATE obtains a higher goodness than that obtained by $GLS^+$ at the same time. In general, we can see that ACTIVATE obtains higher energy faster.

# 6 Conclusion

In this work, we have established the basis to the notion of tree-inducing cycle-cutsets and the transformation of a general cutset to such a cutset. We have shown that in grids we can always transform a cycle-cutset to a tree-inducing cutset with no more vertices than the original one. This results laid the foundation to a more elaborate method of analyzing and bounding the size of minimal cutset, thus allowing us to improve its lower bound in some cases. In other cases, a gap between the lower and the upper bounds remains, and more meticulous research should be undertaken in order to characterize better the classes in which the lower bound can be raised.

In addition, we presented an algorithm which combines the notion of cycle-cutset with the well known Belief Propagation algorithm to achieve an approximate optimum of a sum of unary and binary potentials. This is done by the rather novel concept of traversal from one cutset to another and updating the induced forest, thus creating a local search algorithm, whose update phase spans over many variables. We have presented experiments indicating that this algorithm is on-par with the state-of-the-art in this domain (if not surpasses it) on some restricted problems of grids and when both algorithms use a elementary method of initialization. In this regard the algorithm should be further investigated in order to understand more fully the parameters governing its behavior. Additionally, the algorithm should be extended to handle potentials of higher arity than 2.

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