Inference and Search for Graphical Models

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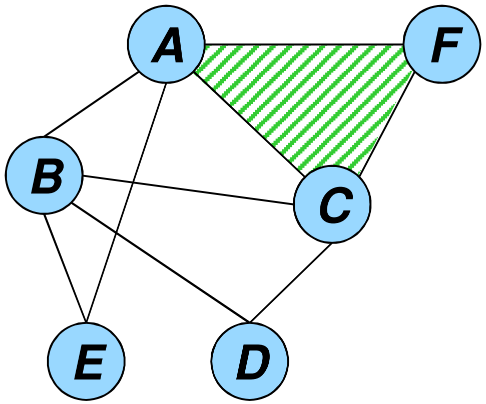


Los Alamos

Sample Applications for Graphical Models

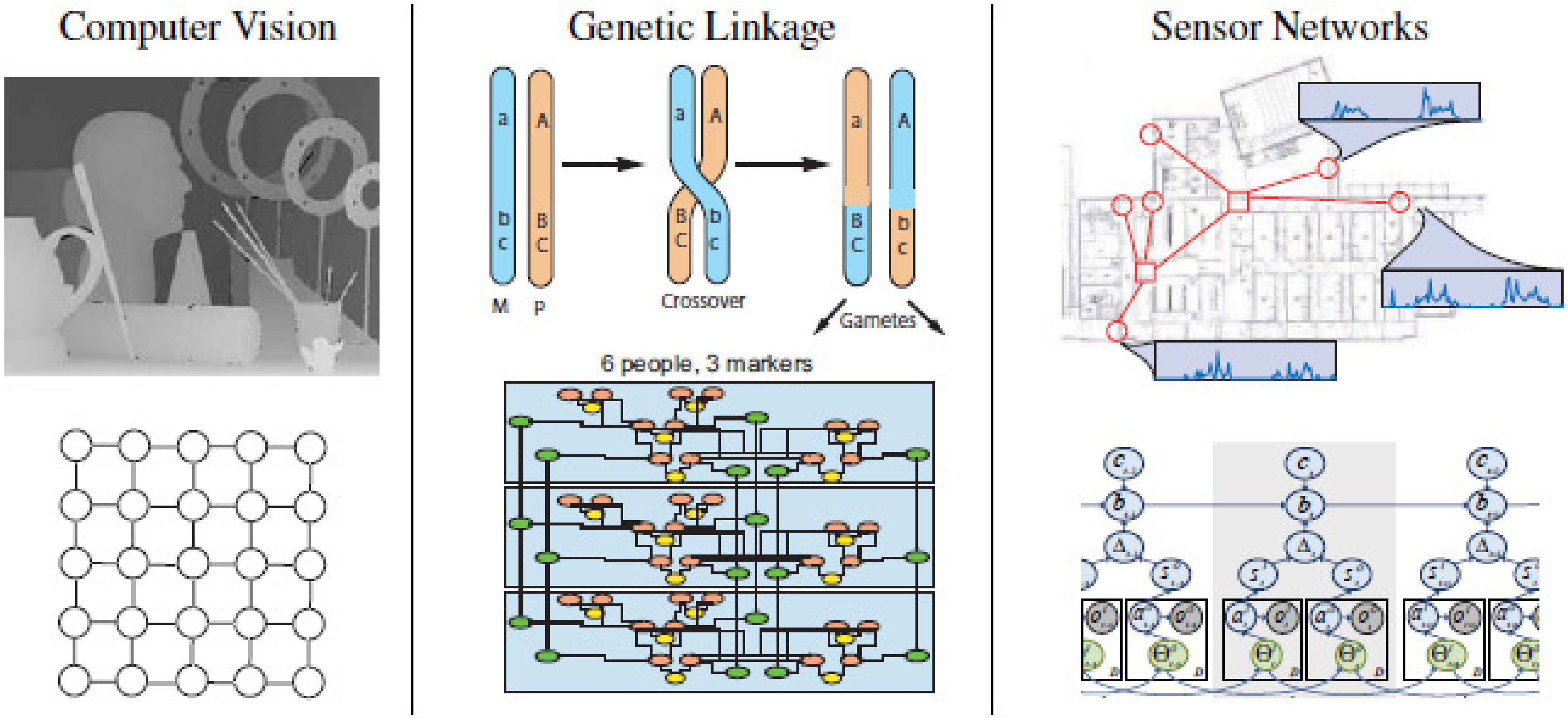


Figure 1: Application areas and graphical models used to represent their respective systems: (a) Finding correspondences between images, including depth estimation from stereo; (b) Genetic linkage analysis and pedigree data; (c) Understanding patterns of behavior in sensor measurements using spatio-temporal models.

Sample Applications for Graphical Models

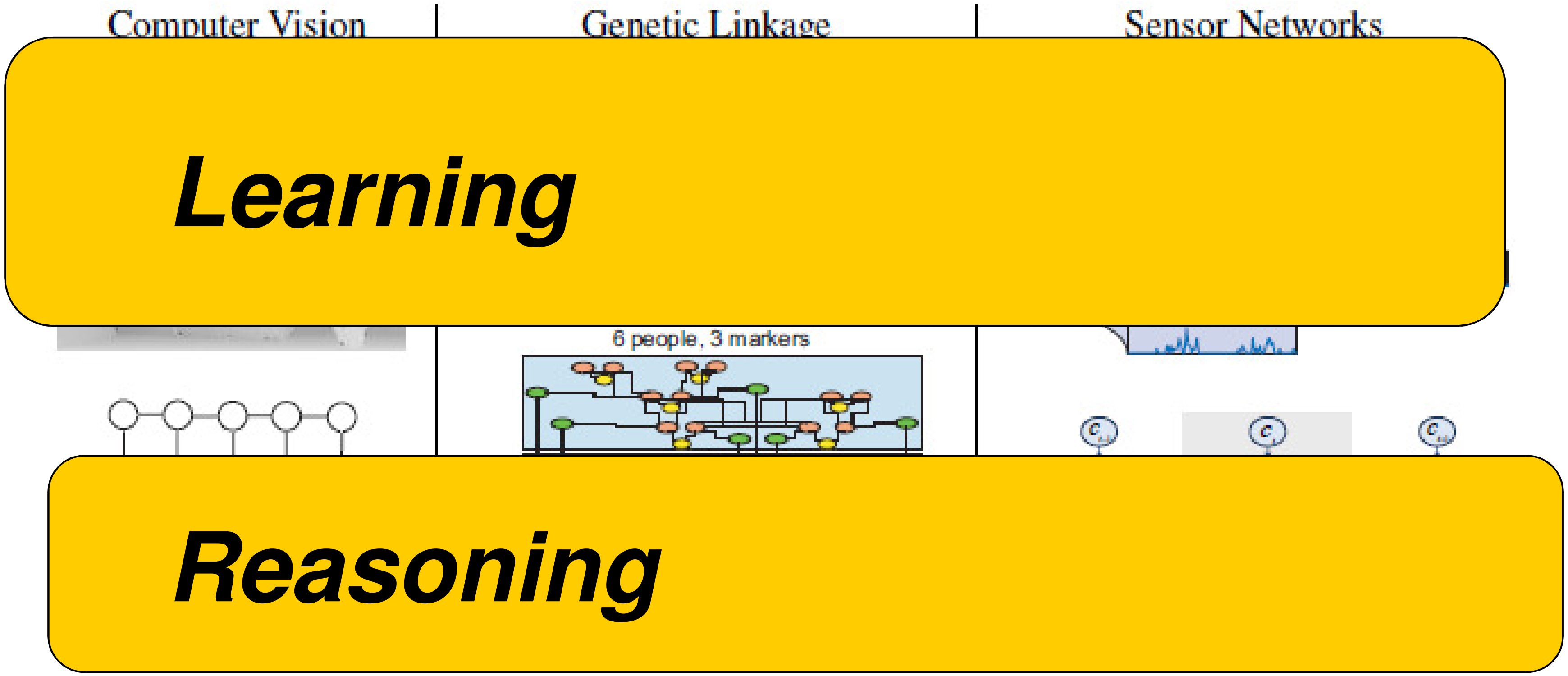


Figure 1: Application areas and graphical models used to represent their respective systems: (a) Finding correspondences between images, including depth estimation from stereo; (b) Genetic linkage analysis and pedigree data; (c) Understanding patterns of behavior in sensor measurements using spatio-temporal models.

Outline

- What are graphical models?
- Inference
- Search; via AND/OR search
- Time vs space, search vs inference
- Bounding inference
- Anytime AND/OR branch and bound
- Experiments, Competitions

and conclusions

Exact algorithms

Approximations

Back to exact: Anytime schemes

Constraint Networks

Map coloring

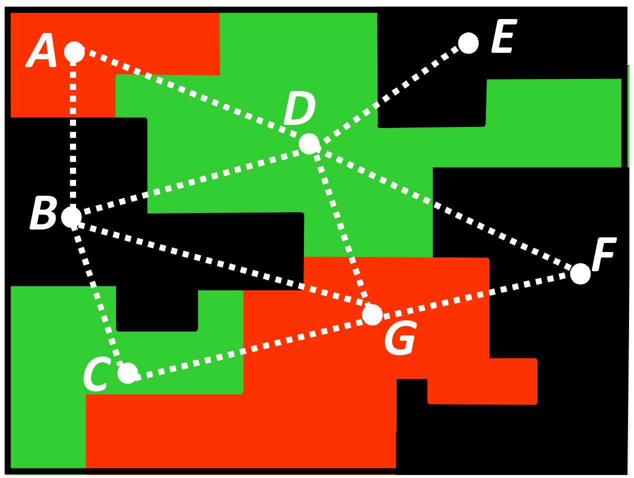
Combination = join
Marginalization = projection

Variables: *countries (A B C etc.)*

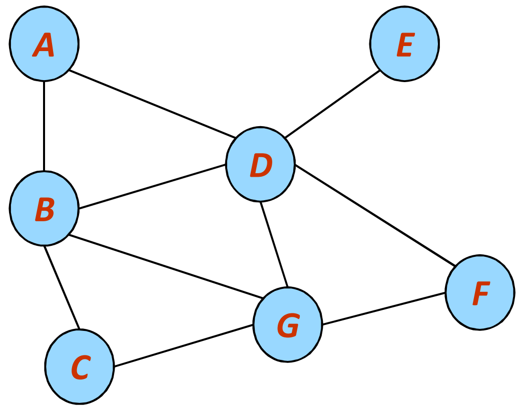
Values: *colors (red green blue)*

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, …

A	*B*
red	*green*
red	*yellow*
green	*red*
green	*yellow*
yellow	*green*
yellow	*red*

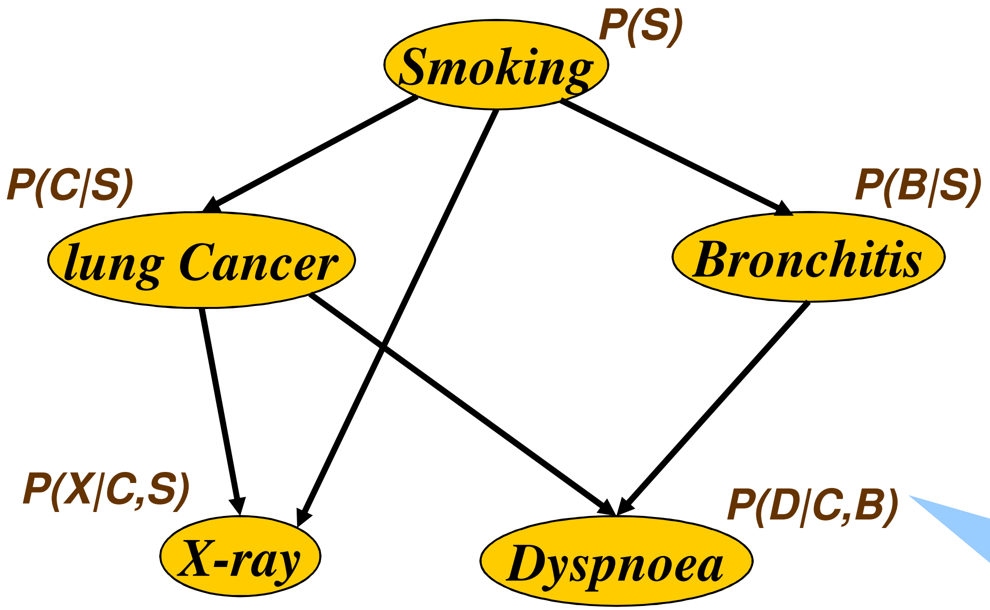


Constraint graph



Queries: Find one solution, all solutions, counting

Bayesian Networks (Pearl 1988)



$$BN = (G, \Theta)$$

CPD:

C	B	P(D|C,B)	
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

$$P(S, C, B, X, D) = P(S)\, P(C|S)\, P(B|S)\, P(X|C,S)\, P(D|C,B)$$

$$P(x_1 \ldots x_n) = \prod_i p(x_i \mid pa(x_i))$$

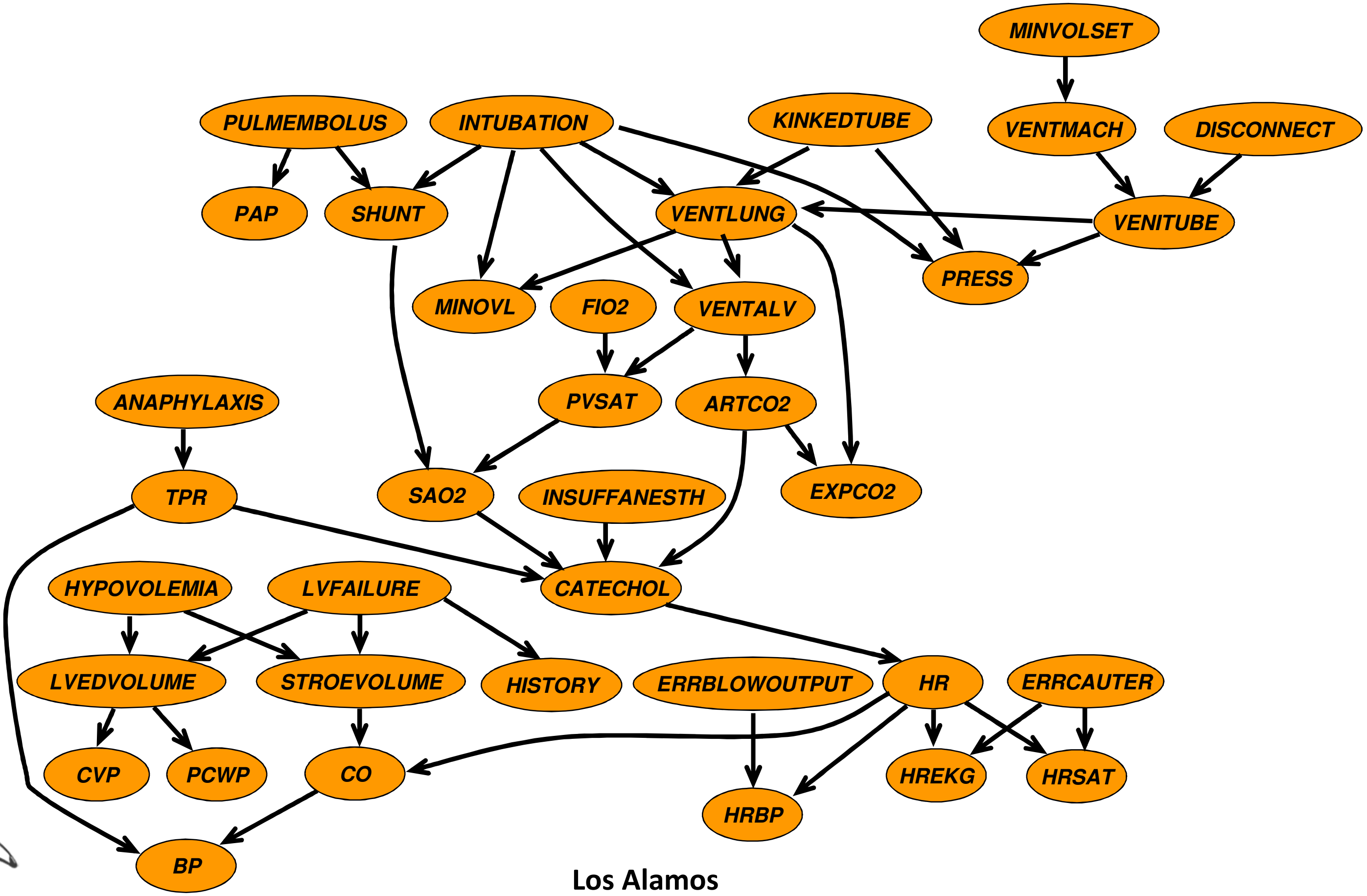
$$P(e) = \sum_{X-E} \prod_i p(x_i \mid pa(x_i))$$

$$mpe = \max_x \prod_i p(x_i \mid pa(x_i))$$

Combination: Product
Marginalization: sum/max

Monitoring Intensive-Care Patients

The “alarm” network - 37 variables, 509 parameters (instead of 2^{37})



Los Alamos

Constraint Optimization Problems
for Graphical Models

A *finite* COP is a triple $R = \langle X, D, F \rangle$ where:

$X = \{X_1, ..., X_n\}$ - variables

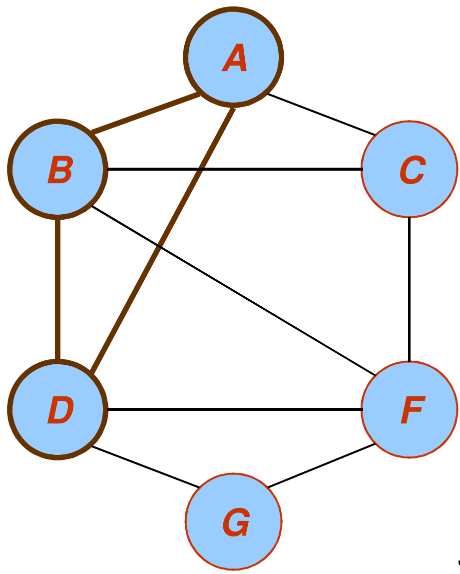
$D = \{D_1, ..., D_n\}$ - domains

$F = \{f_1, ..., f_m\}$ - cost functions

f(A,B,D) has scope {A,B,D}

A	B	D	Cost
1	2	3	3
1	3	2	2
2	1	3	∞
2	3	1	0
3	1	2	5
3	2	1	0

Primal graph =
Variables --> nodes
Functions, Constraints -→ arcs



F(a,b,c,d,f,g)= f_1(a,b,d)+f_2(d,f,g)+f_3(b,c,f)

Global Cost Function

$$F(X) = \sum_{i=1}^{m} f_i(X)$$

Graphical Models

- A graphical model M = (X,D,F,⊗)
  - X = $\{X_1,\ldots,X_n\}$ variables
  - D = $\{D_1,\ldots,D_n\}$ domains
  - F = $\{f_1,\ldots,f_t\}$ functions over $\{S_1,\ldots,S_t\}$
  - Global function ⊗ Combine

$$F(x) = \otimes_f f(x_f)$$

$$G(y) = \Downarrow_{x-y} F(x)$$

$$G(y) = \Downarrow_{x-y} \otimes_f f(x_f)$$

f(x)

- Queries:
  - **Belief updating**: $\Sigma_{X-y} \prod_j P_i$
  - **MPE**: $\max_X \prod_j P_j$
  - **CSP**: $\prod_X \times_j C_j$
  - **Max-CSP**: $\min_X \Sigma_j F_j$

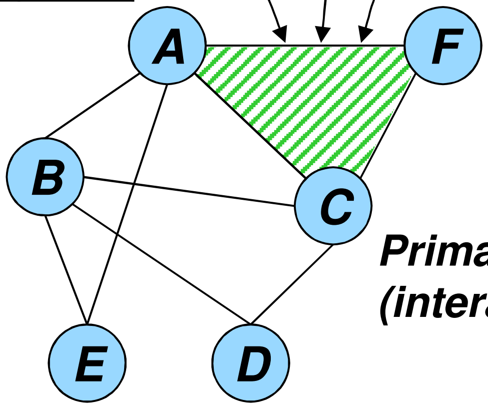
Conditional Probability Table (CPT)

A	C	F	P(F|A,C)
0	0	0	0.14
0	0	1	0.96
0	1	0	0.40
0	1	1	0.60
1	0	0	0.35
1	0	1	0.65
1	1	0	0.72
1	1	1	0.68

Relation

A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue

$f_i := (F = A + C)$



Primal graph (interaction graph)

When combine and marginalize obey Some properties they can be solved By the same algorithms

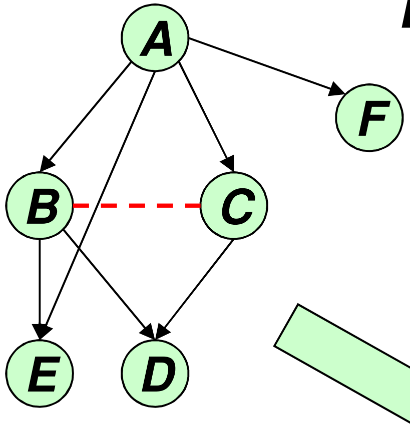
(Bistareli , Rossi and Montanari, 1995, Shenoy , Shafer, 1990, Kasket. Al., 2005 et. Al.)

Mixed Networks

(Mateescu and Dechter, 2004)

Examples: NLP, Linkgage, Software verification, probabilistic languages

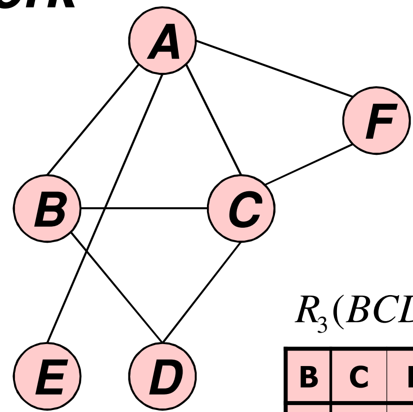
Belief Network



$P(D|B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

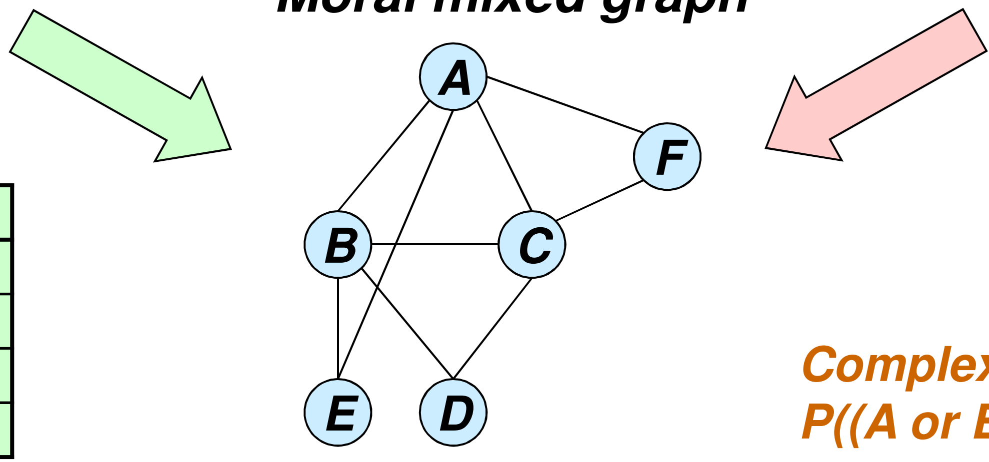
Constraint Network



$R_3(BCD)$

B	C	D
0	0	1
0	1	0
1	1	0

Moral mixed graph



Complex cnf queries: P((A or B) and (~CVD))

$$P_M(\bar{x}) = \begin{cases} P_B(\bar{x} \mid \bar{x} \in \rho) = \dfrac{P_B(\bar{x})}{P_B(\bar{x} \in \rho)}, \text{if } \bar{x} \in \rho \\ 0, \text{otherwise} \end{cases}$$

Graphical Models

- A graphical model M = (X,D,F,$\otimes$)

$$F(x) = \otimes_f f(x_f)$$

$$G(y) = \Downarrow_{x-y} F(x)$$

$$G(y) = \Downarrow_{x-y} \otimes_f f(x_f)$$

Reasoning Queries:
Satisfaction
Optimization
Counting
Hybrids: optimization-counting
Hybrid: Minimize expected utility

Conditional Probability Table (CPT)

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0	0	0	0.14
0	0	1	0.96
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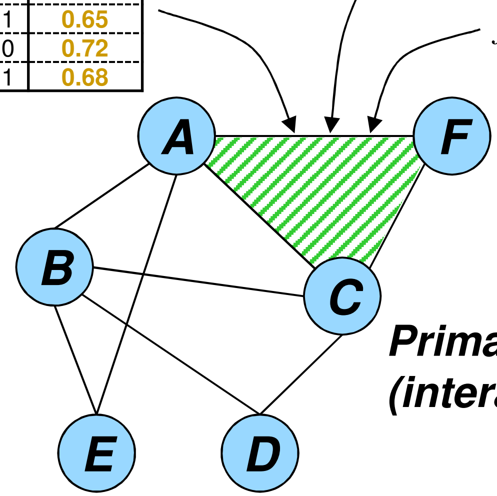
Relation

A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue

$f_i \;:=\; (F = A + C)$



Primal graph (interaction graph)

All these tasks are NP-hard

- ***exploit problem structure***
- ***identify special cases***
- ***approximate***

Reasoning Algorithms

- **By Inference (thinking)**
  - Transform into a single, equivalent (tree) of sub-problems
  - Atomic operation: inference
    - $f_1(X) \otimes f_2\,(Y) \longrightarrow f_3(X,Y)$
- **By Conditioning (guessing)**
  - Transform into many (tree-like) sub-problems.
  - Atomic operation: assign a value $f_1(X=5)$

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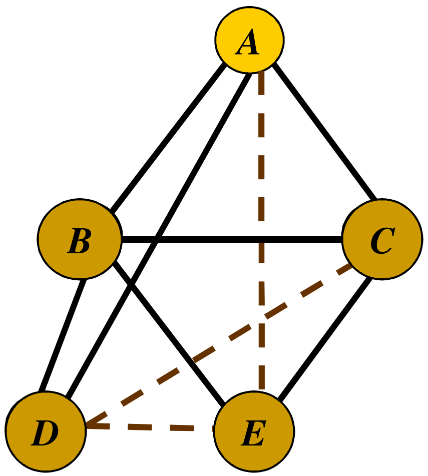
Exact algorithms

Approximations

Back to exact:
Anytime schemes

Query 1: Belief updating: P(X|evidence)=?



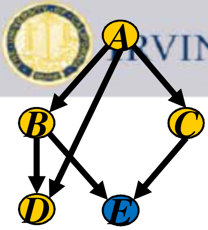
"Moral" graph

$$P(a|e=0) \propto P(a,e=0)=$$

$$\sum_{e=0,d,c,b} P(a)\underbrace{P(b|a)}P(c|a)\underbrace{P(d|b,a)P(e|b,c)}$$

$$P(a)\sum_{e=0}\sum_{d}\sum_{c}P(c|a)\underbrace{\sum_{b}P(b|a)P(d|b,a)P(e|b,c)}_{h^{B}(a,d,c,e)}$$

Variable Elimination



Bucket elimination Algorithm *BE-bel* (Dechter 1996)

$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A)\cdot P(B \mid A)\cdot P(C \mid A)\cdot P(D \mid A,B)\cdot P(E \mid B,C)$$

$\sum_b \prod$ ← ***Elimination operator***

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $\lambda^B(a, d, c, e)$

bucket D: $\lambda^C(a, d, e)$

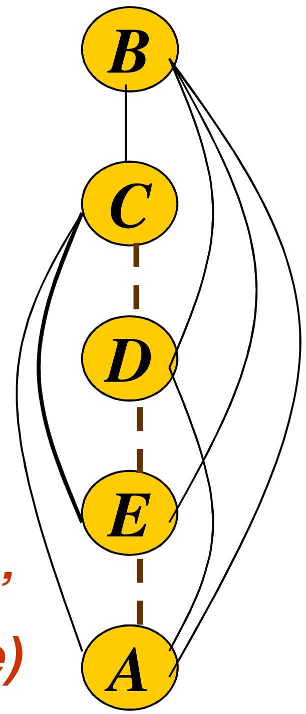
bucket E: $e=0$ $\lambda^D(a, e)$

bucket A: $P(a)$ $\lambda^E(a)$

$P(e=0)$

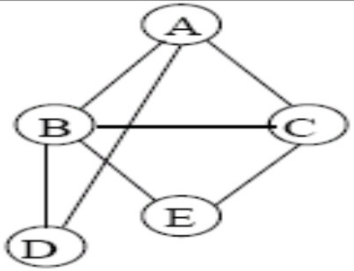
$P(a|e=0)$

W*=4
"induced width"
(max clique size)



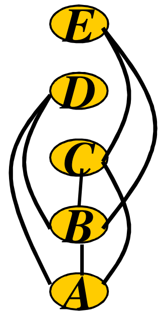
$$P(a|e=0) = \frac{P(a,e=0)}{P(e=0)}$$

Bucket Elimination and Induced Width



Ordering: a, b, c, d, e

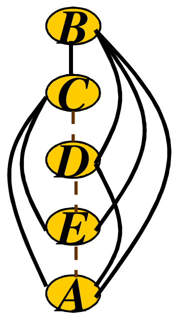
$bucket(E) = \quad P(e|b,c),\ e = 0$
$bucket(D) = \quad P(d|a,b)$
$bucket(C) = \quad P(c|a) \quad || \ P(e = 0|b,c)$
$bucket(B) = \quad P(b|a) \quad || \quad \lambda_D(a,b), \lambda_C(b,c)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_B(a)$



W*=2

Ordering: a, e, d, c, b

$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a)$
$bucket(C) = \quad P(c|a) \quad || \quad \lambda_B(a,c,d,e)$
$bucket(D) = \quad || \ \lambda_C(a,d,e)$
$bucket(E) = \quad e = 0 \quad || \ \lambda_D(a,c)$
$bucket(A) = \quad P(a) \quad || \quad \lambda_E(a)$



W*=4

Query 2: Finding MPE by Bucket Elimination

Algorithm BE-mpe (Dechter 1996, Bertele and Briochi, 1977)

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)P(d|a,b)P(e|b,c)$$

$$\max_X \prod$$

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $h^B(a,d,c,e)$

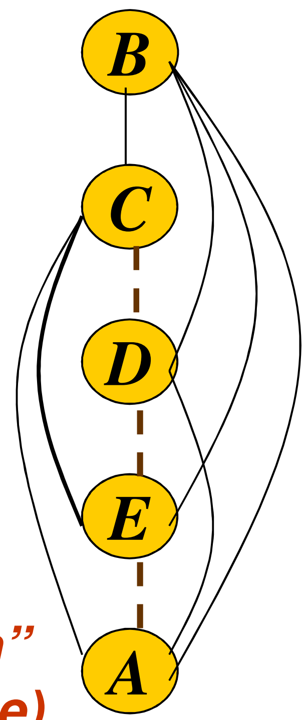
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $h^D(a,e)$

bucket A: $P(a)$ $h^E(a)$

OPT

W*=4
"induced width"
(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_c P(c \mid a') \times h^B(a', d', c, e')$

3. $d' = \arg\max_d h^C(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_a P(a) \cdot h^E(a)$

B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

C: $P(c|a)$ $h^B(a,d,c,e)$

D: $h^C(a,d,e)$

E: $e=0$ $h^D(a,e)$

A: $P(a)$ $h^E(a)$

Return (a', b', c', d', e')

Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times$
$\times P(d' \mid b, a') \times P(e' \mid b, c')$

B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

Time and space exponential in the induced-width / treewidth

$$O(nk^{w^*+1})$$

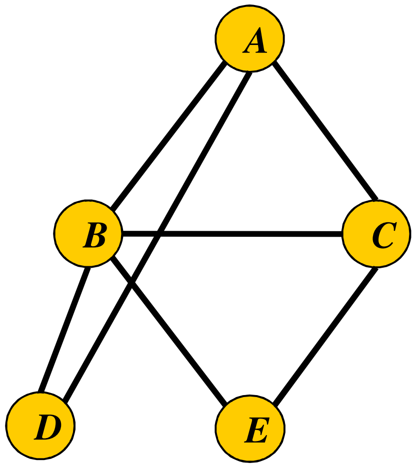
1. $a' = \arg\max_a P(a) \cdot h^E(a)$

A: $P(a)$ $h^E(a)$

Return (a', b', c', d', e')

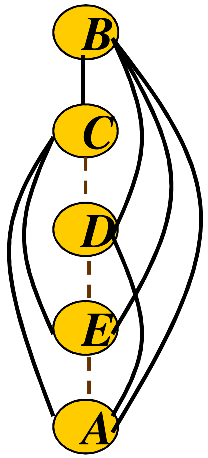
The Induced-width/treewidth

$w^*(d)$ – the induced width of graph along ordering d

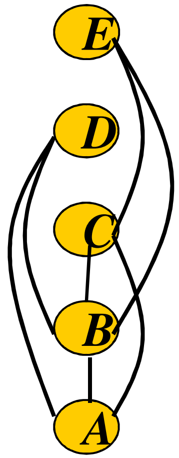


"Moral" graph

The effect of the ordering:



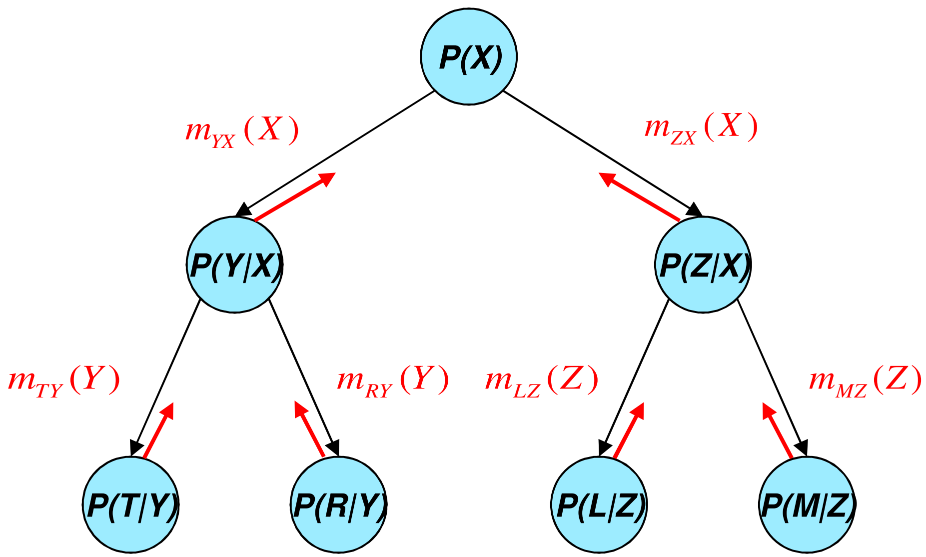
$w^*(d_1) = 4$



$w^*(d_2) = 2$

Complexity of Bucket-elimination

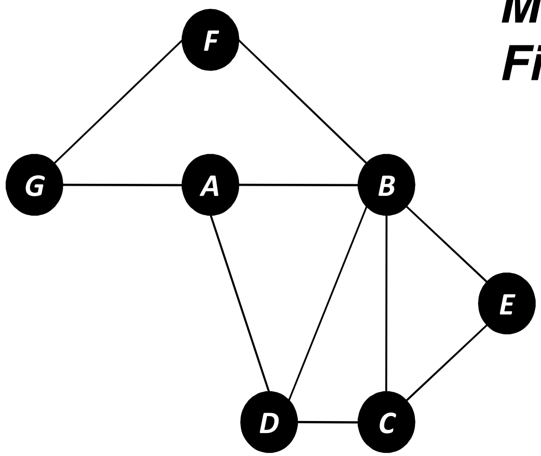
- Theorem: Bucket-elimination is O(r•k^{w*+1}) time and O(nk^{w*}) space.
- When w=1 then w*=1 → trees
- When we have a tree of functions w=w* .

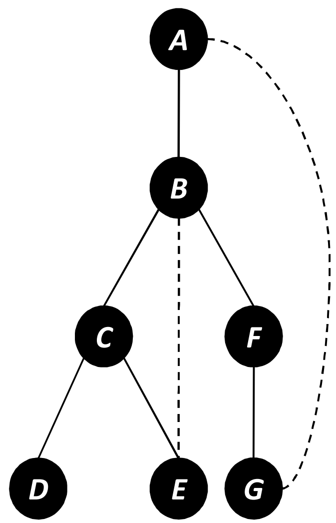


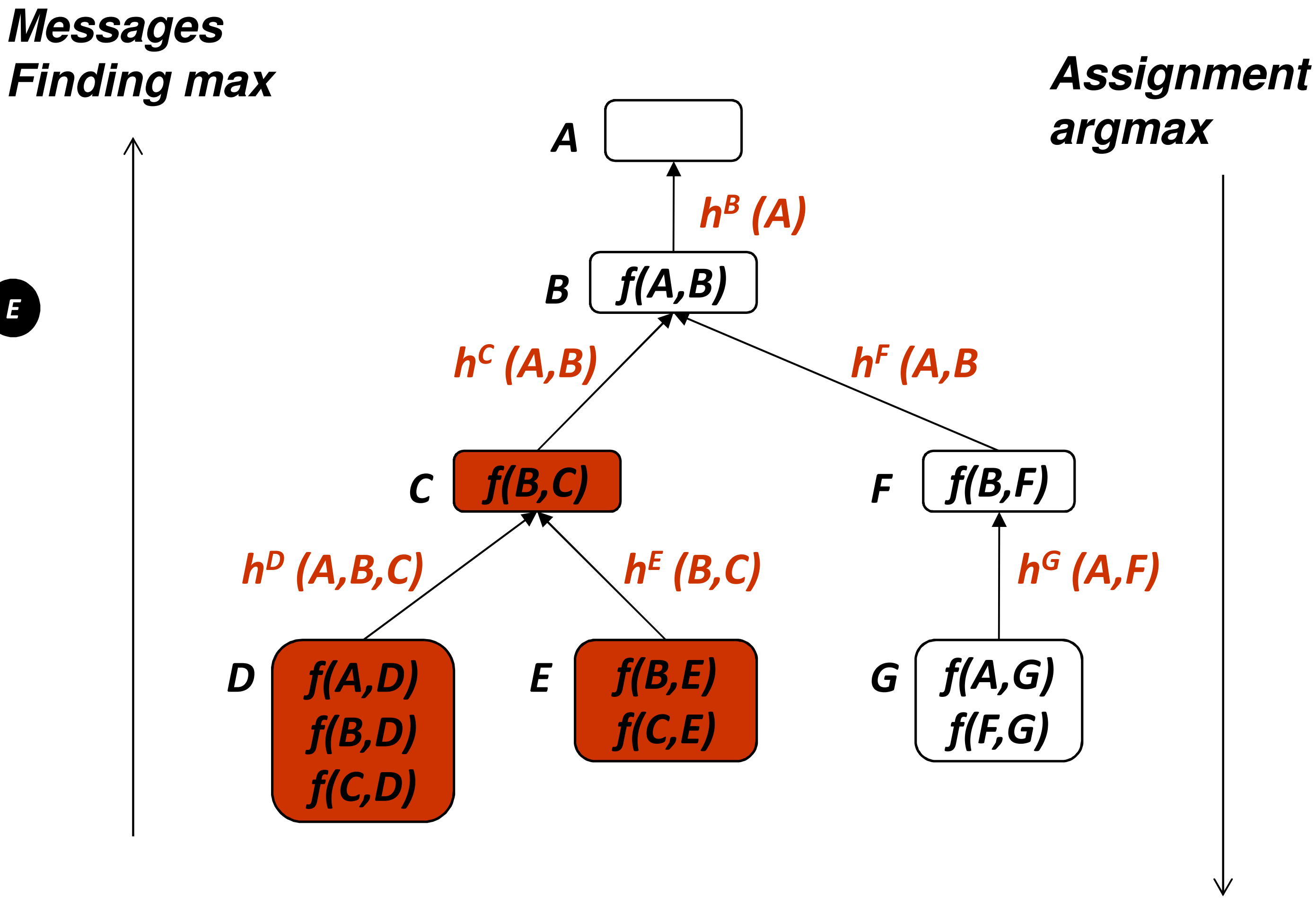
bucket-elimination Sends messages From leaves to root

Bucket-Tree Elimination

$$\min_{a,b,c,d,e,f,g} f(a,b)+f(a,d)+f(b,c)+f(a,d)+f(b,d)+$$
$$f(c,d)+f(b,e)+f(c,e)+f(b,f)+f(a,g)+f(f,g)=$$

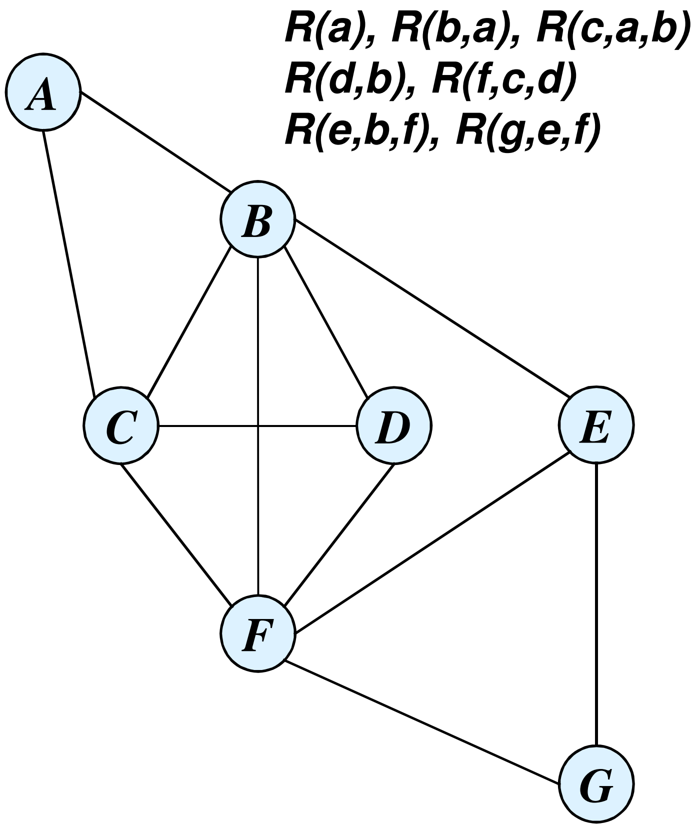


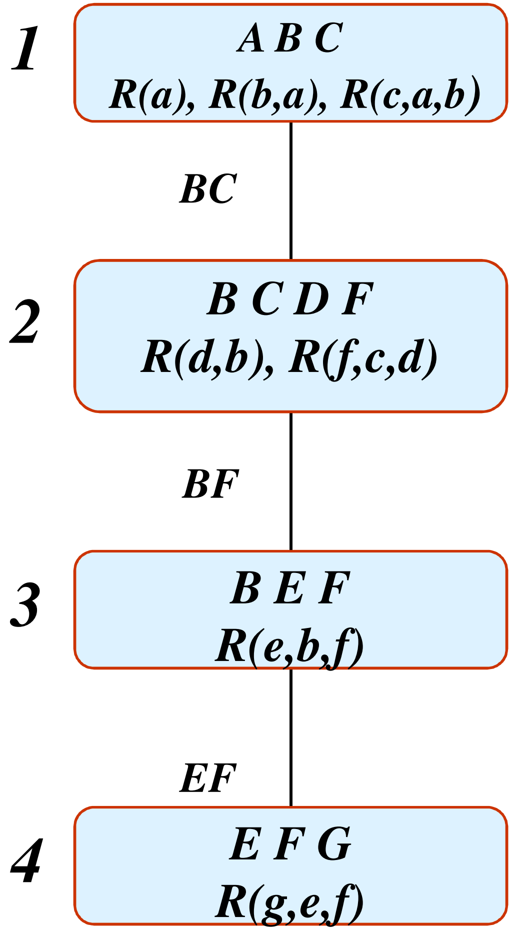




Ordering: (A, B, C, D, E, F, G)

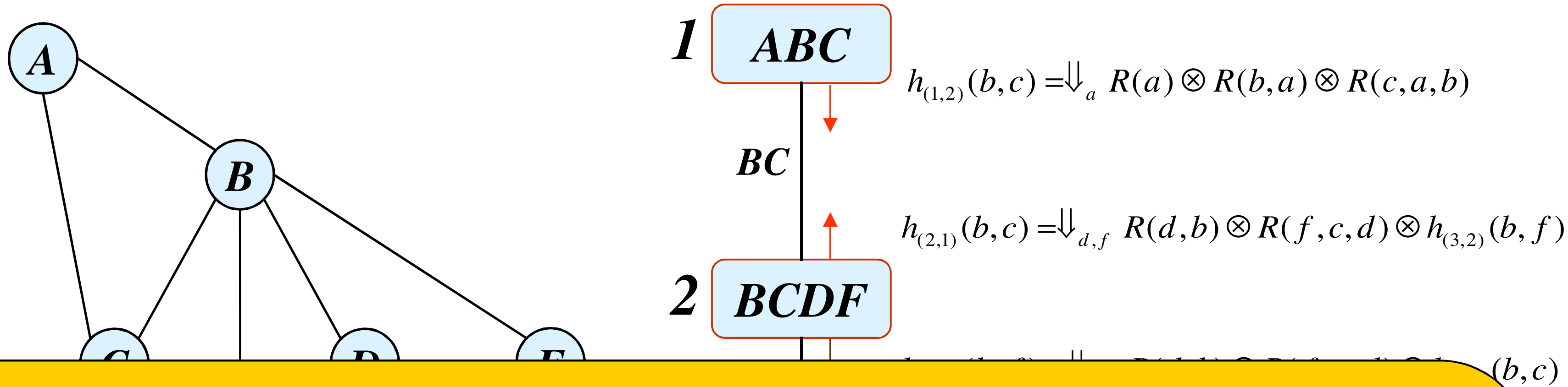
Tree Decomposition





- Each function in a cluster
- Satisfy running intersection property
- Then infer a function in a cluster and send to neighbors

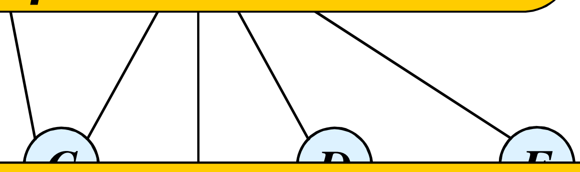
Junction-tree Clustering

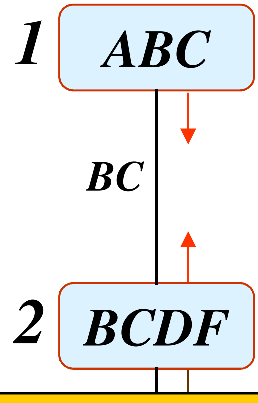


Time exponential in the induced-width / treewidth O(nk^{w*+1}), space exponential in the separator width along ordering d

Junction-tree Clustering

Each cluster is perfectly explicit.
Inference re-parameterizes the graphical model



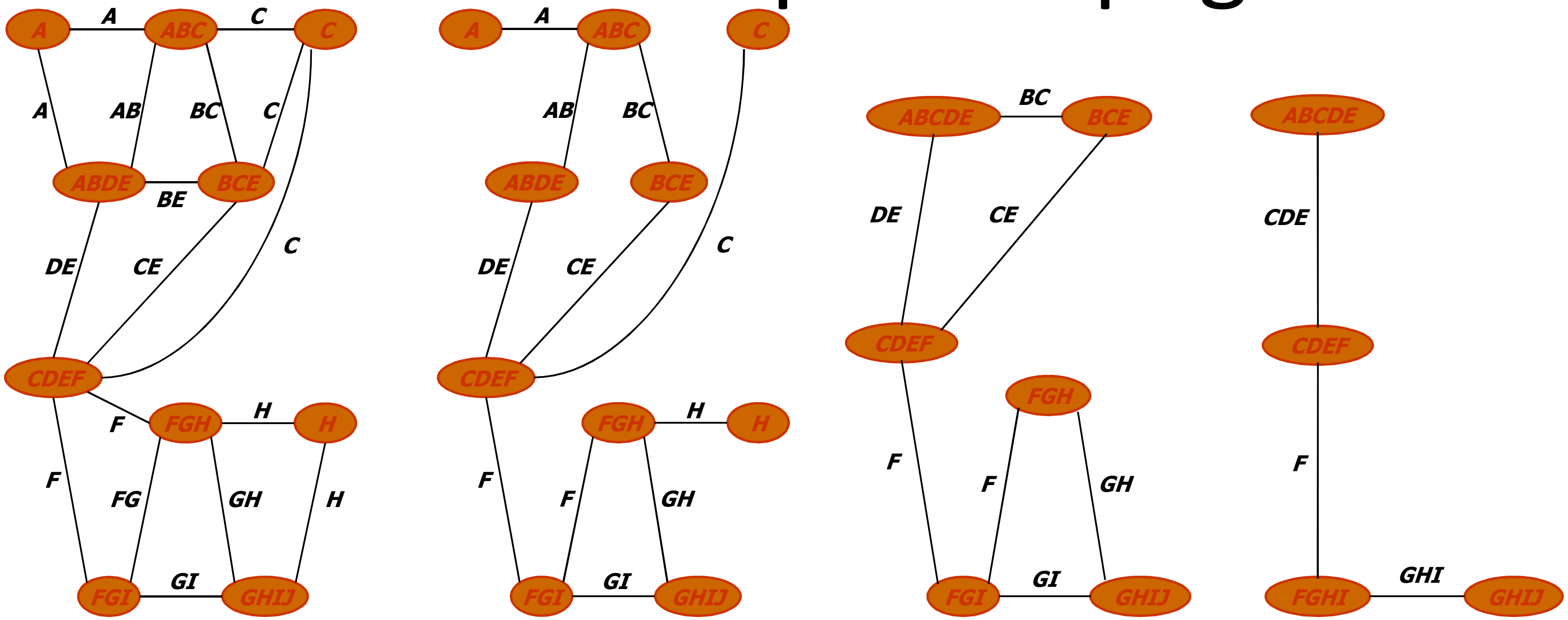


$$h_{(1,2)}(b,c) = \Downarrow_a R(a) \otimes R(b,a) \otimes R(c,a,b)$$

$$h_{(2,1)}(b,c) = \Downarrow_{d,f} R(d,b) \otimes R(f,c,d) \otimes h_{(3,2)}(b,f)$$

Time exponential in the induced-width / treewidth O(nk^{w*+1}), space exponential in the separator width along ordering d

Iterative Join-Graphs Propagation



more accuracy

less complexity

(Kask, Mateescu,Gogate,Dechter, 2002,2009)

Summary of Inference

- Inference is time and space exponential in w*(d)
- Finding the best ordering is NP-complete but good orderings can be achieved
- Inference is a **compilation** scheme; a re-parameterization scheme: each cluster possesses local information to answer whatever query without looking outside.
- Main limitation: memory complexity

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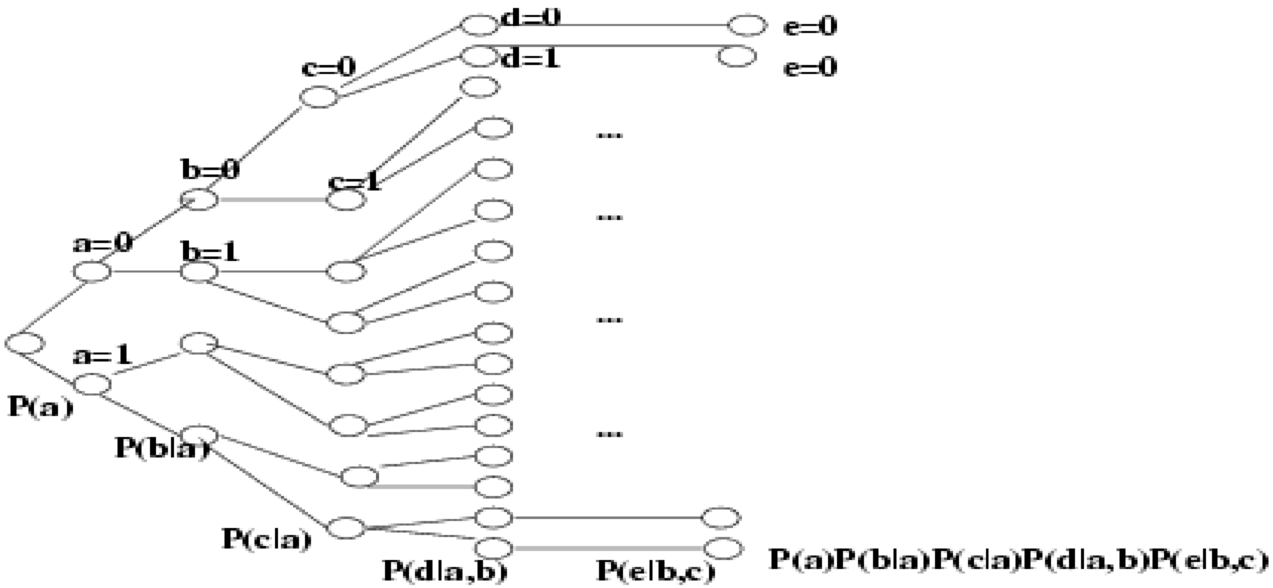
Approximations

Back to exact: Anytime schemes

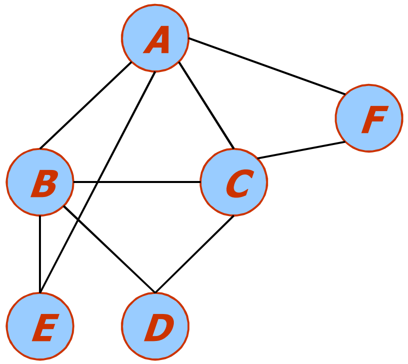
Enumeration of the the probability tree

$$P(a, e=0) = P(a)\sum_{b} P(b \mid a)\sum_{c} P(c \mid a)\sum_{b} P(d \mid a, b)\sum_{e=0} P(e \mid b, c)$$

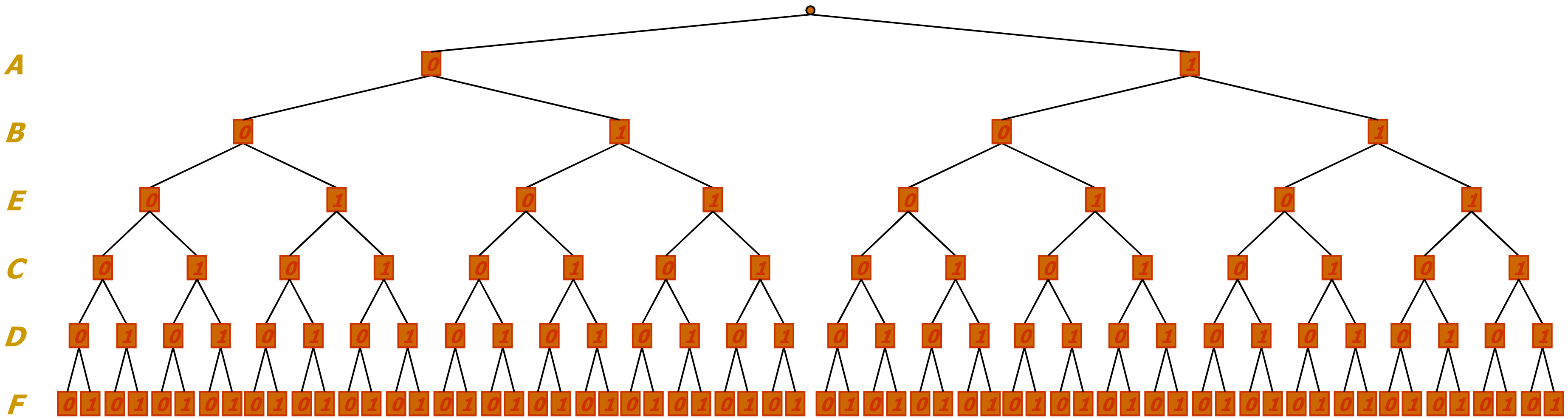


Complexity of searching the probability tree: exponential time. But can use linear space

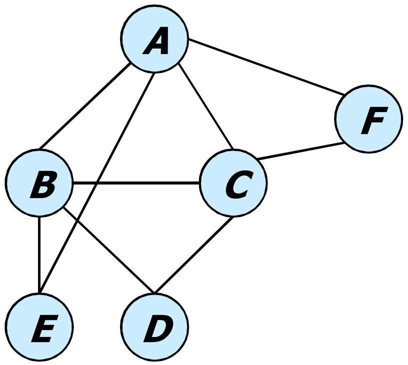
Classic OR Search Space



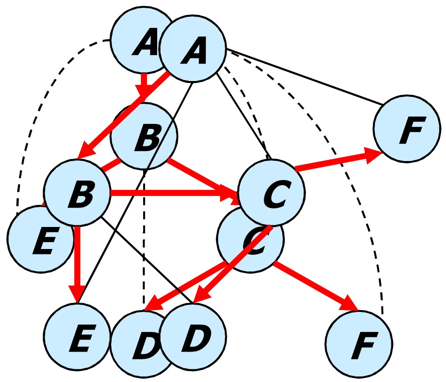
Ordering: A B E C D F



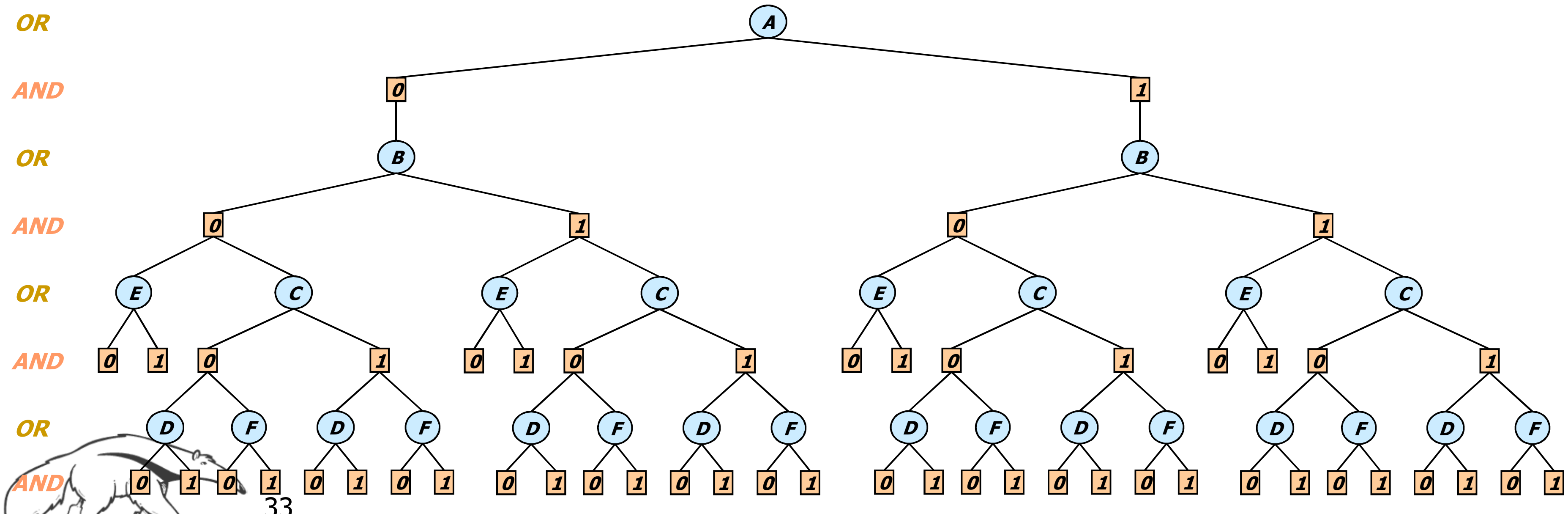
AND/OR Search Space



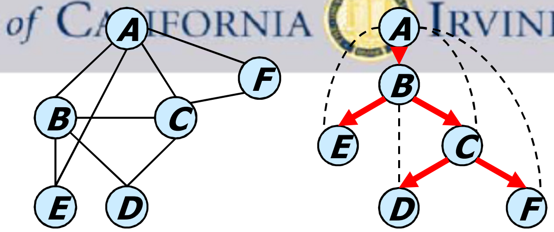
Primal graph

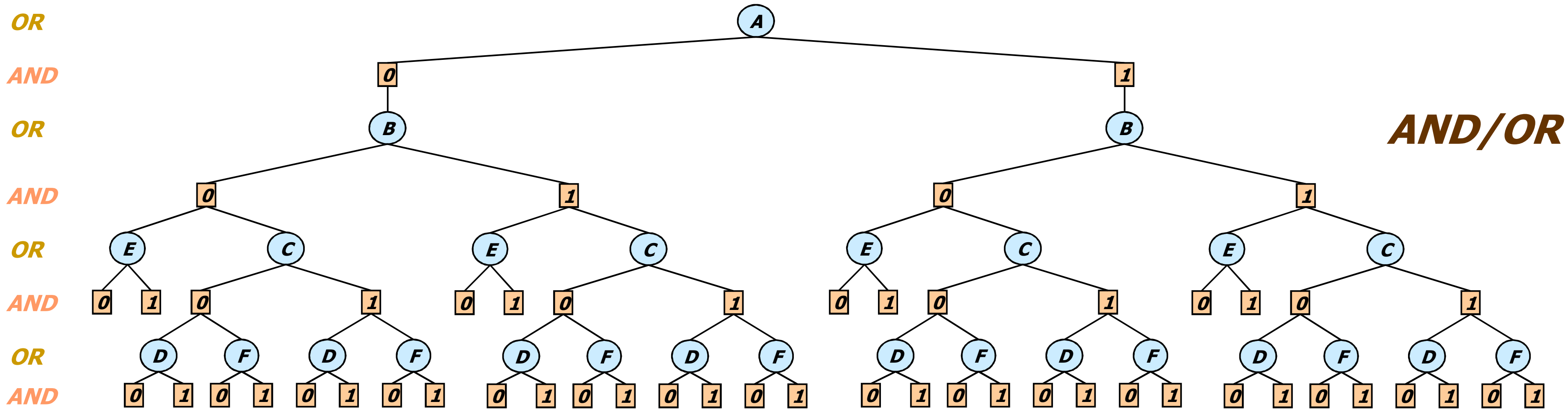


DFS tree

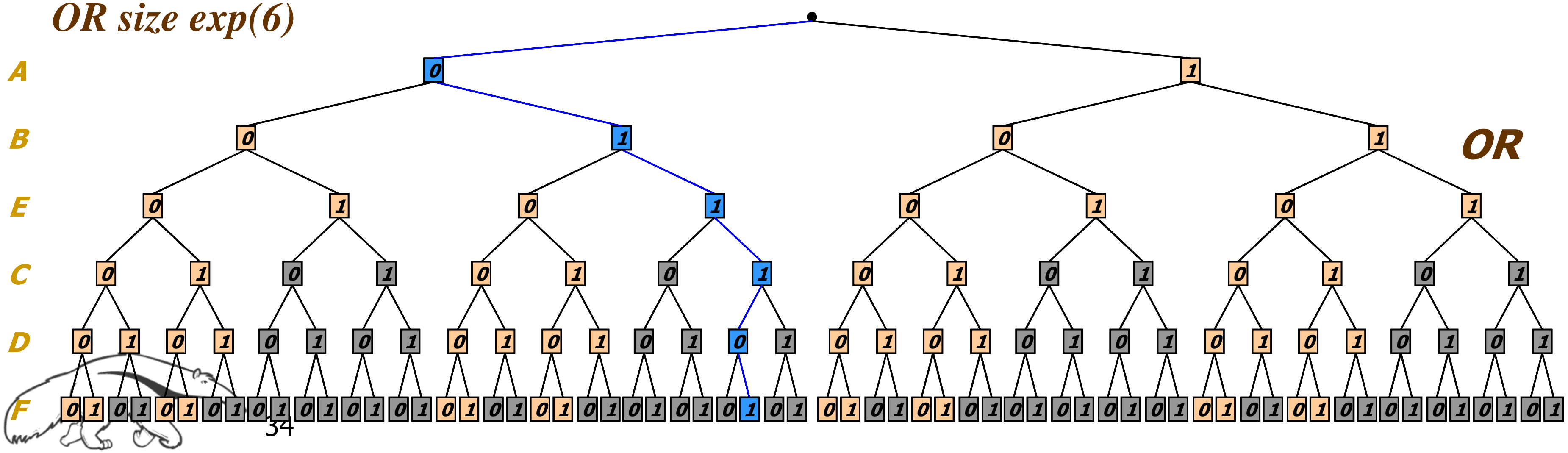


AND/OR vs. OR





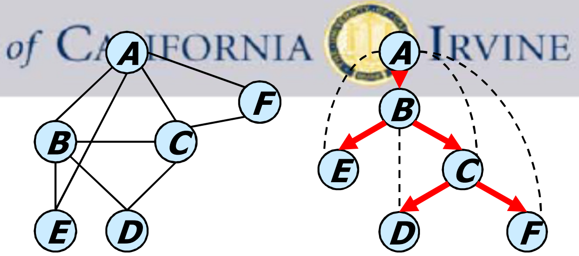
AND/OR size: exp(4),
OR size exp(6)

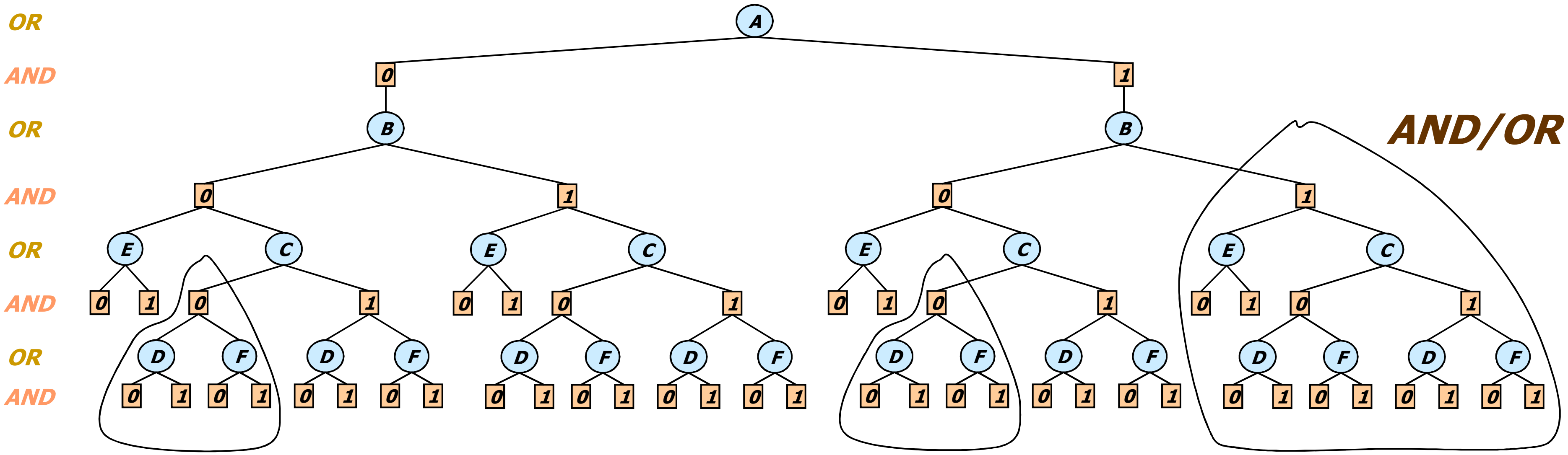


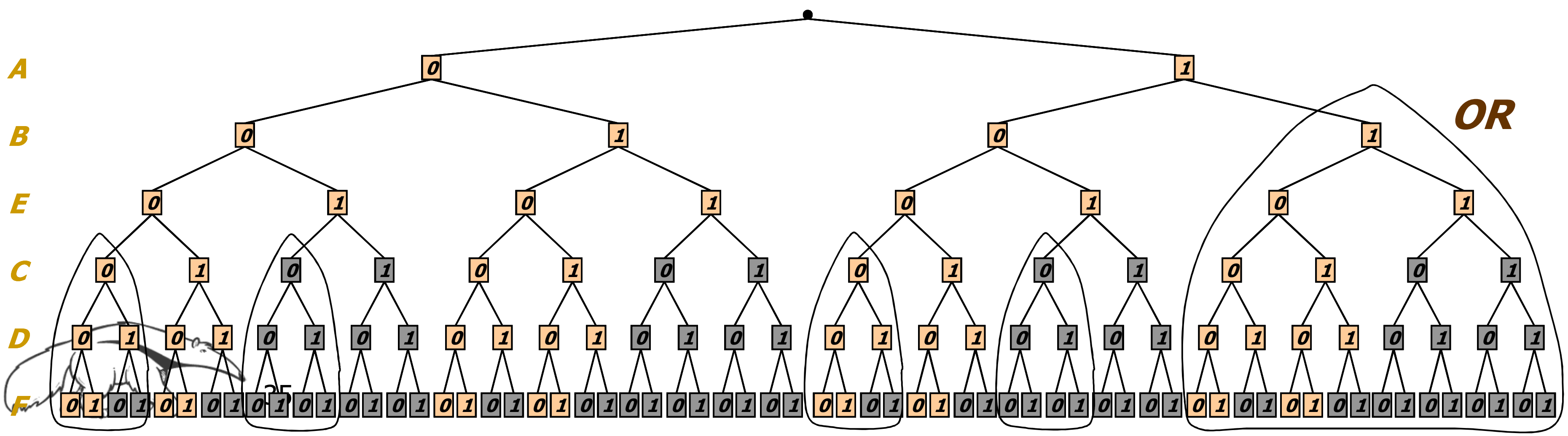
AND/OR vs. OR

with Constraints

No-goods
(A=1,B=1)
(B=0,C=0)



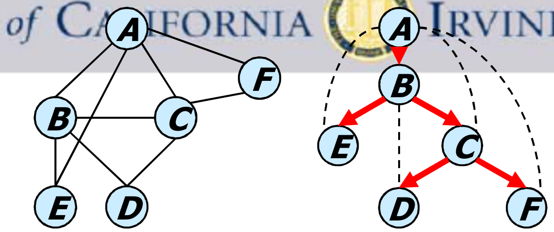


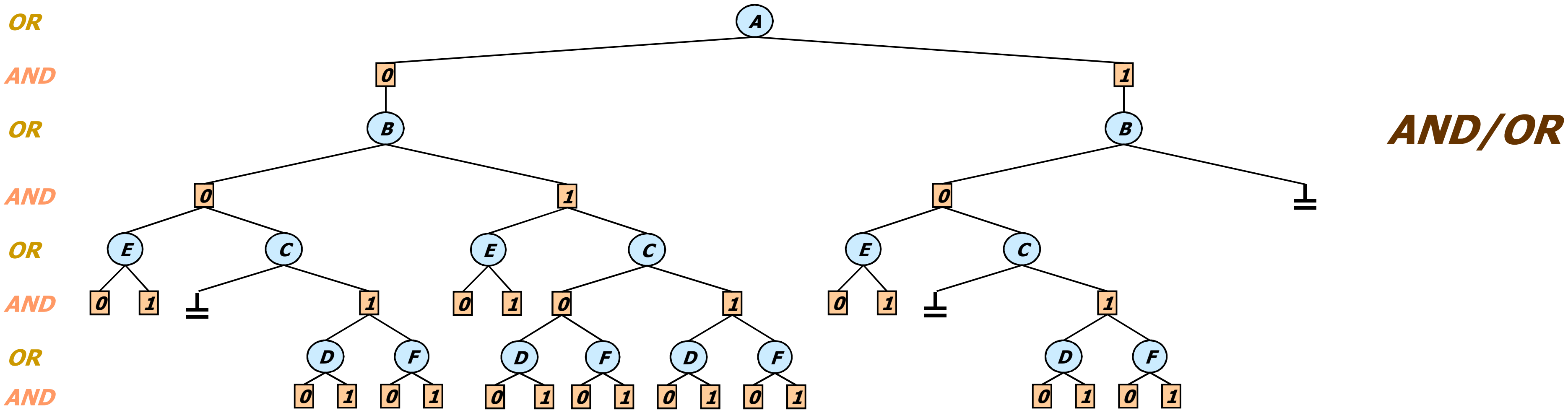


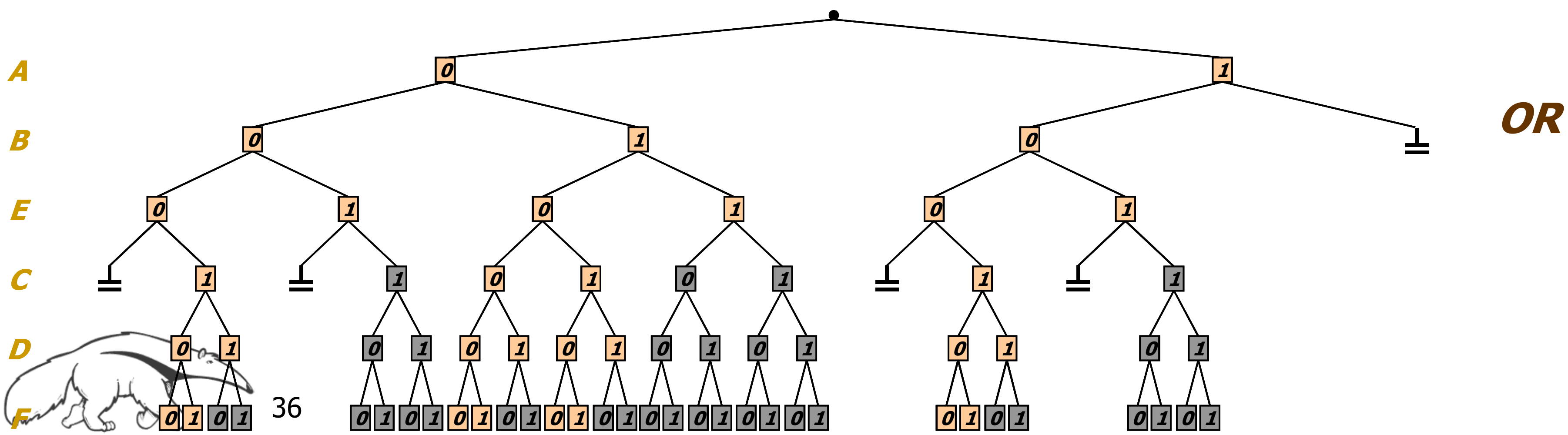
AND/OR vs. OR

with Constraints

No-goods
(A=1,B=1)
(B=0,C=0)

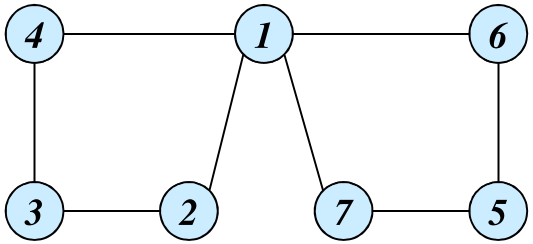






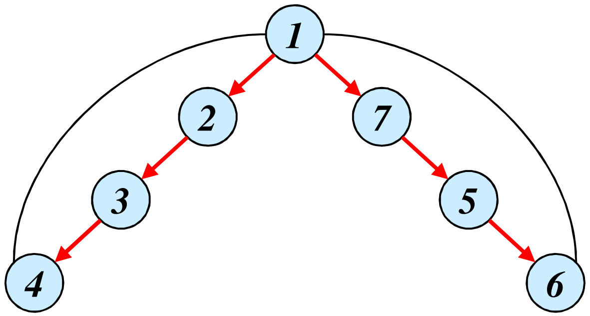
Pseudo-Trees

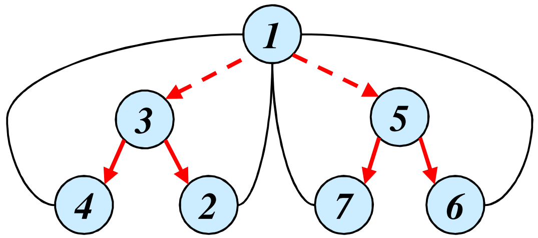
(Freuder 85, Bayardo 95, Bodlaender and Gilbert, 91)

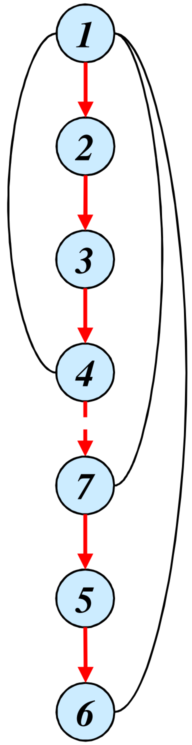


(a) Graph

$$m <= w^* \log n$$





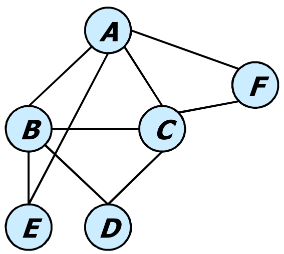


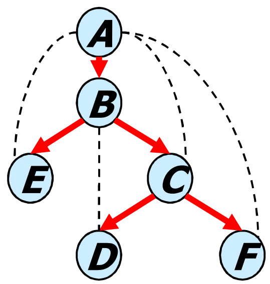
(b) DFS tree
depth=3

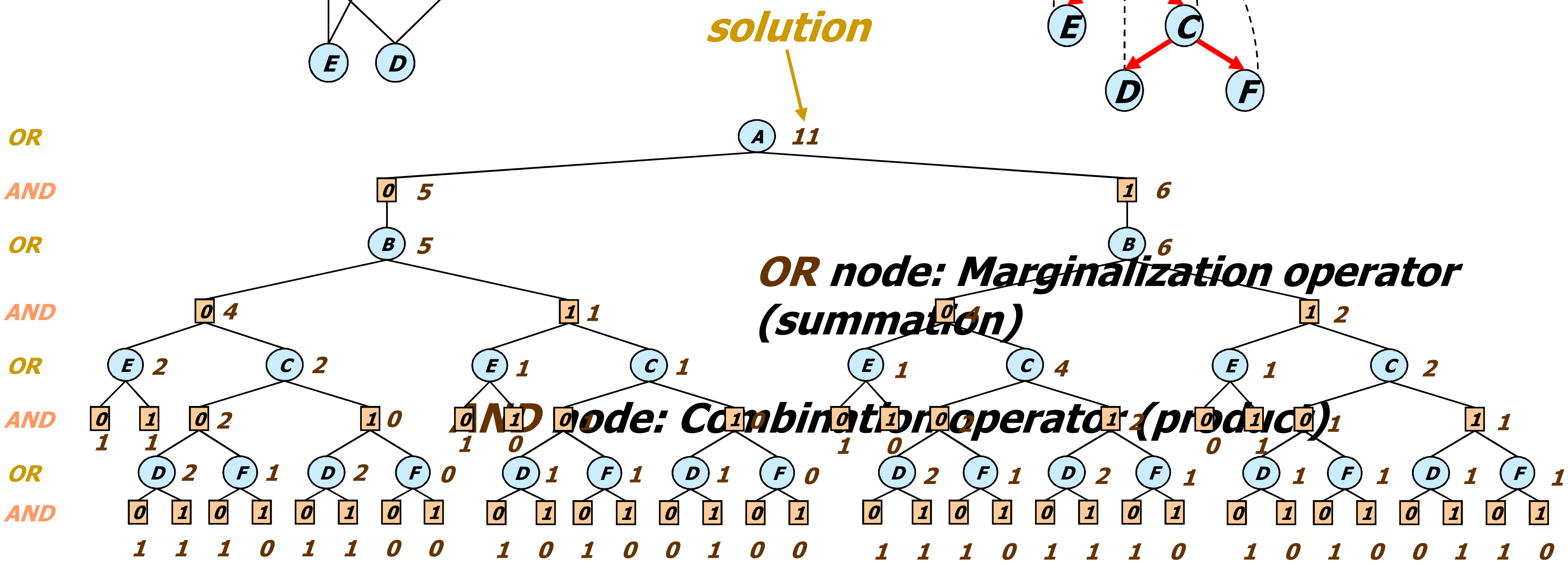
(c) pseudo- tree
depth=2

(d) Chain
depth=6

DFS algorithm (#CSP example)







Value of node = number of solutions below it

The recursive value rule

$$\mathbf{G}(\boldsymbol{Z}) = \Downarrow_{\boldsymbol{X}-\boldsymbol{Z}} \otimes_{\boldsymbol{f}\in\boldsymbol{F}} \mathrm{f(x)}$$

$$\mathrm{v(n)} = \mathrm{G}|_{n=(x_1,\ldots,x_i})$$

$$v(n) = \bigotimes_{n'\in children(n)} v(n'), \qquad \textit{if } n = \langle X, x\rangle \textit{ is an AND node,}$$

$$v(n) = \Downarrow_{n'\in children(n)} (w_{(n,n')} \bigotimes v(n')), \qquad \textit{if } n = X \textit{ is an OR node.}$$

AND/OR Tree DFS Algorithm (Belief Updating)

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

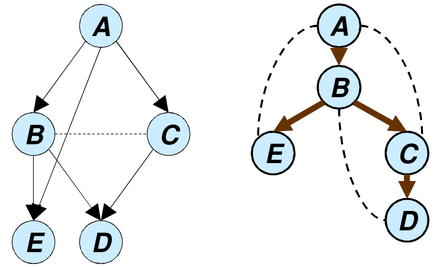
A	B=0	B=1
0	.4	.6
1	.1	.9

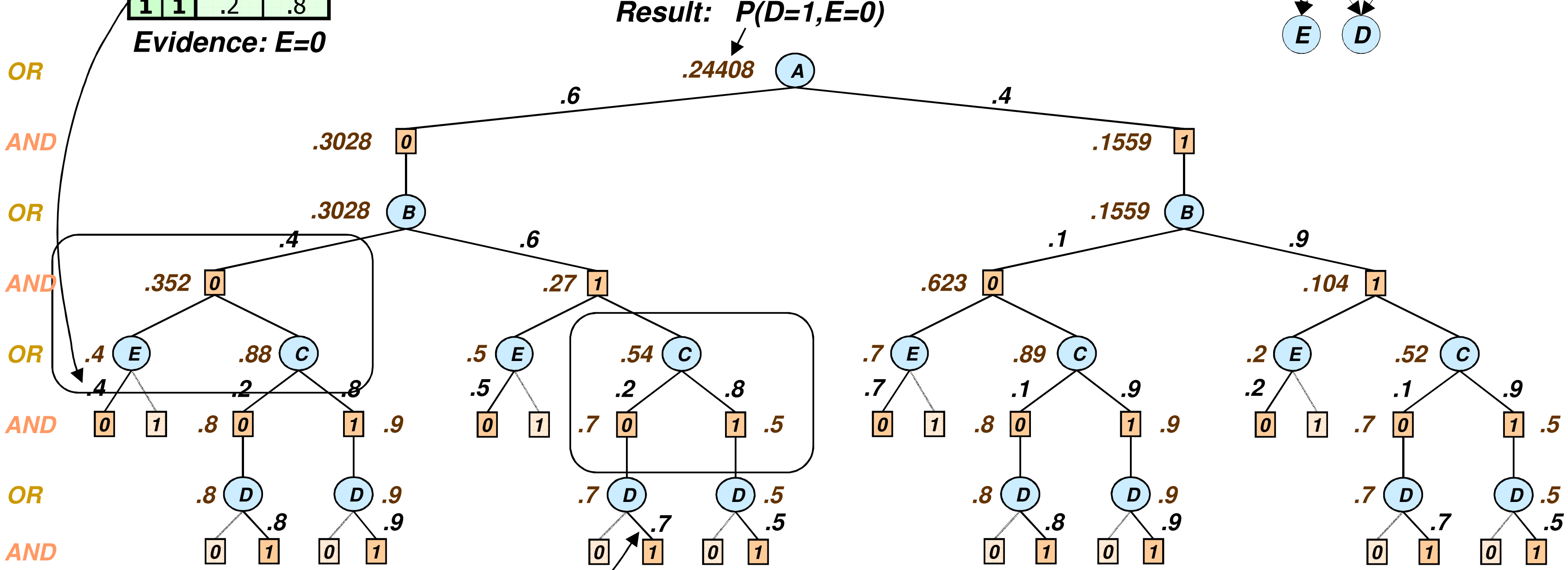
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





$P(D \mid B, C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

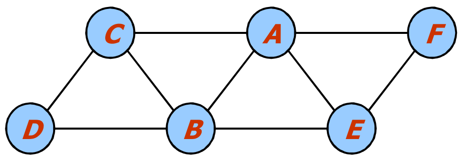
Evidence: D=1

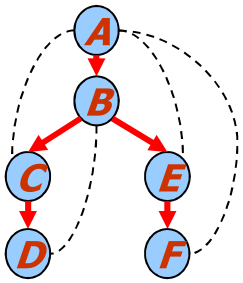
***OR* node: Marginalization operator (summation)**

***AND* node: Combination operator (product)**

***Value* of node = updated belief for sub-problem below**

AND/OR Tree Search for COP





A	B	f_1
0	0	**2**
0	1	**0**
1	0	**1**
1	1	**4**

A	C	f_2
0	0	**3**
0	1	**0**
1	0	**0**
1	1	**1**

A	E	f_3
0	0	**0**
0	1	**3**
1	0	**2**
1	1	**0**

A	F	f_4
0	0	**2**
0	1	**0**
1	0	**0**
1	1	**2**

B	C	f_5
0	0	**0**
0	1	**1**
1	0	**2**
1	1	**4**

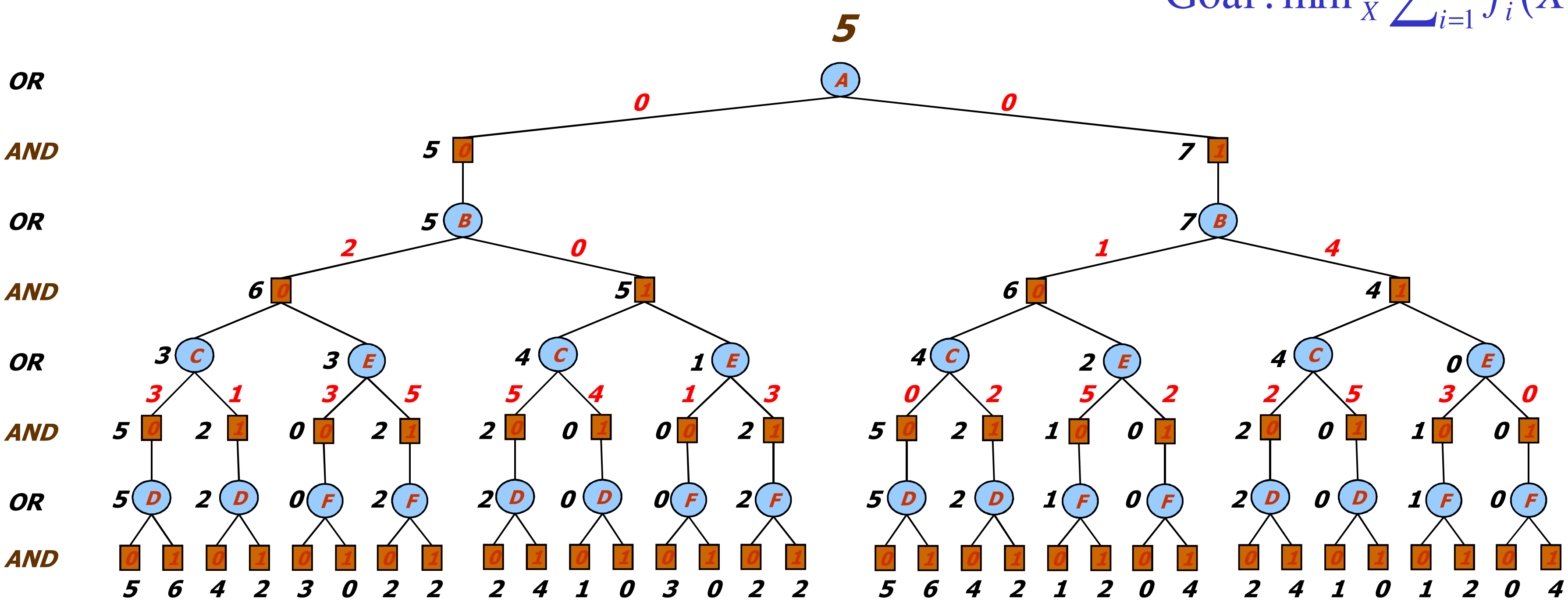
B	D	f_6
0	0	**4**
0	1	**2**
1	0	**1**
1	1	**0**

B	E	f_7
0	0	**3**
0	1	**2**
1	0	**1**
1	1	**0**

C	D	f_8
0	0	**1**
0	1	**4**
1	0	**0**
1	1	**0**

E	F	f_9
0	0	**1**
0	1	**0**
1	0	**0**
1	1	**2**

$$\text{Goal} : \min_X \sum_{i=1}^{9} f_i(X)$$



AND node = Combination operator (summation)

OR node = Marginalization operator (minimization)

Properties of AND/OR Tree Search

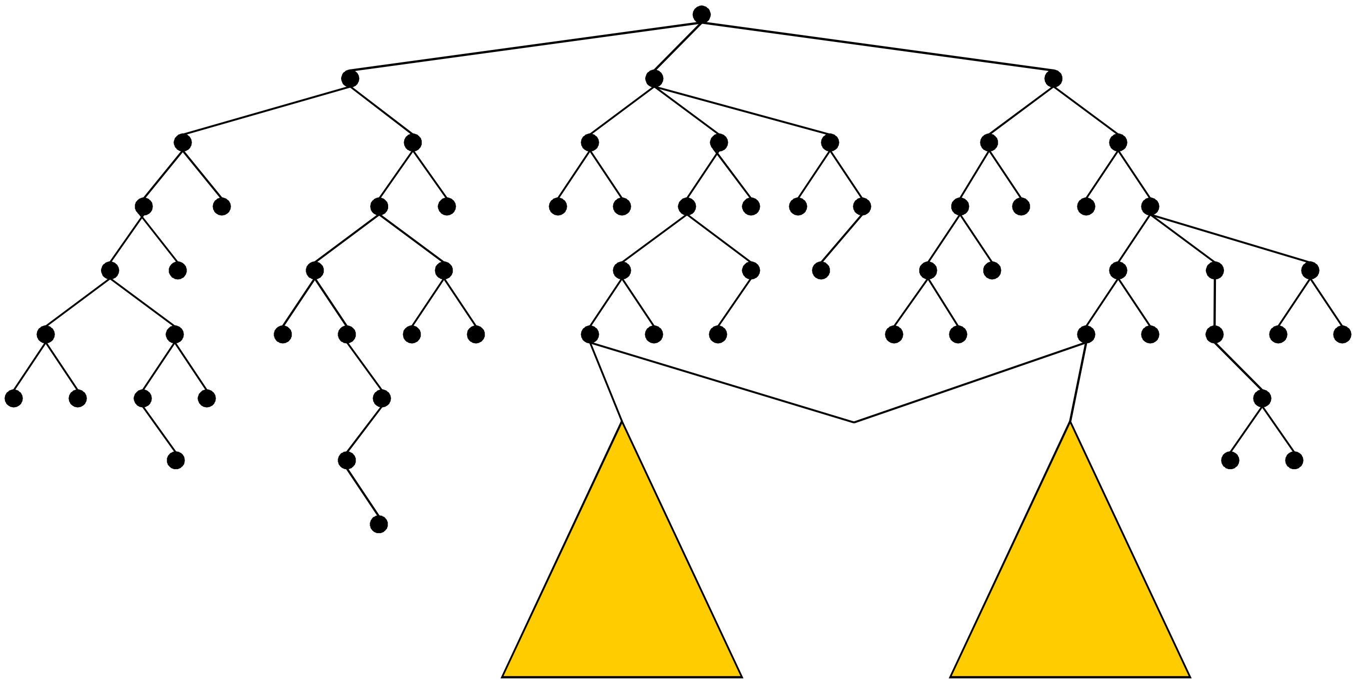
	AND/OR tree	**OR tree**
Space	O(n)	O(n)
Time	$O(n\ k^{h})$ $O(n\ k^{w^* \log n})$ (Freuder & Quinn85), (Collin, Dechter & Katz91), (Bayardo & Miranker95), (Darwiche01)	$O(k^{n})$

***k* = domain size**
***h* = depth of pseudo-tree**
***n* = number of variables**
***w** = treewidth**

***Tasks:* Consistency, Counting, Optimization, Belief updating Max-expected utility, partition function**

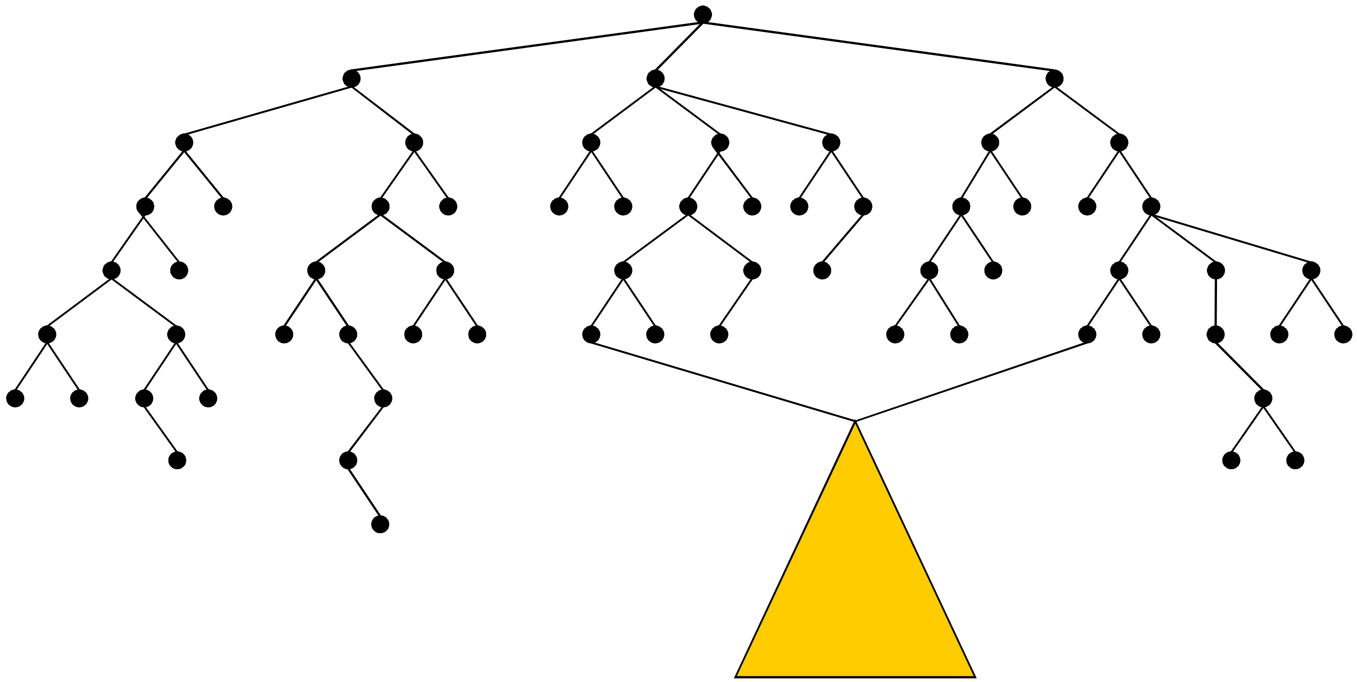
From Search Trees to Search Graphs

- Any two nodes that root identical sub-trees or sub-graphs can be merged

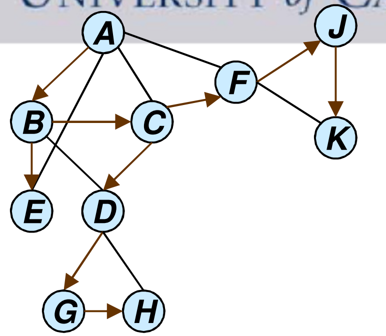


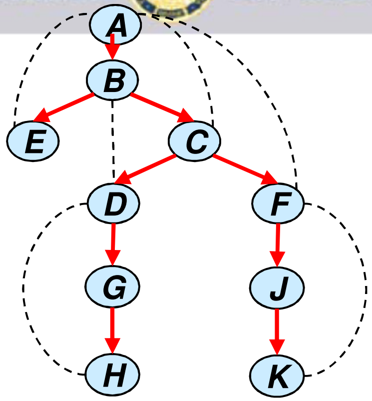
From Search Trees to Search Graphs

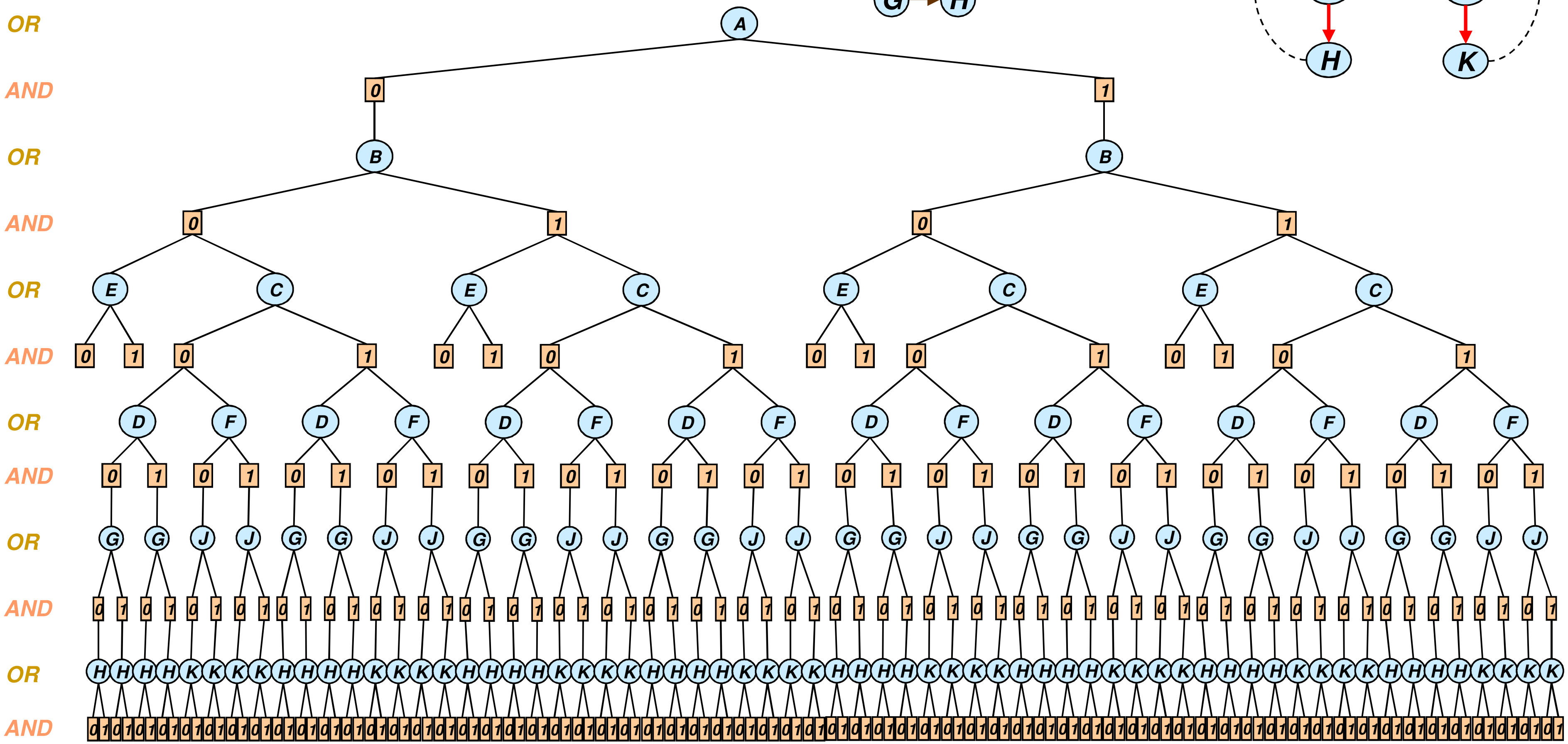
- Any two nodes that root identical sub-trees or sub-graphs can be merged



From AND/OR Tree







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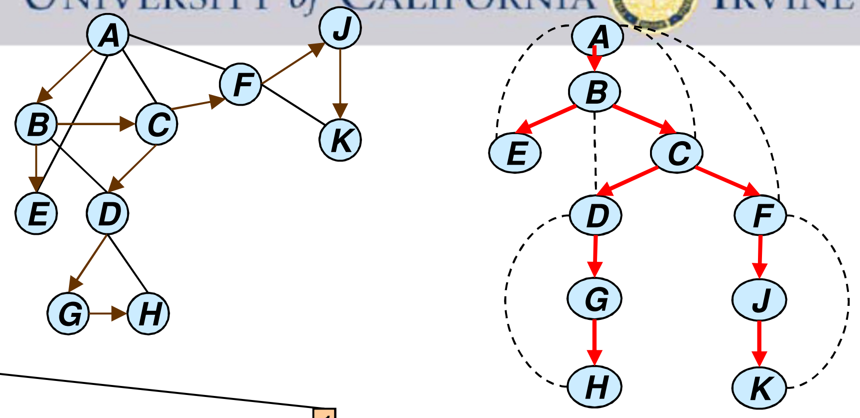
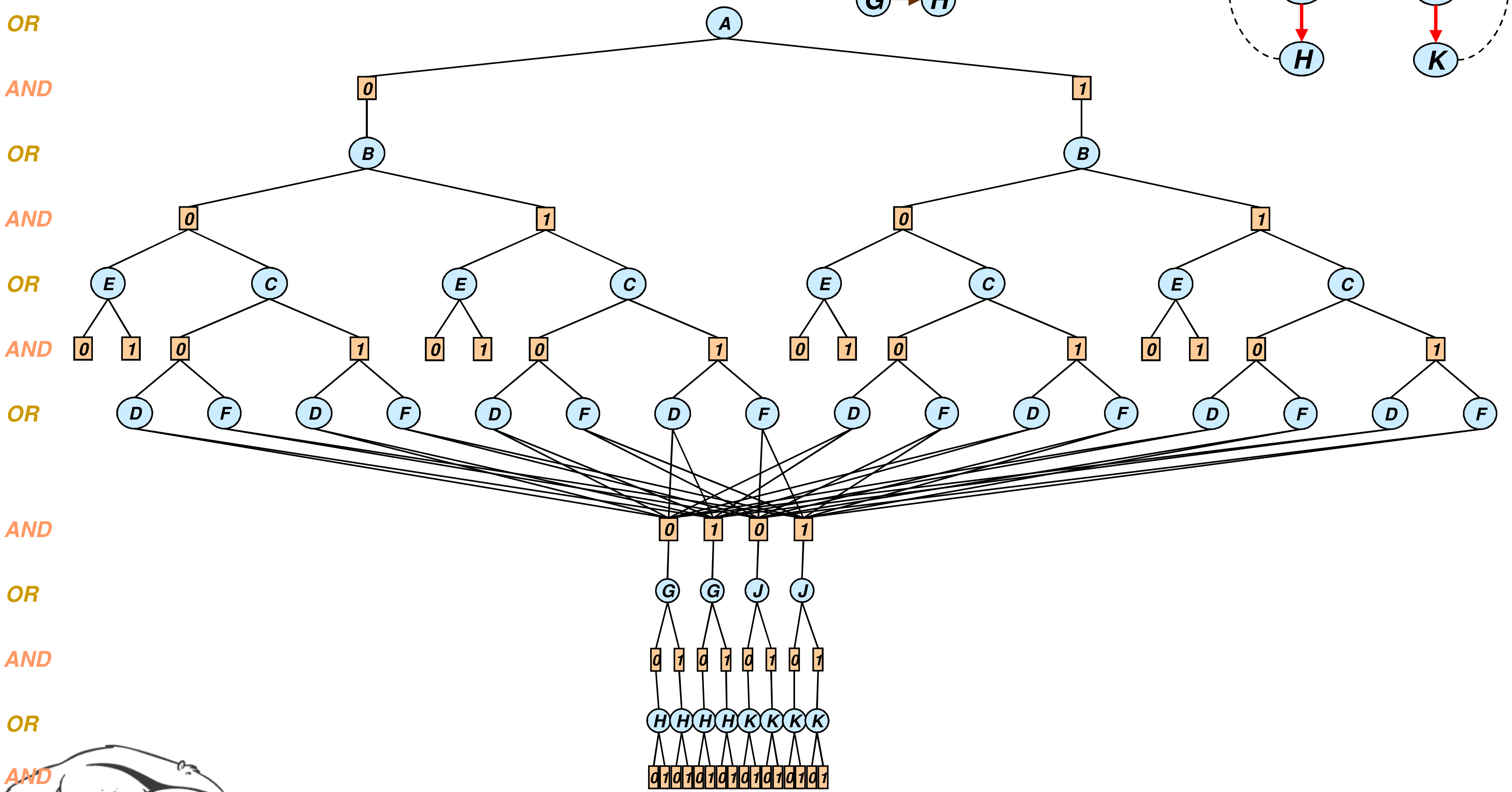
An AND/OR Graph

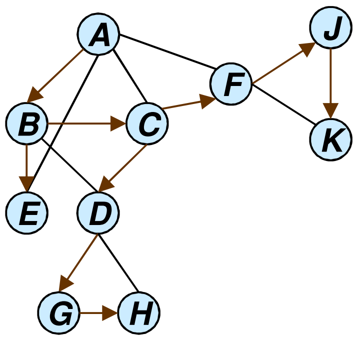


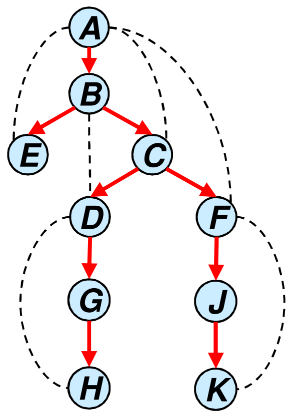


Context-based Caching

- Caching is possible when context is the same

- context = parent-separator set in induced pseudo-graph
  = current variable +
  parents connected to subtree below





context(B) = {A, B}

context(c) = {A,B,C}

context(D) = {D}

context(F) = {F}

AND/OR Tree DFS Algorithm (Belief Updating)

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

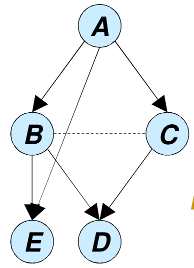
A	B=0	B=1
0	.4	.6
1	.1	.9

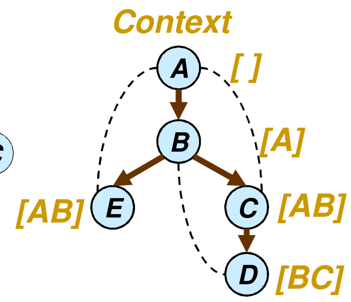
$P(C \mid A)$

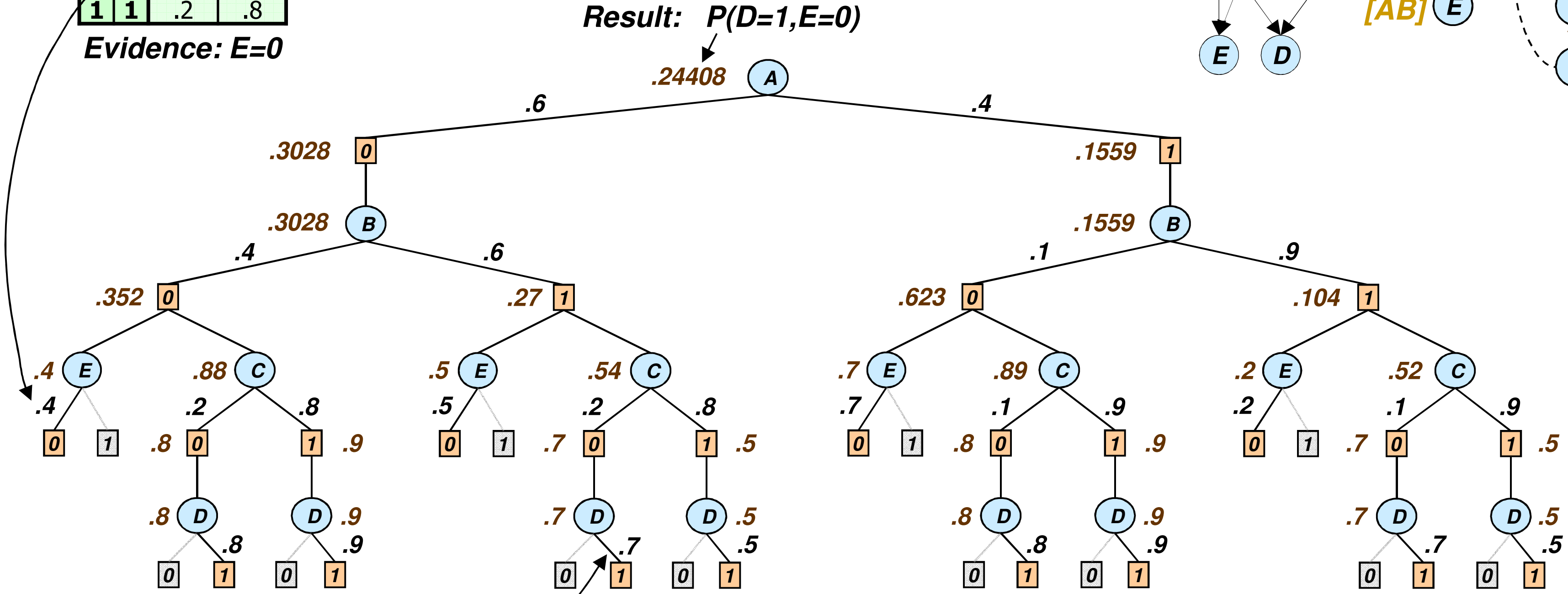
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4







$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

OR node: Marginalization operator (summation)

AND node: Combination operator (product)

Value of node = updated belief for sub-problem below

AND/OR Graph DFS Algorithm (Belief Updating)

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

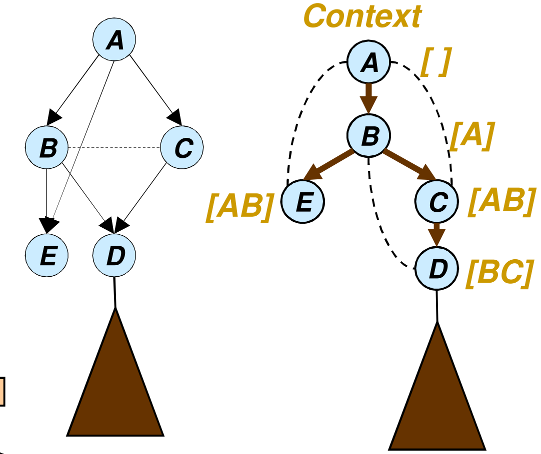
A	B=0	B=1
0	.4	.6
1	.1	.9

$P(C \mid A)$

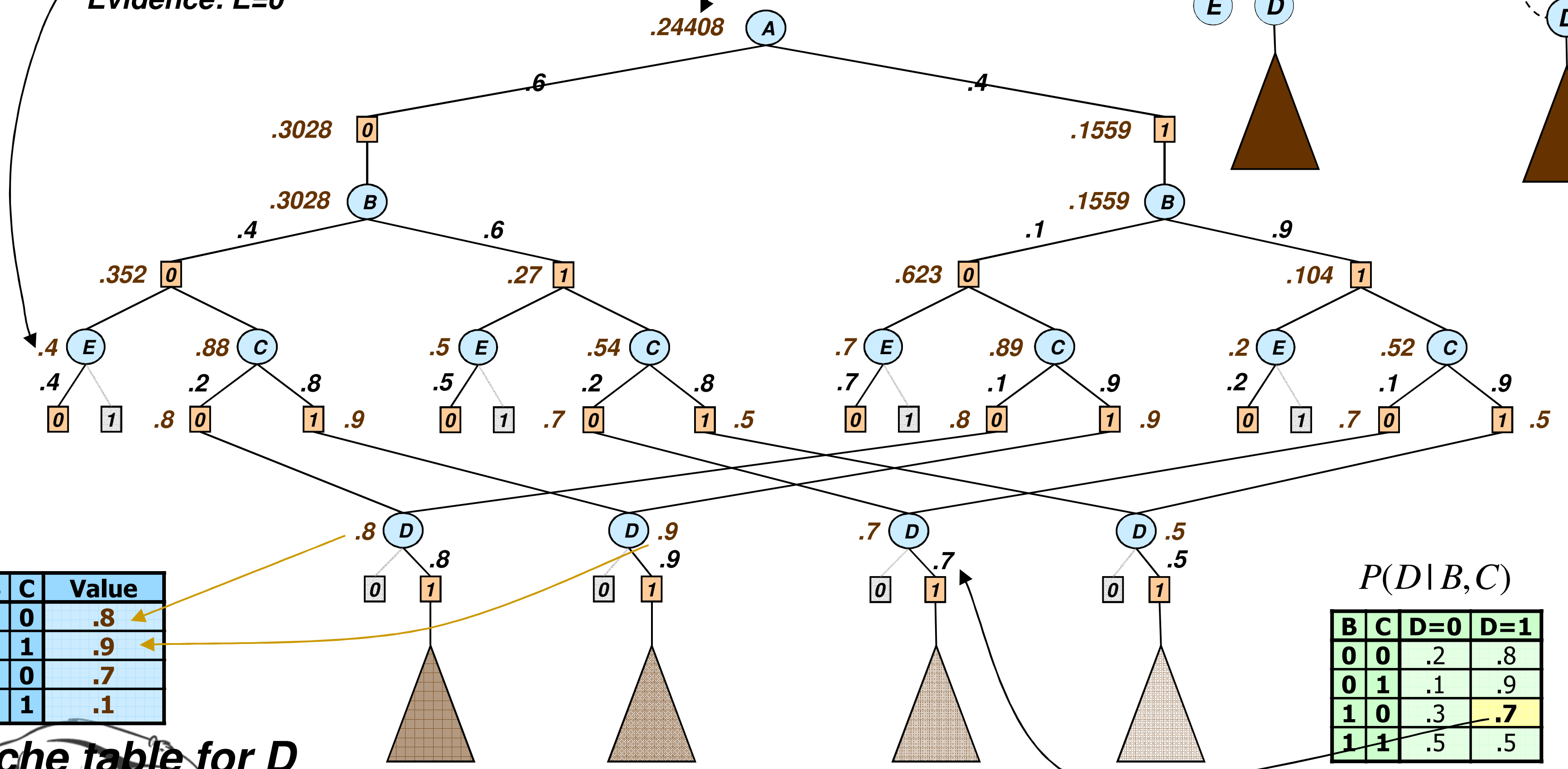
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4



Result: P(D=1,E=0)



B	C	Value
0	0	.8
0	1	.9
1	0	.7
1	1	.1

Cache table for D

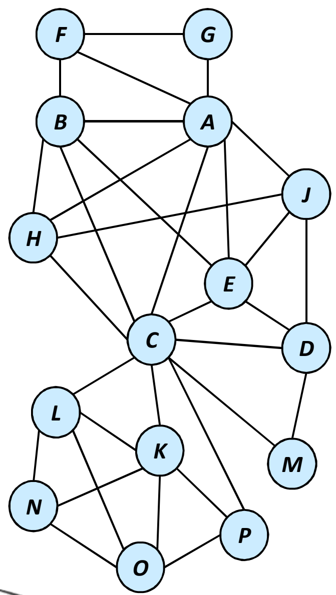
$P(D \mid B, C)$

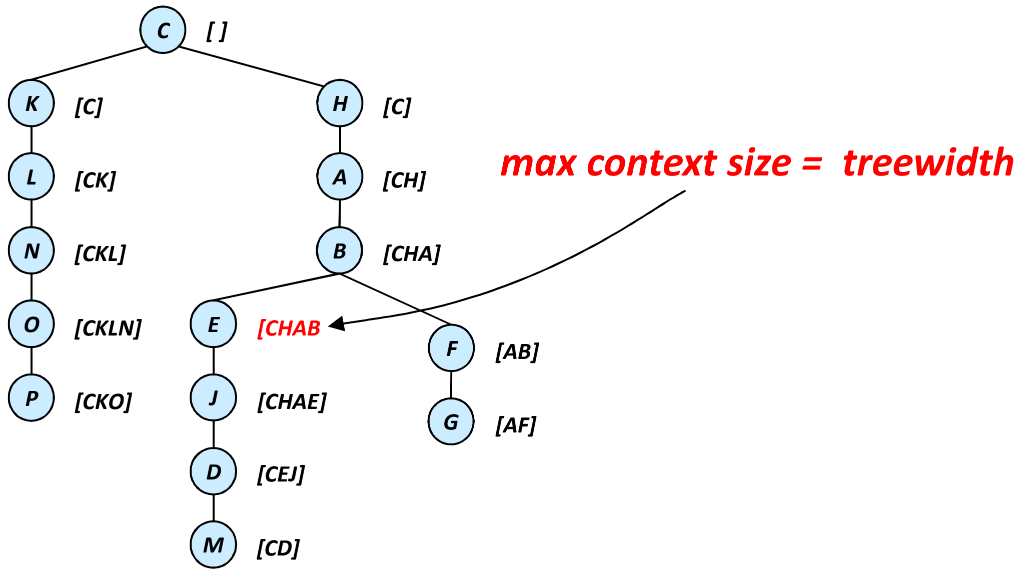
B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

How Big Is The Context?

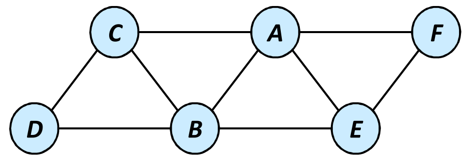
Theorem: The maximum context size for a pseudo tree is equal to the treewidth of the graph along the pseudo tree.



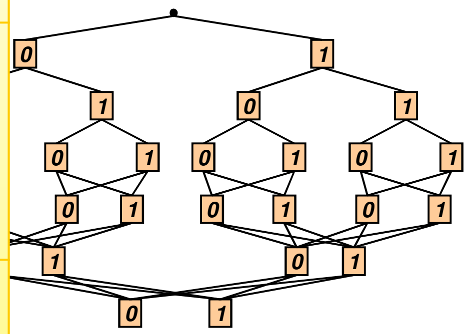


(C K H A B E J L N O D P M F G)

All Four Search Spaces



	AND/OR graph	OR graph
Space	$O(n\,k^{w^*})$	$O(n\,k^{pw^*})$
Time	$O(n\,k^{w^*})$	$O(n\,k^{pw^*})$

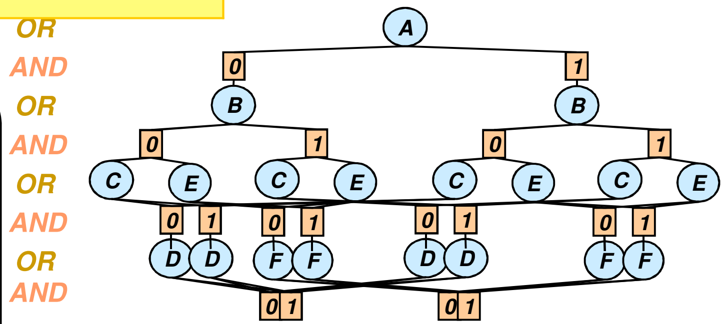


...xt minimal OR search graph

28 nodes

Computes any query:

- *Constraint satisfaction*
- *Optimization*
- *Weighted counting*

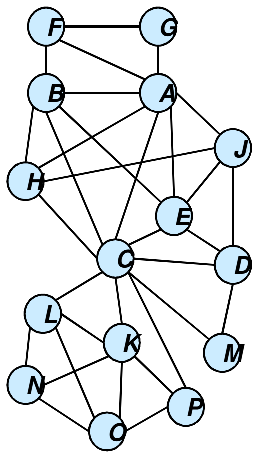


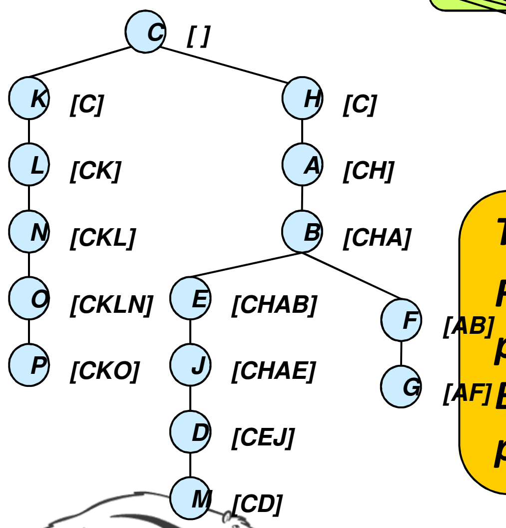
Context minimal AND/OR search graph

18 AND nodes

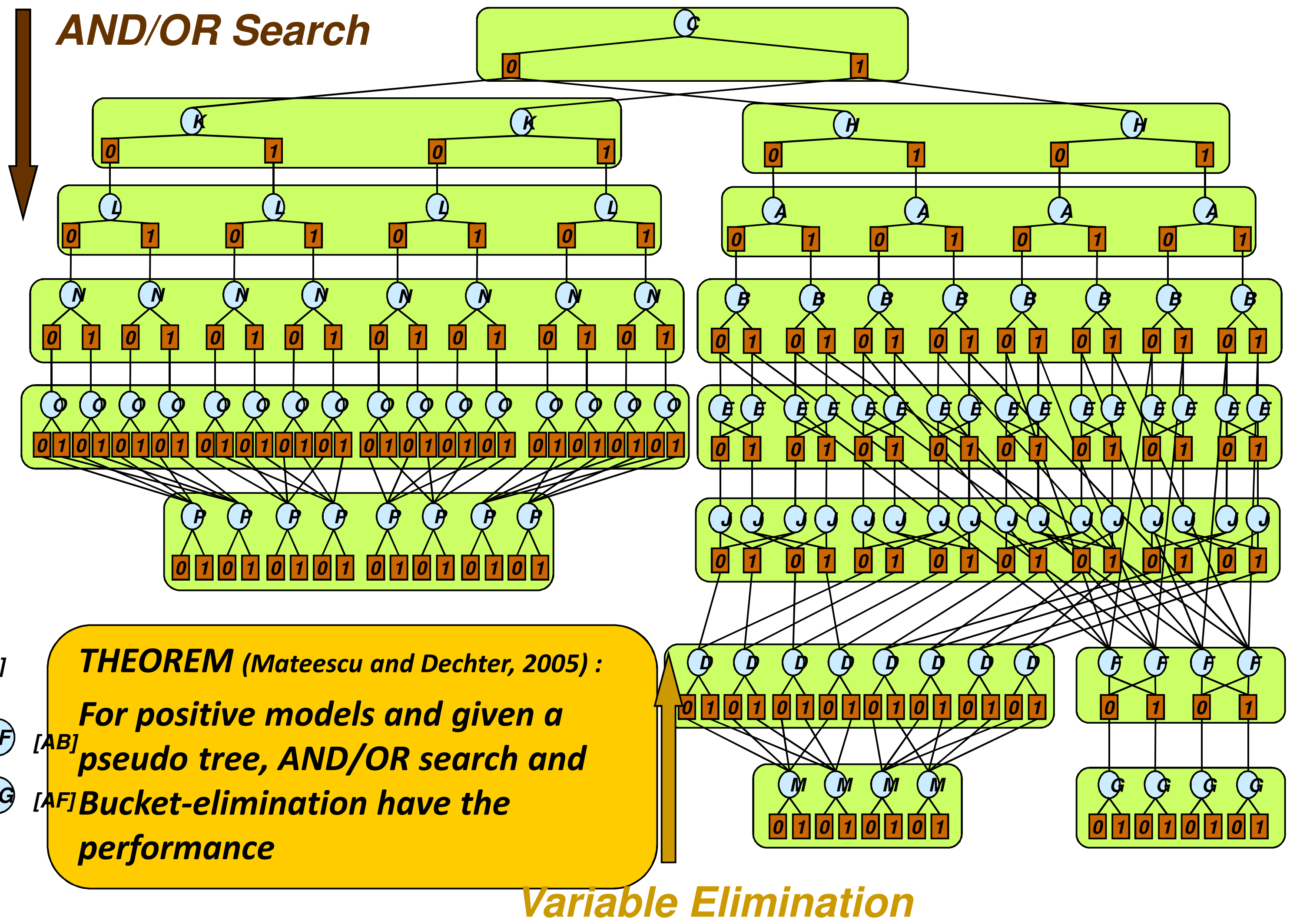
Any query is best computed Over the c-minimal AO space

What is better? Inference or Search





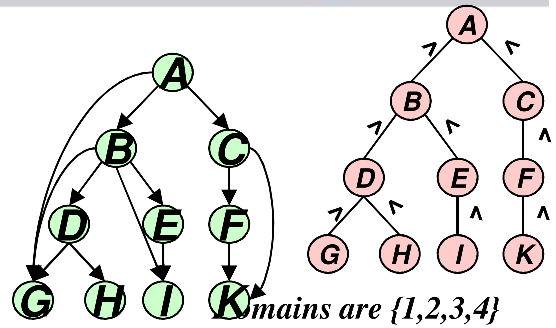
(C K H A B E J L N O D P M F G)

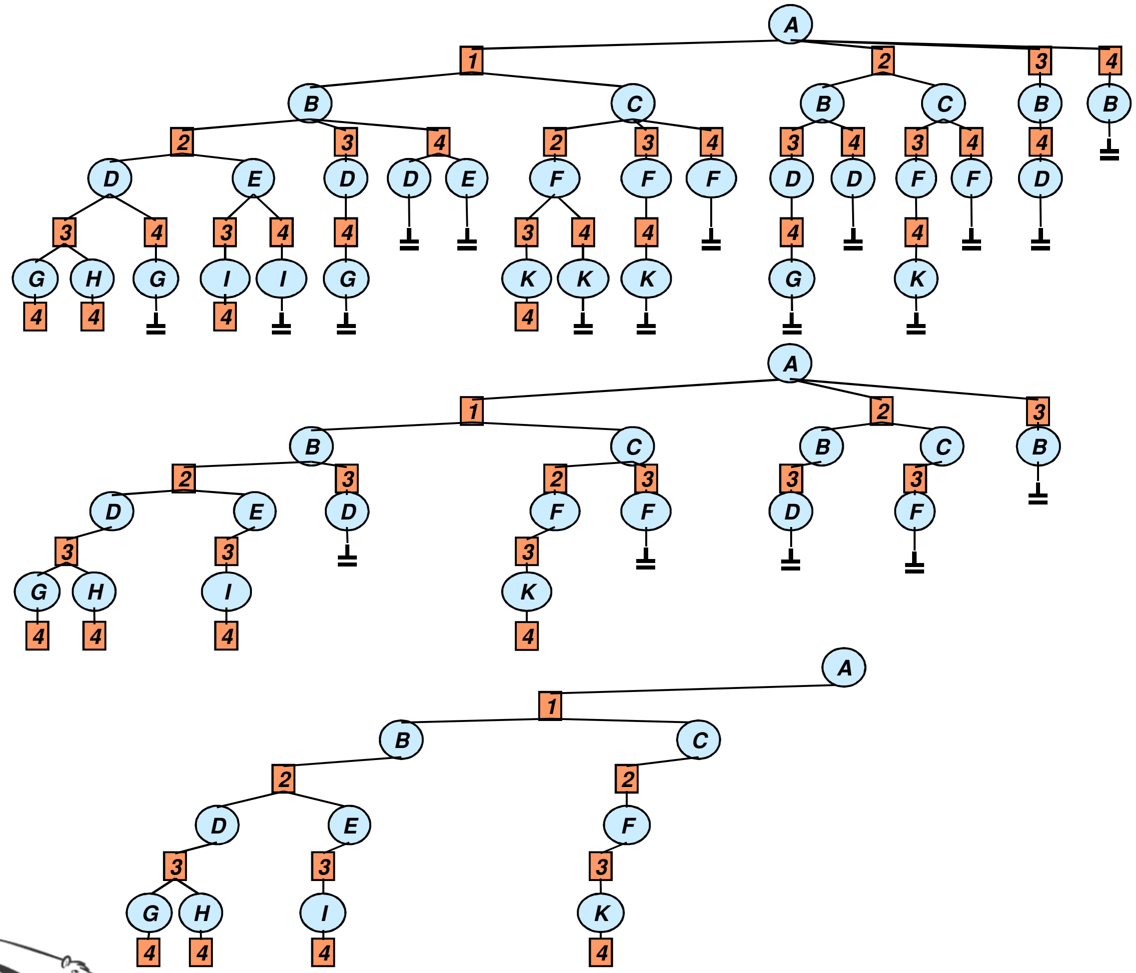


***THEOREM** (Mateescu and Dechter, 2005) :*

For positive models and given a pseudo tree, AND/OR search and Bucket-elimination have the performance

The Effect of Constraint Propagation





CONSTRAINTS ONLY

FORWARD CHECKING

MAINTAINING ARC CONSISTENCY

Outline

- What are graphical models?
- Inference
- Search; via AND/OR search
- Time vs space, search vs inference
- Bounding inference
- Anytime AND/OR branch and bound
- Experiments, Competitions

and conclusions

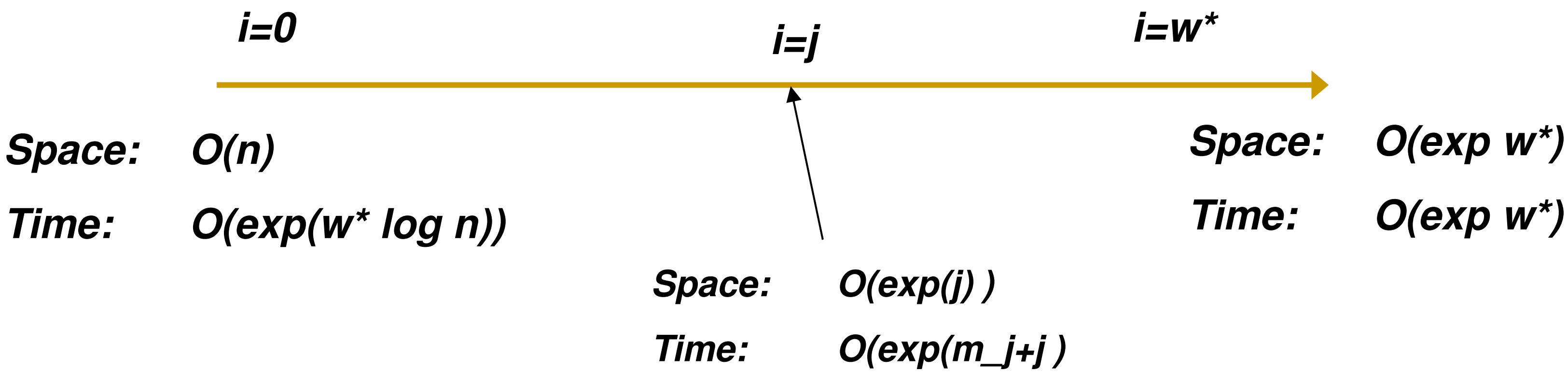
Exact algorithms

Approximations

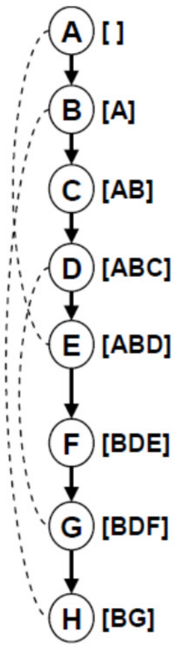
Back to exact: Anytime schemes

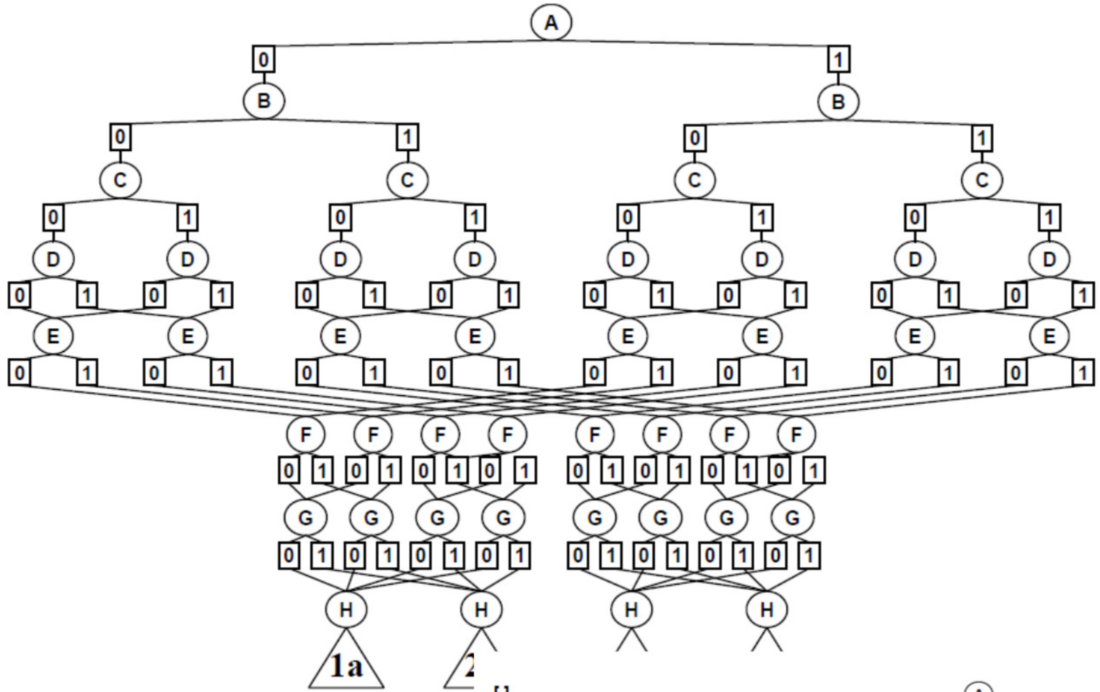
Searching AND/OR Graphs trading space for time

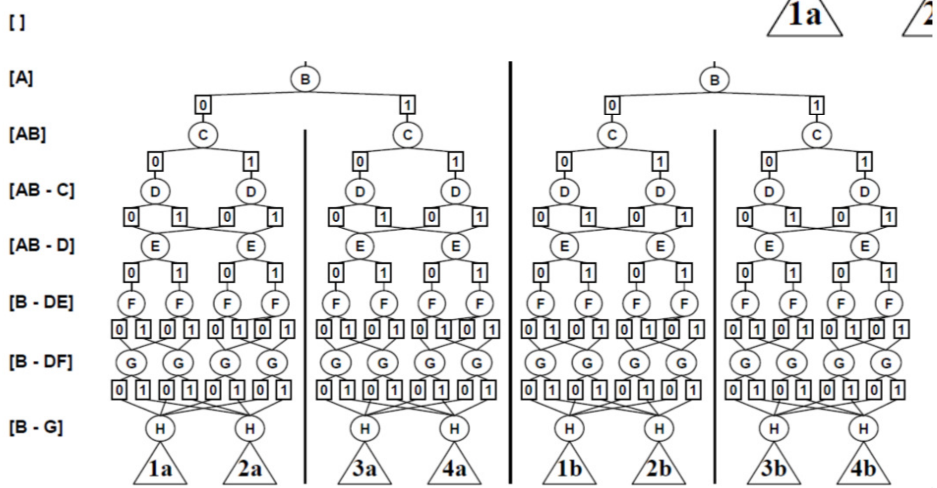
- AO(j): searches depth-first, cache i-context
  - j = the max size of a cache table (i.e. number of variables in a context)

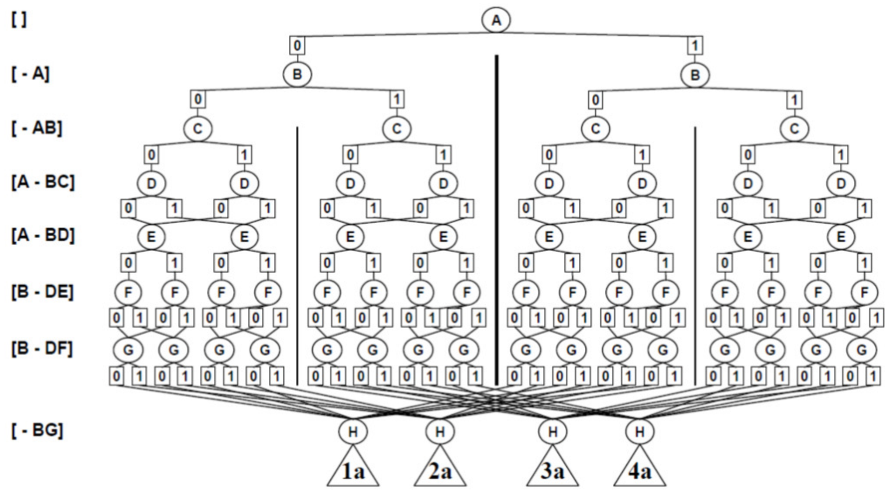


AOC(j) Adaptive Caching









Outline

- What are graphical models?
- Inference
- Search; via AND/OR search
- Time vs space, search vs inference
- Bounding inference (mini-bucket, cost-shifting, belief propagation)
- Anytime AND/OR branch and bound
- Experiments and conclusions

Exact algorithms

Approximations

Back to exact: Anytime schemes

Mini-bucket approximation:MPE task
(Dechter and Rish, 1997, 2003)

Split a bucket into mini-buckets =>bound complexity

$$\textbf{bucket}(X) = \{ h_1, \ldots, h_r, h_{r+1}, \ldots, h_n \}$$

$$h^X = \max_X \prod_{i=1}^{n} h_i$$

$$\{ h_1, \ldots, h_r \} \qquad \{ h_{r+1}, \ldots, h_n \}$$

$$g^X = (\max_X \prod_{i=1}^{r} h_i) \cdot (\max_X \prod_{i=r+1}^{n} h_i)$$

$$h^X \leq g^X$$

Exponential complexity decrease : $O(e^n) \rightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Element

$max_B \Pi$ $max_B \Pi$

Bucket B P(E|B,C) P(B|A) P(D|A,B)

Bucket C P(C|A) h^B (C,E)

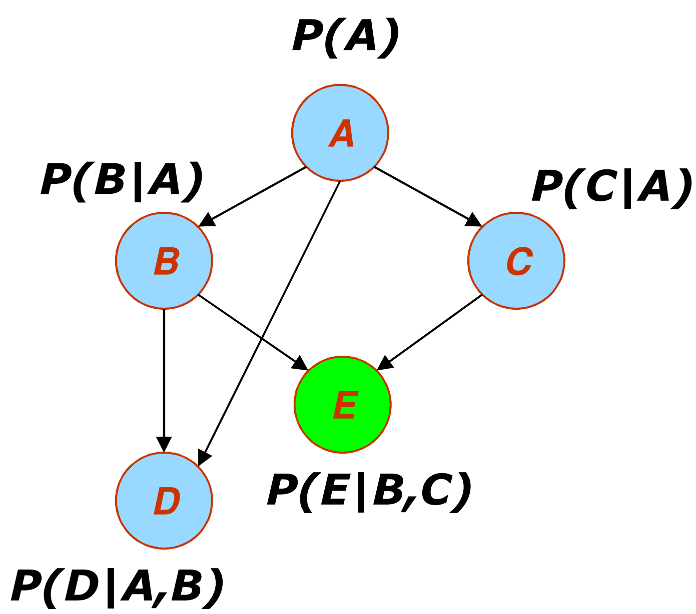
Bucket D h^B (A,D)

Bucket E E = 0 h^C (A,E)

Bucket A P(A) h^E (A) h^D (A)

W=2

Node duplication, renaming



MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

Mini-Clustering (for Sum-Product)

Split a cluster into mini-clusters => bound complexity

$$\{h_1, \dots, h_r, h_{r+1}, \dots, h_n\}$$

APPROXIMATE algorithm

$$\{h_1, \dots, h_r\} \qquad \{h_{r+1}, \dots, h_n\}$$

$$\sum_{elim} \prod_{i=1}^{n} h_i \leq \left(\sum_{elim} \prod_{i=1}^{r} h_i \right) \cdot \left(\sum_{elim} \prod_{i=r+1}^{n} h_i \right)$$

$$O(e^{n}) \rightarrow O(e^{\text{var}(r)}) + O(e^{\text{var}(n-r)})$$

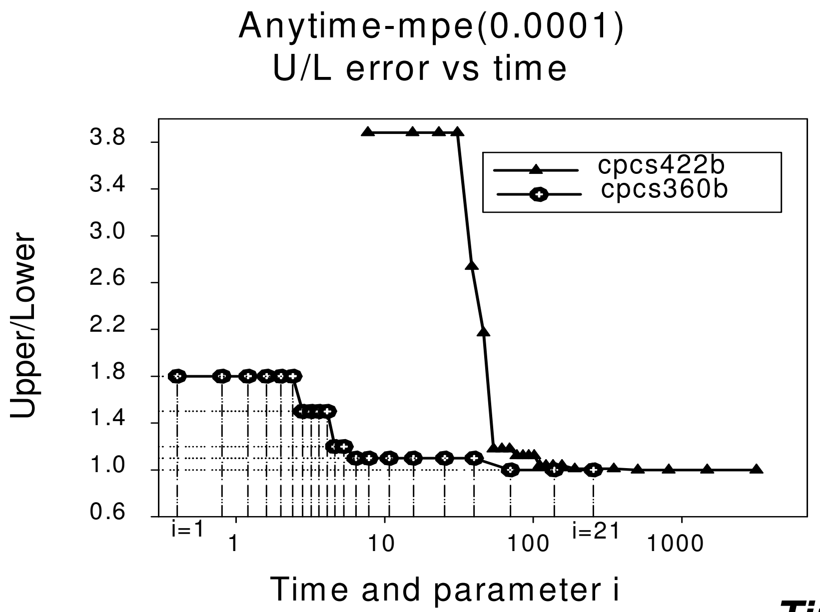
Exponential complexity decrease

Properties of MBE(i)

- **Complexity:** *O(r exp(i))* time and *O(exp(i)) space*.
- Yields an upper-bound and a lower-bound.
- **Accuracy:** determined by upper/lower (U/L) bound.
- As ***i*** increases, both accuracy and complexity increase.
- Possible use of mini-bucket approximations:
  - As anytime algorithms
  - As heuristics in search
- Other tasks: similar mini-bucket approximations for: belief updating, MAP and MEU (Dechter and Rish, 1997)

CPCS networks – medical diagnosis
(noisy-OR CPD’s)

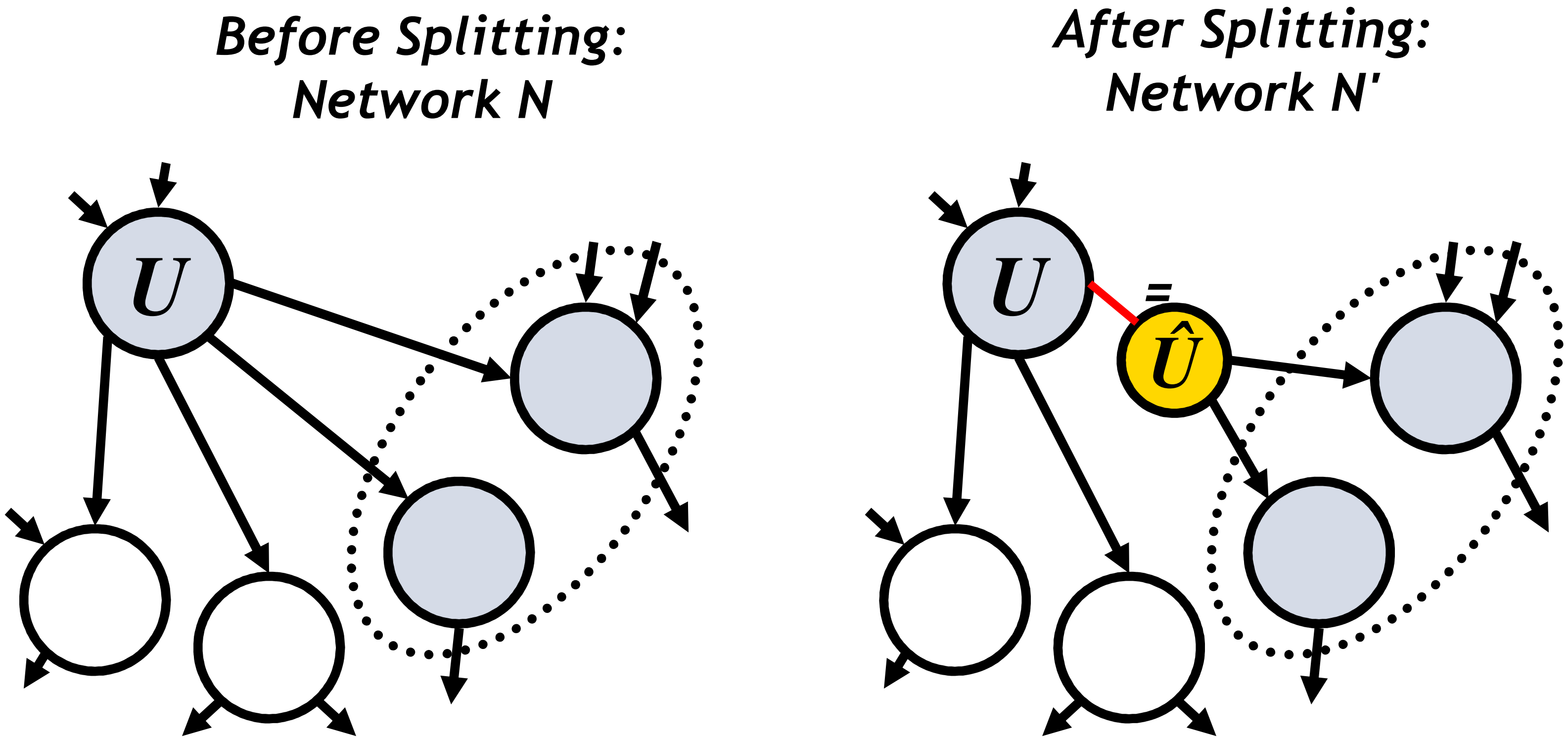
Test case: no evidence



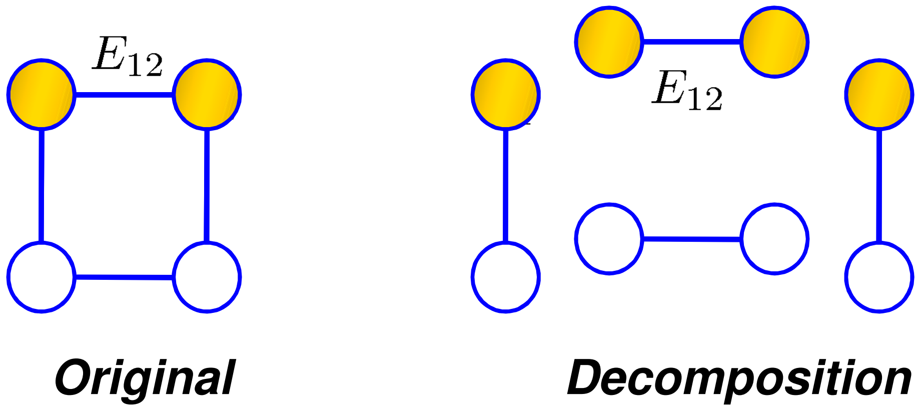
Algorithm	Time (sec) cpcs360	cpcs422
elim-mpe	115.8	1697.6
anytime-mpe(ε), $\varepsilon = 10^{-4}$	70.3	505.2
anytime-mpe(ε), $\varepsilon = 10^{-1}$	70.3	110.5

Mini-Bucket: duplicating/Splitting a Node

Variables in different buckets are renamed and duplicated
(Kask et. al., 2001), (Geffner et. al., 2007), (Choi, Chavira, Darwiche , 2007)



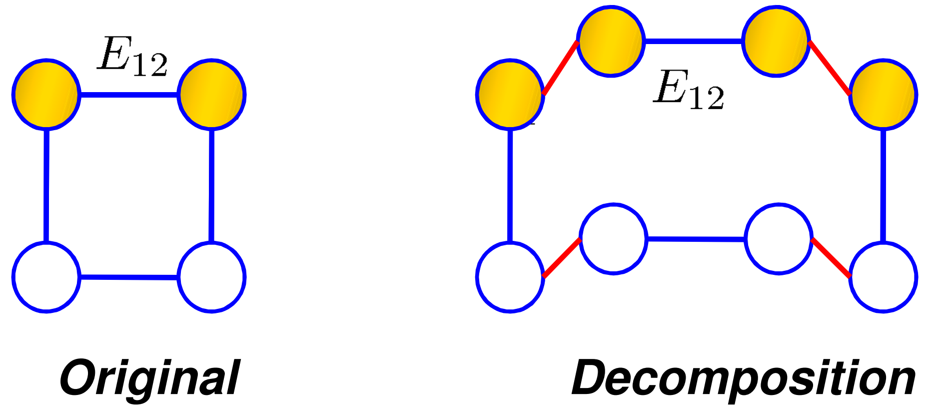
Tightening Bounds via cost-shifting



$$\max_{\underline{x}} \sum_{ij} E_{ij}(x_i, x_j) \leq \sum_{ij} \max_{\underline{x}} E_{ij}(x_i, x_j)$$

- Decompose graph into smaller subproblems
- Solve each independently; optimistic bound
- Exact if all copies agree

Decomposition view

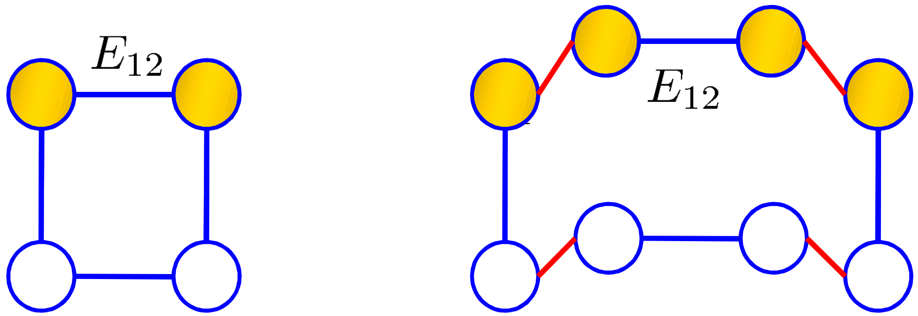


$$\forall i \sum_j \lambda_{ij}(x_i) = 0$$

$$\max_{\underline{x}} \sum_{ij} E_{ij}(x_i, x_j) \le \min_{\lambda} \sum_{ij} \max_{\underline{x}} E_{ij}(x_i, x_j) + \lambda_{ij}(x_i) + \lambda_{ji}(x_j)$$

- Decompose graph into smaller subproblems
- Solve each independently; optimistic bound
- Exact if all copies agree
- Enforce lost equality constraints via Lagrange multipliers

Decomposition view



Original ***Decomposition***

$$\forall i \sum_j \lambda_{ij}(x_i) = 0$$

$$\max_{\underline{x}} \sum_{ij} E_{ij}(x_i, x_j) \le \min_{\lambda} \sum_{ij} \max_{\underline{x}} E_{ij}(x_i, x_j) + \lambda_{ij}(x_i) + \lambda_{ji}(x_j)$$

Same bound by different names

- Dual decomposition (Komodakis et al. 2007)
- TRW, MPLP (Wainwright et al. 2005; Globerson & Jaakkola 2007)
- Soft arc consistency (Cooper & Schiex 2004)

Join-graph based cost-shifting

m_{11}, m_{12}- moment-matching messages

Bucket B: $\max_B \prod$ P(E|B,C) ⟷ (m_{11} →, ← m_{12}) $\max_B \prod$ P(B|A) P(D|A,B)

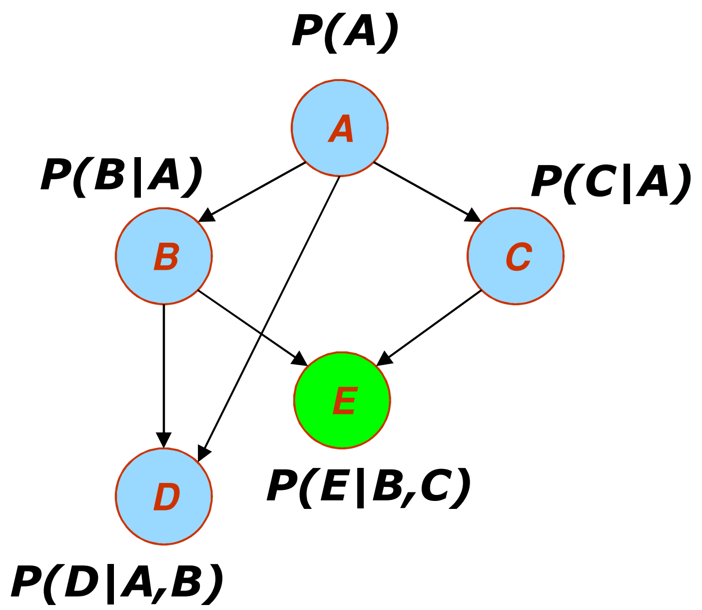
Bucket C: P(C|A) h^B (C,E)

Bucket D: h^B (A,D)

Bucket E: E = 0 h^C (A,E)

Bucket A: P(A) h^E (A) h^D (A)

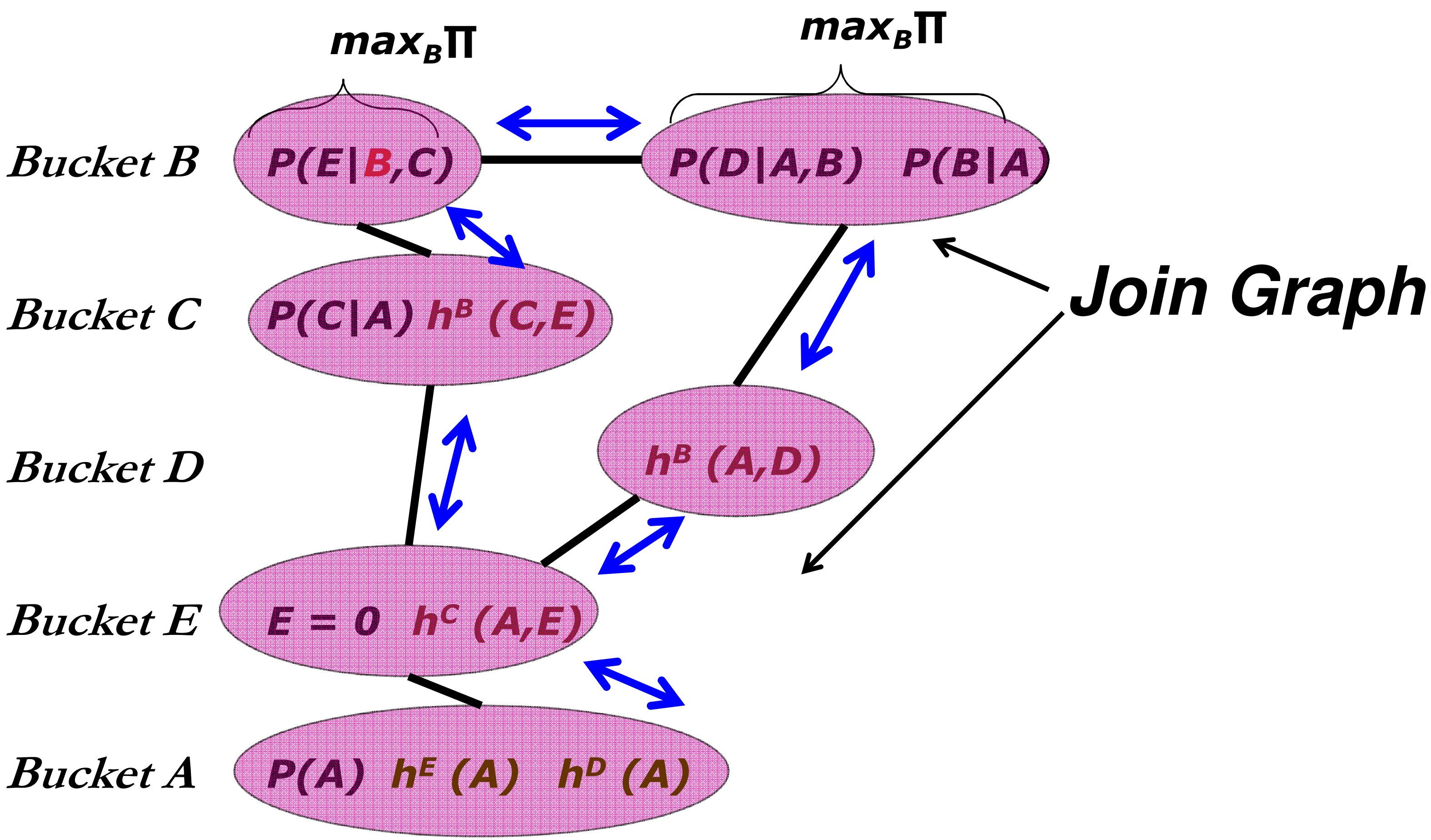
W=2



MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

Join-graph based cost-shifting

(Ihler, Flerova, Dechter, Otten, UAI 2012)



AOBB + MBE+ Join-graph based cost-shifting

Alexander Ihler, Natalia Flerova, Rina Dechter and Lars Otten (UAI '12)

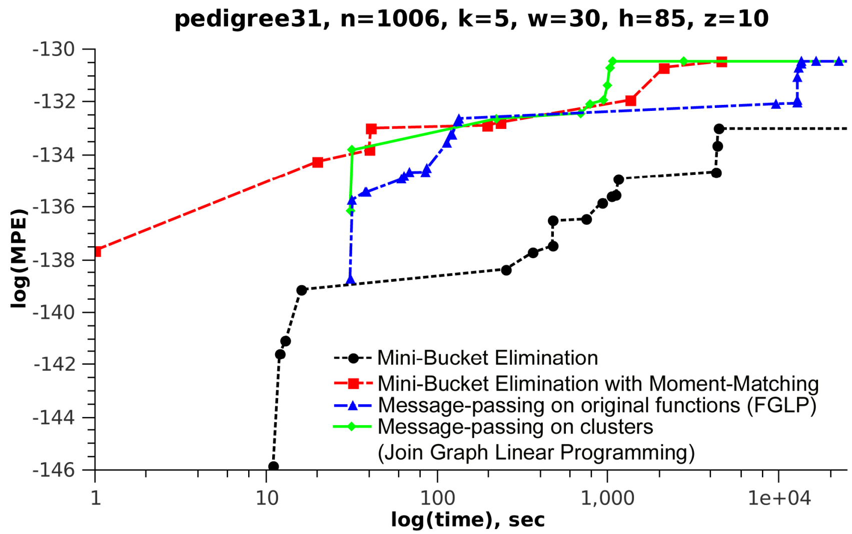
Mini-Bucket Elimination(i):
- *single-pass*
- *operates on clusters of size i*

\+

Linear Programming:
- *iterative*
- *operates on original factors*

Join Graph Linear Programming

MBE with Moment-Matching



Outline

- What are graphical models?
- Inference
- Search; via AND/OR search
- Time vs space, search vs inference
- Bounding inference
- Anytime AND/OR branch and bound
- Experiments, Competitions

and conclusions

Exact algorithms

Approximations

Back to exact: Anytime schemes

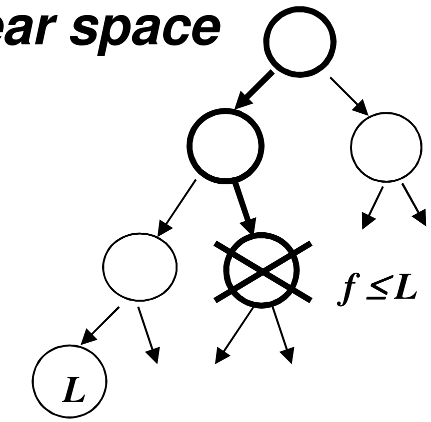
Basic Heuristic Search Schemes

Heuristic function $f(x^p)$ computes a lower bound on the best extension of x^p and can be used to guide a heuristic search algorithm. We focus on:

1. Branch-and-Bound

Use heuristic function $f(x^p)$ to prune the depth-first search tree

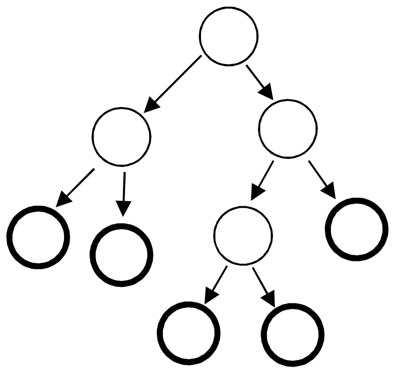
Linear space



2. Best-First Search

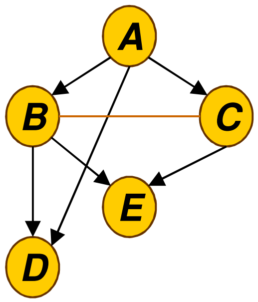
Always expand the node with the highest heuristic value

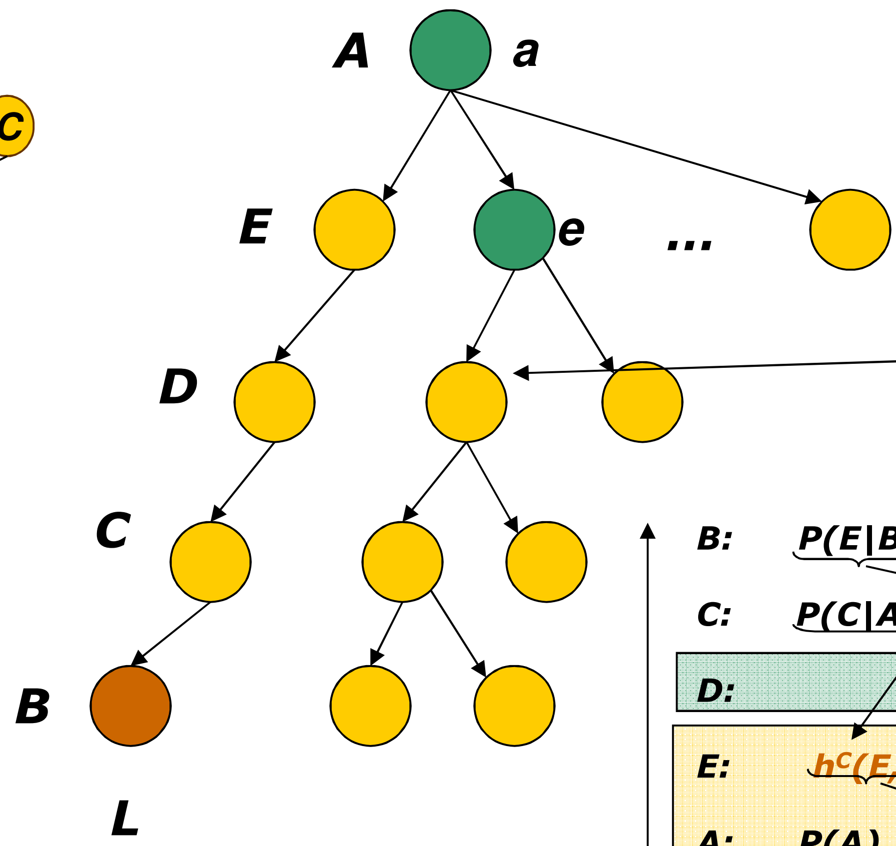
$f(x^p)$ needs lots of memory



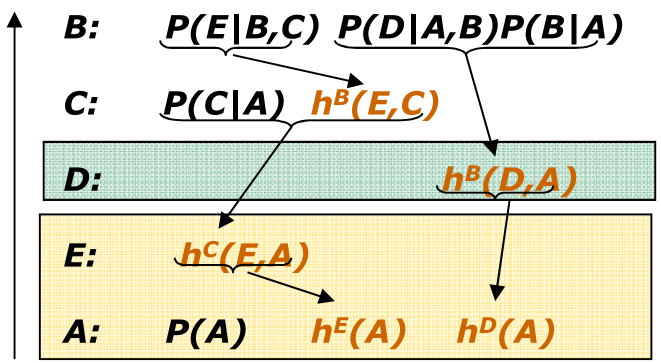
Mini-bucket Heuristics for BB search

(Kask and dechterAIJ, 2001, Kask, Dechter and Marinescu 2004, 2005, 2009, Otten 2012)



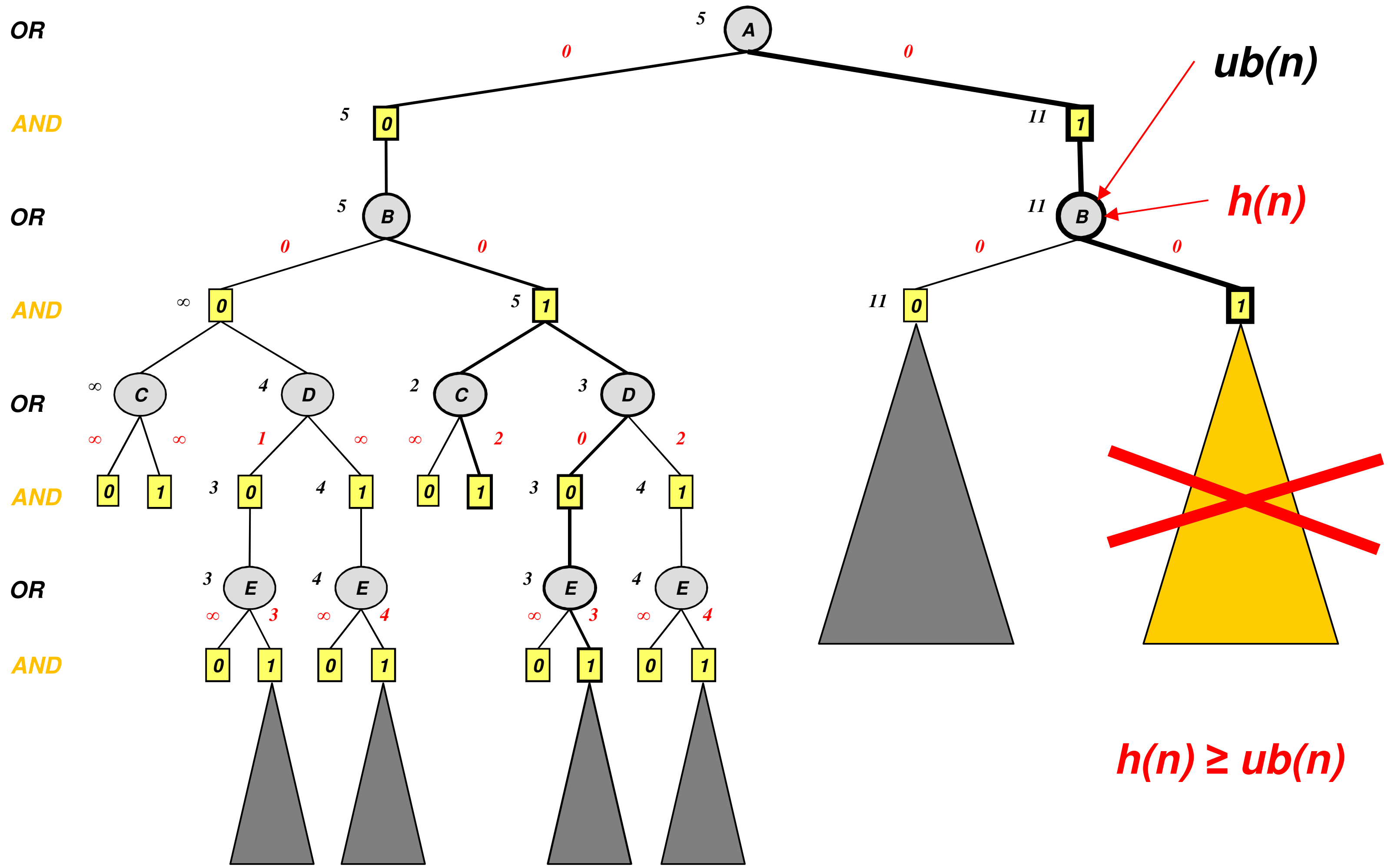


h(x) computed by MB(i) before or during search



$$f(a,e,D) = P(a) \cdot h^B(D,a) \cdot h^C(e,a)$$

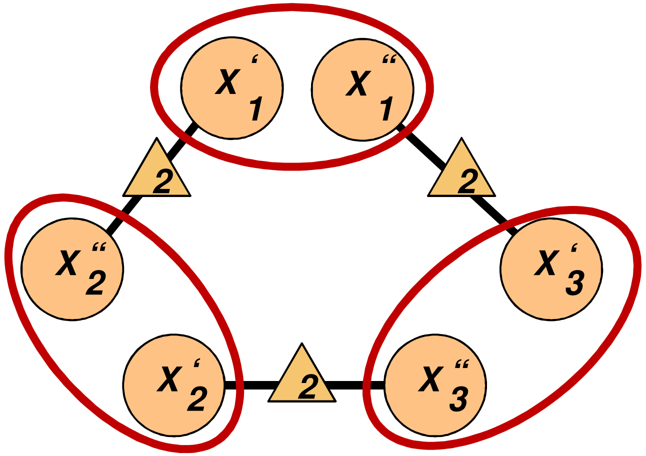
AND/OR Branch-and-Bound



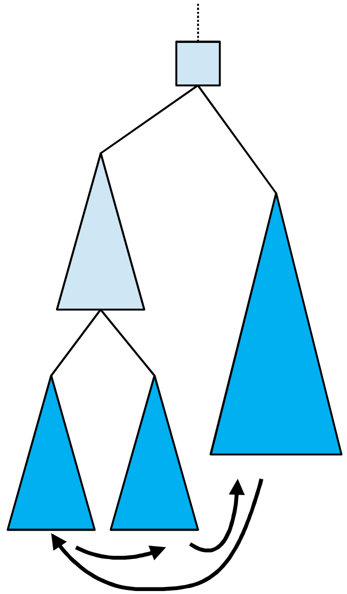
PASCAL 2011 Probabilistic Inference Challenge

- http://www.cs.huji.ac.il/project/PASCAL/
- Evaluates solvers in three categories:
  - PR: Probability of evidence / partition function
  - MAR: Posterior node marginals
  - MPE: Most probable explanation (our entry)
- Three tracks each: 20 sec, 20 min, 1 hour.
- Variety of benchmark domains:
  - CSPs, Deep Belief Nets, Image Alignment and Segmentation, Object Detection, Protein Folding, …

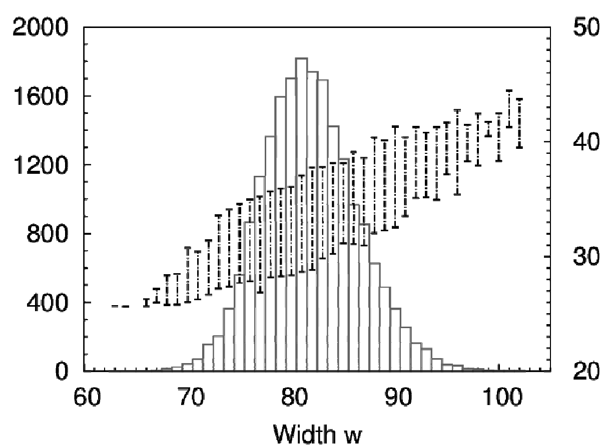
AOBB + Central Enhancements



$$\min_{\lambda} \sum_{(ij)} \max_{X}(f_{ij}(X_i, X_j) + \lambda_{ij}(X_i), \lambda_{ji}(X_j))$$



+ SLS *(Hutter et. Al 2005)*



Cost-shifting (MPLP) Re-parametrization

Tighter bounds by iteratively solving linear programming relaxations and message passing on join graph.

Breadth-First Subproblem Rotation

Improved anytime performance through interleaved processing of independent subproblems.

Enhanced Variable Ordering Schemes

Highly efficient, stochastic minfill / mindegree implementations for lower-width orderings.

(Ihler ,Flerova, Dechter, Otten, 2012, Otten and Dechter 2011, Kask, Gelfand, Otten, Dechter 2010)

Competition:
Gurobi vs AOBB on Pascal2 instances (Junkyu and Lam, ongoing)

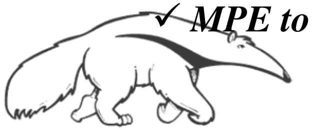
Benchmark

Problem	PEDIGREE	WCSP	Protein Folding
Total #. Instances	22 From UAI'08 Competition	61 From PASCAL2Competition	10 From PASCAL2Competition
Compared Instances	22	15 Exclude 46 Indeterminate Cases*	7 Exclude 3 Indeterminate Cases*
BRAOBB (G.M of Time)	51.07 sec G.M of 20 instances terminated for both	15.96 sec G. M of 13 instances terminated for both	1004.25 sec G. M of 7 instances, terminated for AOBB
GUROBI (G.M of Time)	7.15 sec G.M of 20 instances terminated for both	72.65 sec G. M of 13 instances terminated for both	NA 9/10 Memory Out
BRAOBB VS .GUROBI	0 VS. 22	12 VS. 3	6 Vs. 1 **AOBB won by memory out of Gurobi**

Indeterminate Cases :

- ✓ ***Both Time out, or BRAOBB Time out and Gurobi Memory out (4GB),***
- ✓ ***One of the solver time out earlier than the other's running time***
- ✓ ***MPE to 0/1 ILP Conversion was not available for experiments***

Link to more results



WCSP

	name (n,f,k,s,i,w,h)	Gurobi Time	Tg/Ta	AOBB Time
1	SPOT5 1502 209,412,4,3,6,6,15	1.72	24.57%	7
2	SPOT5 29 82,463,4,2,14,14,24	25.31	76.70%	33
3	SPOT5 404 100,711,4,3,19,19,45	310.58	2218.43%	14
4	SPOT5 503 143,636,4,3,9,9,44	622.49	8892.71%	7
5	SPOT5 54 67,272,4,3,11,11,19	28.15	402.14%	7
6	SPOT5 42 190,1395,4,3,13,26,87	3600 (TO)	NA	289
7	bwt3ac 45,686,11,2,12,16,27	164.4	342.50%	48
8	bwt4ac 179,7110,18,2,7,42,90	(MO)	NA	697

	name (n,f,k,s,i,w,h)	Gurobi Time	Tg/Ta	AOBB Time
9	driverlog01ac 71,619,4,2,9,9,30	7.74	96.75%	8
10	GEOM30a_3 30,82,3,2,6,6,15	42.42	707.00%	6
11	GEOM30a_4 30,82,4,2,6,6,15	6.76	112.67%	6
12	GEOM30a_5 30,82,5,2,6,6,15	6.69	111.50%	6
13	myciel5g_3 47,237,3,2,14,21,24	1012.8	1875.56%	54
14	queen5_5_3 25,161,3,2,15,18,20	2744.57	8072.26%	34
15	queen5_5_4 25,161,4,2,12,18,20	977.24	534.01%	183

BRAOBB : Tout 60 min (MBE-MM, MPLP 2ec, JGLP 2 sec, SLS 2x2 sec, Ordering from Kalev's Code 3sec)
GUROBI : Tout 60 min(Default, Single Processor, Dual Simplex at the Root)
Memory Limit : Both 4 GB

Indeterminate Cases Total 46 (Not presented Above)

- ✓***Both Time Out : 5/46***
- ✓***AOBB Time Out, Gurobi Memory out 15/46***
- ✓***ILP Conversion NA : 26/46***

PROTEIN FOLDING

	name (n,f,k,s,i,w,h)	Gurobi Time	Tg/Ta	AOBB Time
1	pdb1b25 1972,8817,81,2,3,51,178	(MO)	NA	21600 (TO)
2	pdb1d2e 1328,5220,81,2,3,22,136	(MO)	NA	557
3	pdb1fmj 614,2760,81,2,3,35,118	(MO)	NA	21600 (TO)
4	pdb1i24 337,1360,81,2,3,33,58	78.54	21.76%	361
5	pdb1iqc 1040,4042,81,2,3,26,107	(MO)	NA	1801

	name (n,f,k,s,i,w,h)	Gurobi Time	Tg/Ta	AOBB Time
6	pdb1jmx 739,2943,81,2,3,37,80	(MO)	NA	21600 (TO)
7	pdb1kgn 1060,4715,81,2,3,38,164	(MO)	NA	536
8	pdb1kwh 424,1881,81,2,3,27,93	(MO)	NA	16638
9	pdb1m3y 1364,5037,81,2,3,29,93	(MO)	NA	647
10	pdb1qks 926,3712,81,2,3,36,124	(MO)	NA	493

BRAOBB : Tout 6Hr (MBE-MM, MPLP 60 sec, JGLP 60 sec, SLS 20x10 sec, Ordering from Kalev's Code 3min)
GUROBI : Tout 6Hr (Default, Single Processor, Dual Simplex at the Root)
Memory Limit : Both 4 GB

Indeterminate Cases Total 3

- ✓***AOBB Time Out, Gurobi Memory Out 3***

Pedigree

name (n,f,k,s,i,w,h)		Gurobi Time	Tg/Ta	AOBB Time
1	Pedigree1 (298,335,4,5,15,15,60)	0.7	10.00%	7
2	Pedigree13 (888,1078,3,4,20,32,163)	15.8	2.33%	679
3	Pedigree18 (931,1185,5,5,19,19,98)	6.45	53.75%	12
4	Pedigree19 (693,794,5,5,14,24,99)	173.66	NA	1800 (TO)
5	Pedigree20 (387,438,5,4,19,22,73)	11.81	18.75%	63
6	Pedigree23 (309,403,5,4,18,25,60)	2.72	4.77%	57
7	Pedigree25 (993,1290,5,5,20,24,70)	3.1	16.32%	19
8	Pedigree30 (1015,1290,5,5,20,20,121)	7.25	55.77%	13
9	Pedigree31 (1006,1184,5,5,18,30,116)	20.32	18.81%	108
10	Pedigree33 (581,799,4,5,20,27,139)	4.12	9.36%	44
11	Pedigree34 (922,1161,5,4,17,30,130)	40.06	41.30%	97

name (n,f,k,s,i,w,h)		Gurobi Time	Tg/Ta	AOBB Time
12	Pedigree37 (726,1033,5,4,13,21,59)	3.09	11.44%	27
13	Pedigree38 (581,725,5,4,12,16,62)	5.99	13.31%	45
14	Pedigree39 (953,1273,5,4,20,20,78)	6.01	30.05%	20
15	Pedigree40 (842,1031,7,5,14,28,160)	221.15	NA	1800 (TO)
16	Pedigree41 (885,1063,5,5,18,32,113)	24.5	8.75%	280
17	Pedigree42 (390,449,5,4,15,23,60)	2.36	3.58%	66
18	Pedigree44 (644,812,4,5,20,26,72)	9.83	18.90%	52
19	Pedigree50 (478,515,6,4,10,17,53)	12.05	48.20%	25
20	Pedigree51 (871,1153,5,4,19,38,114)	14.64	3.30%	443
21	Pedigree7 (867,1069,4,4,19,31,119)	8.01	10.27%	78
22	Pedigree9 (935,1119,7,4,20,26,132)	7.85	25.32%	31

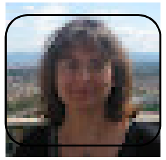
BRAOBB : Tout 30min (MBE-MM, MPLP 2ec, JGLP 2 sec, SLS 2x2 sec, Ordering from Kalev's Code 3sec)
GUROBI : Tout 30min (Default, Single Processor, Dual Simplex at the Root)
Memory Limit : Both 4 GB

Conclusions and Software

- **Conclusion:** Exact Search+Inference algorithms lead to effective anytime schemes and effective approximation and should be exploited.
- Search and inference are not so different; you may do effective inference by search
- Software
  - AND/OR search algorithms
  - Bucket-tree elimination
  - Generalized belief propagation
  - Samplesearch sampling
  - AOBB source code is freely available, under an open-source license.
  - http://graphmod.ics.uci.edu/group/Software

For publication see:

http://www.ics.uci.edu/~dechter/publications.html

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