Advances in Combinatorial Optimization Tasks over Graphical Models

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BGU 2011

Outline

- Graphical models: the primary reasoning principles
- OR Search Trees
- AND/OR Search Trees
- AND/OR Branch-and-Bound and Best-First Search
- Lower Bounding Heuristics
- Experiments
- More recent work

2004-2009

2010-now

Example Constraint Networks

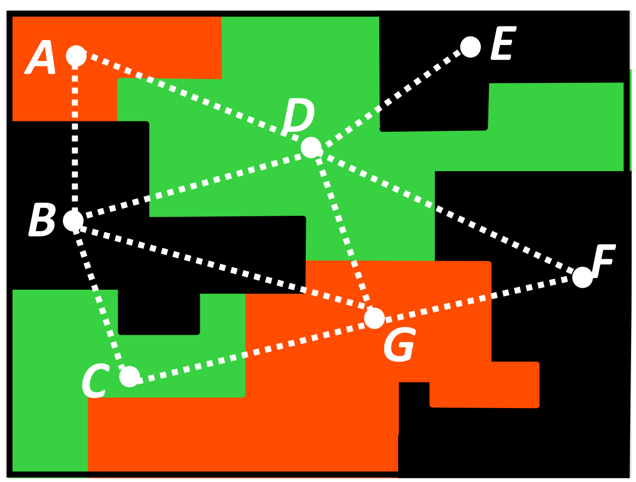
Map coloring

Variables: *countries (A B C etc.)*

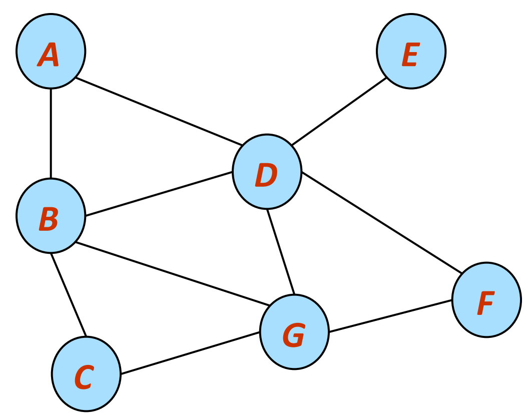
Values: *colors (red green blue)*

Constraints: **A ≠ B, A ≠ D, D ≠ E** , …

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



Constraint graph



Queries: Find one solution, all solutions, counting

Constraint Optimization Problems

for Graphical Models

A *finite COP* is a triple $R = \langle X, D, F \rangle$ where:

$X = \{X_1, ..., X_n\}$ - variables

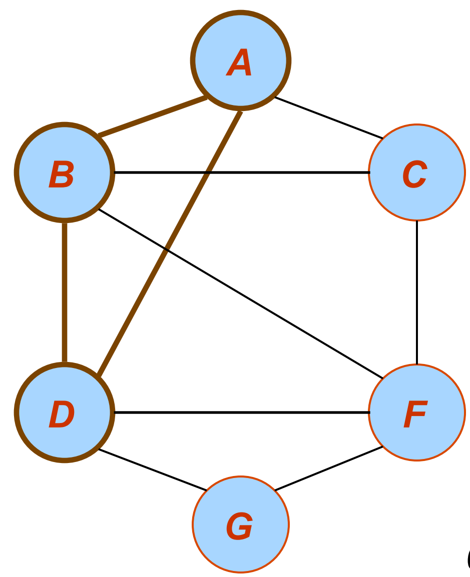
$D = \{D_1, ..., D_n\}$ - domains

$F = \{f_1, ..., f_m\}$ - cost functions

f(A,B,D) has scope {A,B,D}

A	B	D	Cost
1	2	3	3
1	3	2	2
2	1	3	∞
2	3	1	0
3	1	2	5
3	2	1	0

Primal graph =
Variables --> nodes
Functions, Constraints -→ arcs



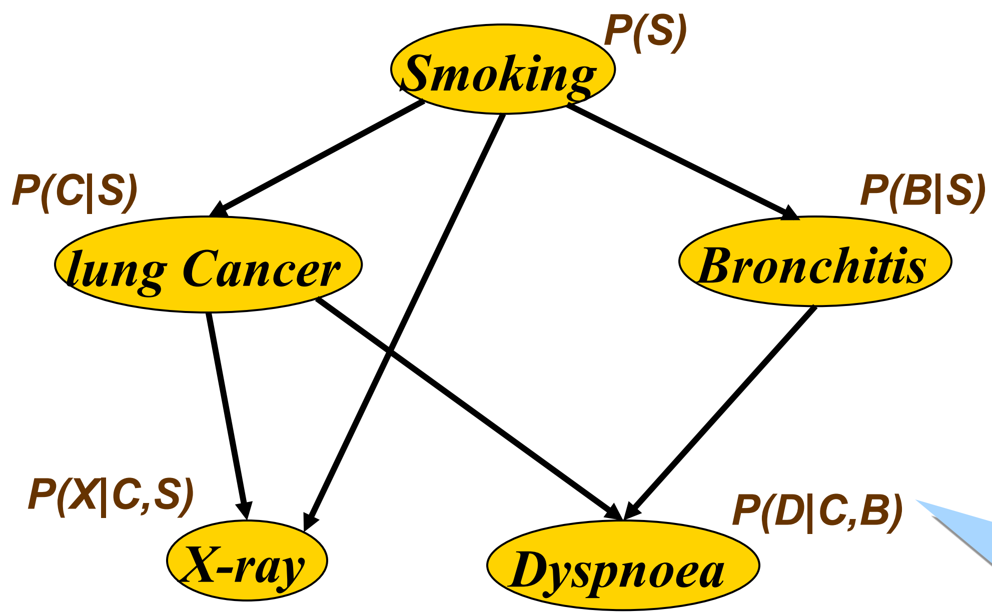
F(a,b,c,d,f,g)= f_1(a,b,d)+f_2(d,f,g)+f_3(b,c,f)

Global Cost Function

$$F(X) = \sum_{i=1}^{m} f_i(X)$$

Bayesian Networks

The MAP/MPE Problem over Bayesian Networks



BN = (G, Θ)

CPD:

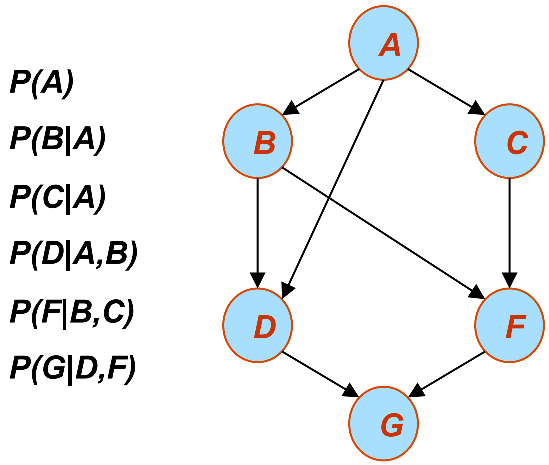
C	B	P(D|C,B)	
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

$P(S, C, B, X, D) = P(S)\, P(C|S)\, P(B|S)\, P(X|C,S)\, P(D|C,B)$

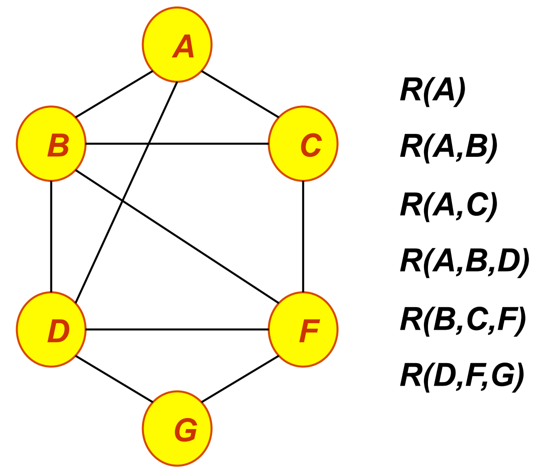
Posterior marginals, Most probable tuple (MPE)

- *MPE/MAP= find argmax $P(S)\cdot P(C|S)\cdot P(B|S)\cdot P(X|C,S)\cdot P(D|C,B)$ =?*

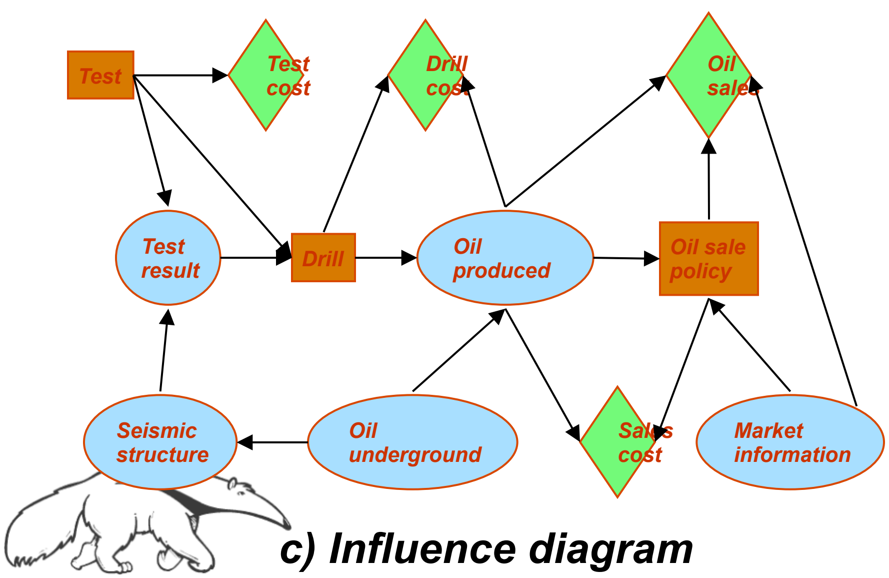
Graphical Models



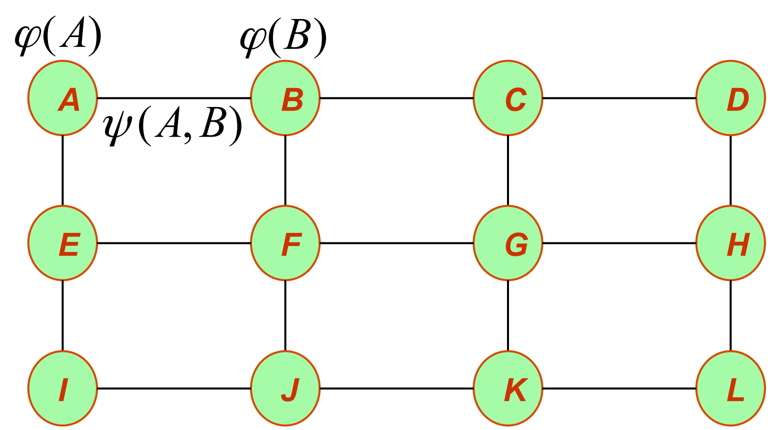
a) Belief network



b) Constraint network



c) Influence diagram



d) Markov network

Graphical Models

- A graphical model (**X**,**D**,**F**):
  - **X** = $\{X_1,\ldots X_n\}$ variables
  - **D** = $\{D_1, \ldots D_n\}$ domains
  - **F** = $\{f_1,\ldots,f_r\}$ functions
    (constraints, CPTS, CNFs …)
- Operators:
  - combination
  - elimination (projection)
- Tasks:
  - **Belief updating**: $\Sigma_{X\text{-}y} \prod_j P_i$
  - **MPE**: $\max_X \prod_j P_j$
  - **CSP**: $\prod_X \times_j C_j$
  - **Max-CSP**: $\min_X \Sigma_j F_j$

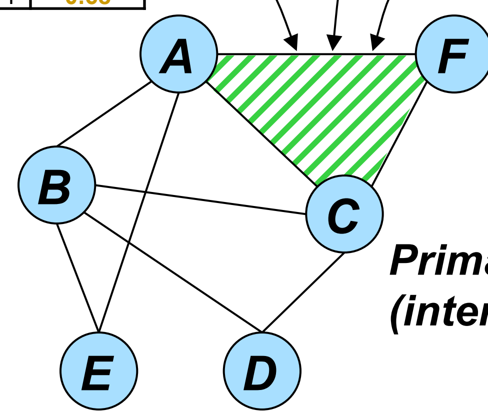
Conditional Probability Table (CPT)

A	C	F	P(F|A,C)
0	0	0	0.14
0	0	1	0.96
0	1	0	0.40
0	1	1	0.60
1	0	0	0.35
1	0	1	0.65
1	1	0	0.72
1	1	1	0.68

Relation

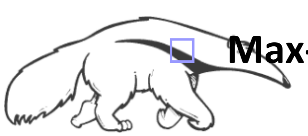
A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue

$f_i := (F = A + C)$



Primal graph (interaction graph)

- ***All these tasks are NP-hard***
  - ***exploit problem structure***
  - ***identify special cases***
  - ***approximate***

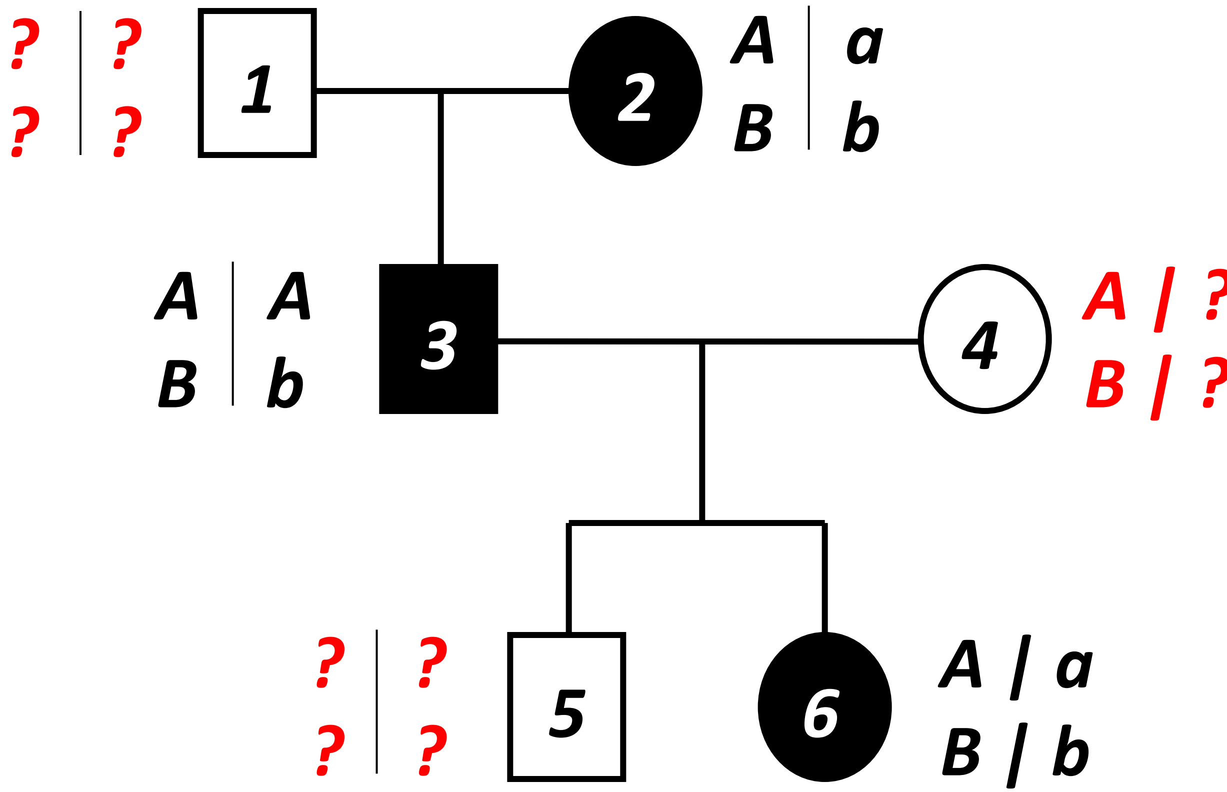


Types of Constraint Optimization

- Valued CSPs, Weighted CSPs, Max-CSPs, Max-SAT
- Most Probable Explanation (MPE/MAP)
- Linear Integer Programs

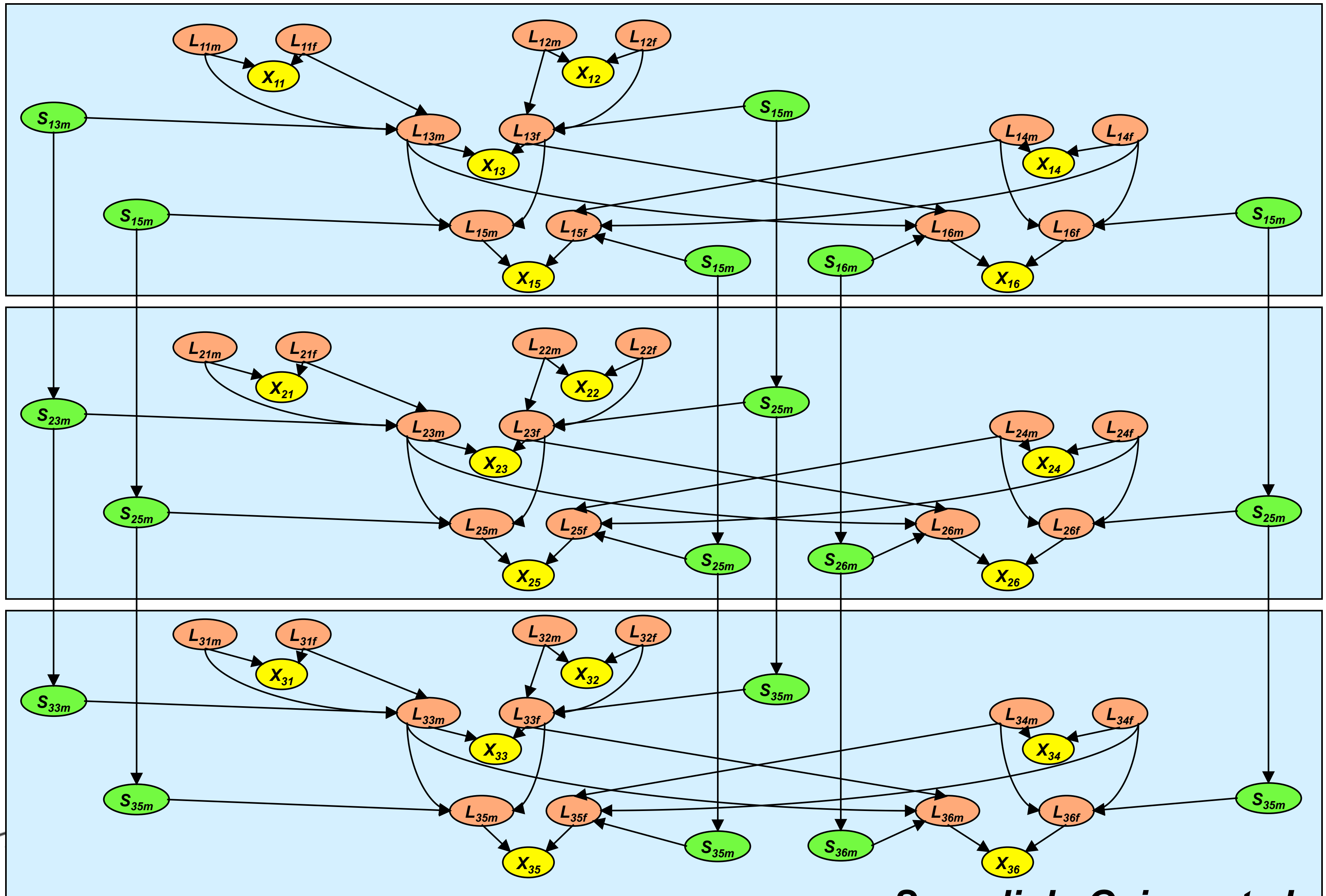
- **Examples**:
  - Problems translated from planning
  - Unit scheduling maintenance
  - Combinatorial auctions
  - Maximum-likelihood haplotypes in linkage

Linkage Analysis



- ***6 individuals***
- ***Haplotype: {2, 3}***
- ***Genotype: {6}***
- ***Unknown***

Pedigree: 6 people, 3 markers



Superlink, Geiger et.al

Solution Techniques, State of the art

AND/OR search

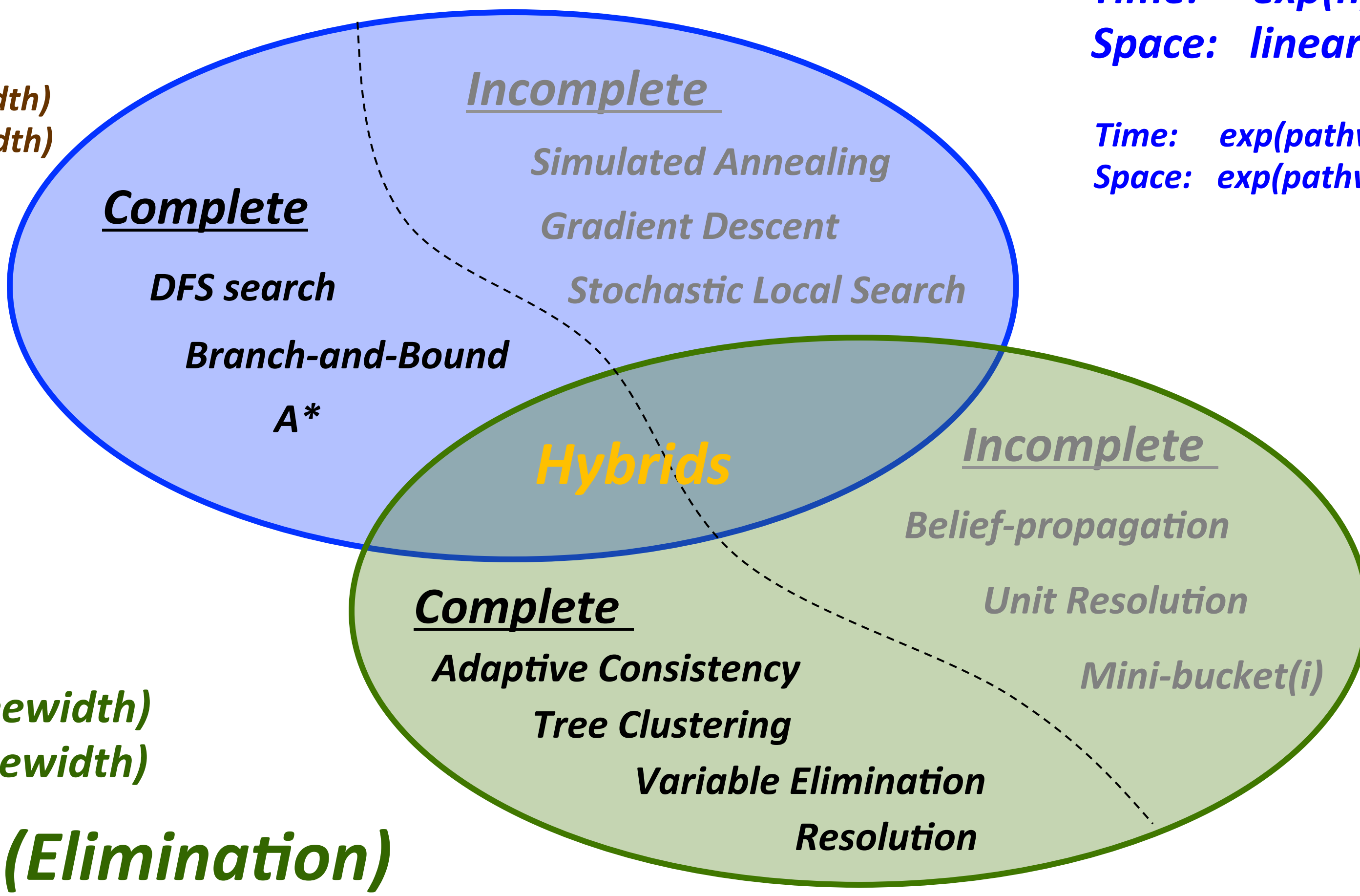
*Time: exp(treewidth * log n)*
Space: linear

Space: exp(treewidth)
Time: exp(treewidth)

Search (Conditioning)

Time: exp(n)
Space: linear

Time: exp(pathwidth)
Space: exp(pathwidth)



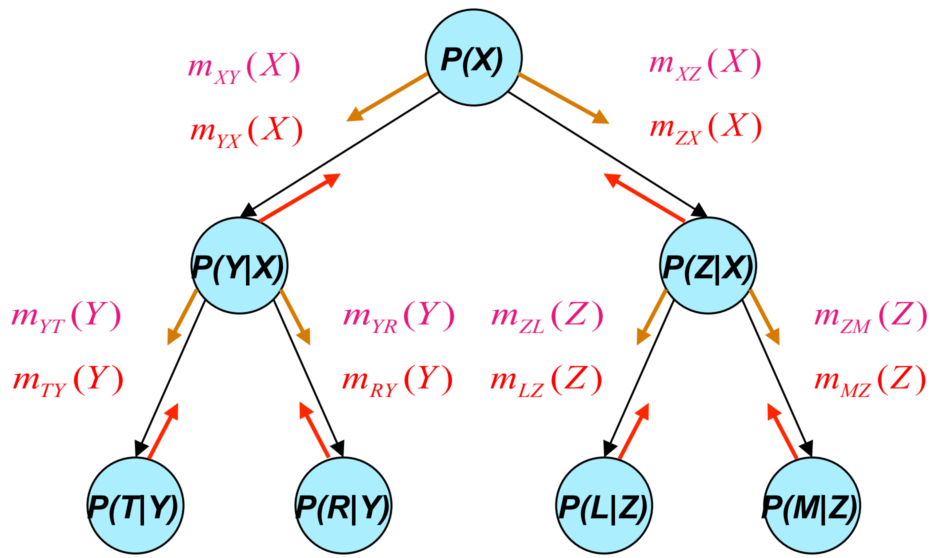
Time: exp(treewidth)
Space: exp(treewidth)

Inference (Elimination)

Tree-solving is Easy

Belief updating (sum-prod)

CSP – consistency (projection-join)



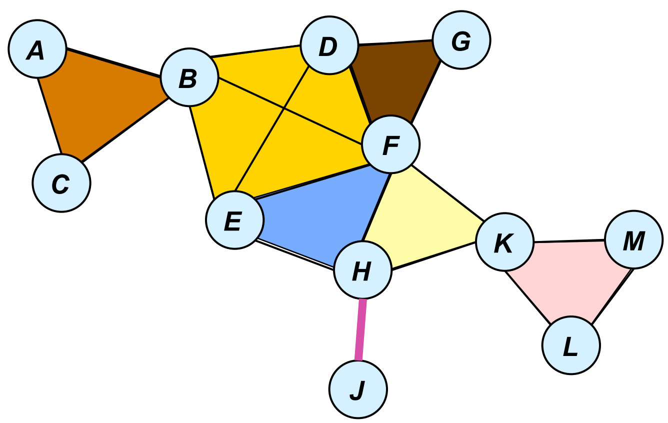
Dynamic Programming

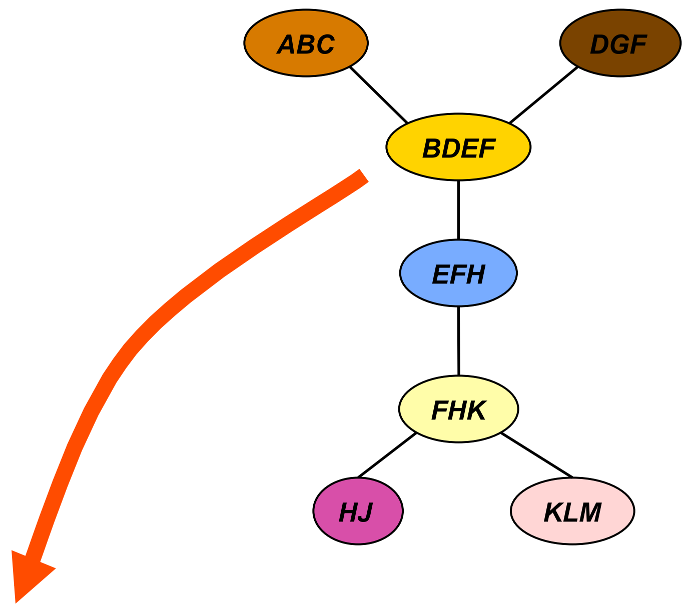
MPE (max-prod)

#CSP (sum-prod)

Trees are processed in linear time and memory
Also Acyclic graphical models

Inference and Treewidth



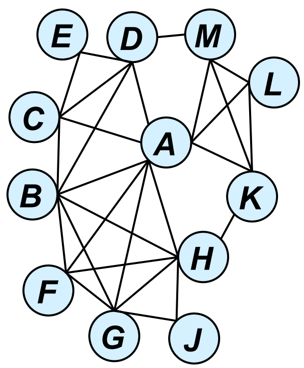


Inference algorithm:
Time: *exp(tree-width+1)*
Space: *exp(separator-width)*

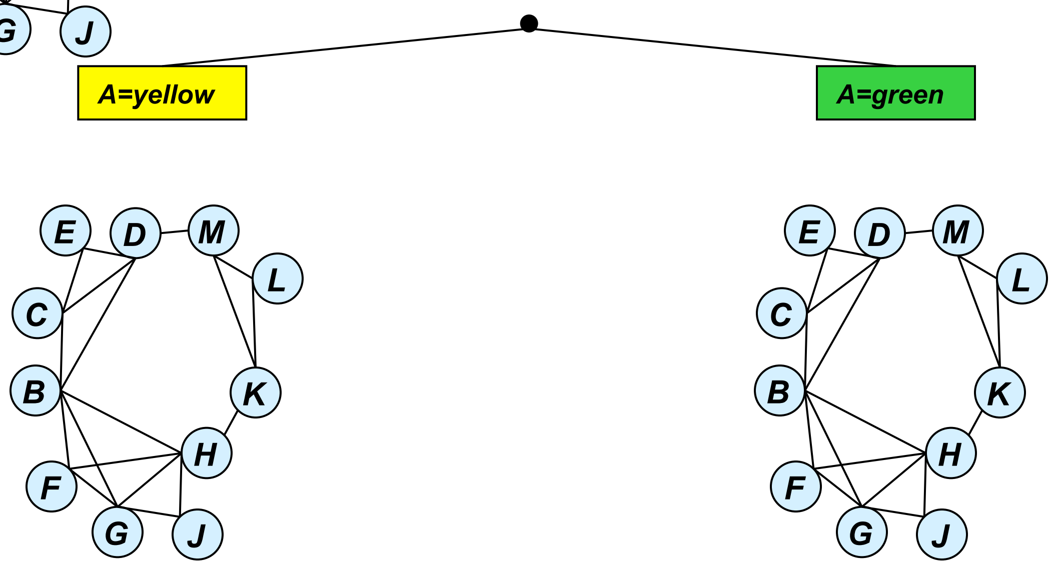
treewidth = 4 - 1 = 3
treewidth = (maximum cluster size) – 1
Separator-width=2

Search over the Cutset

Graph Coloring problem

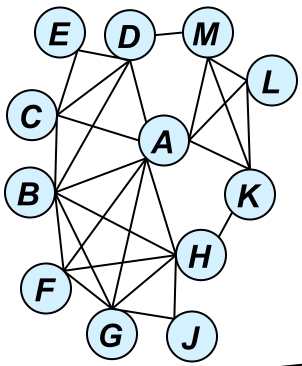


- ***Inference may require too much memory***
- ***Condition (guessing) on some of the variables***

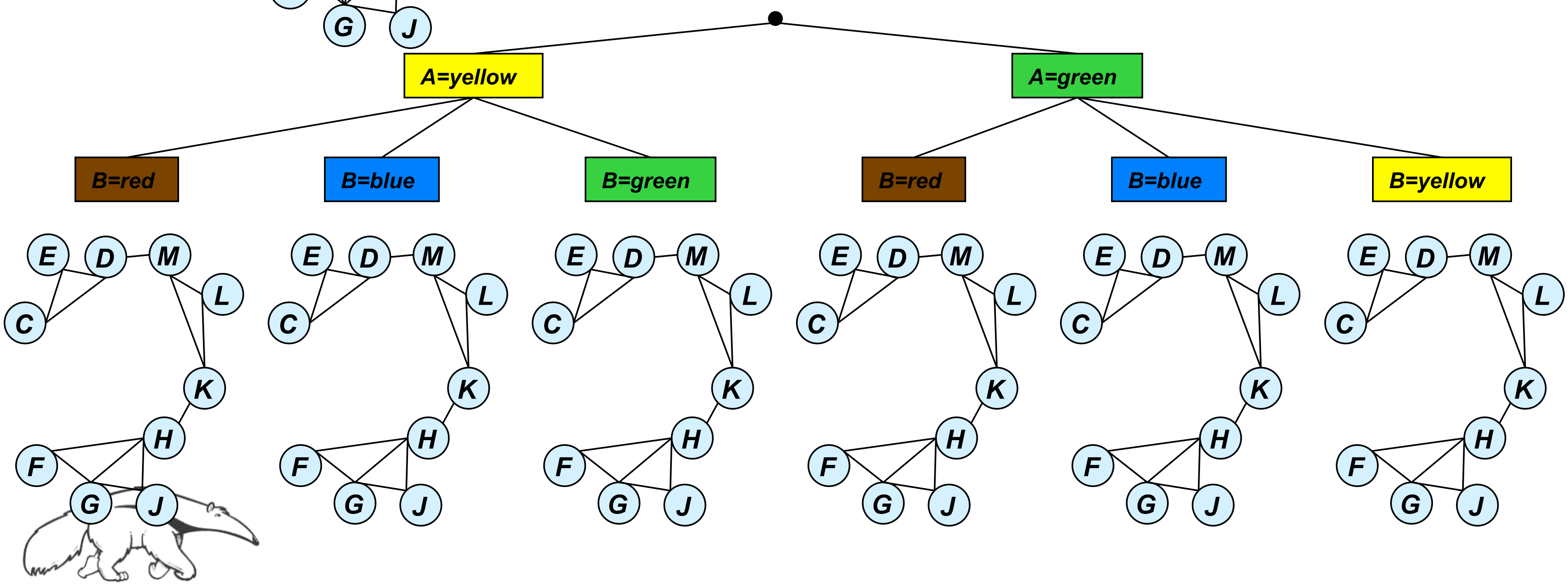


Search over the Cutset (cont)

Graph Coloring problem

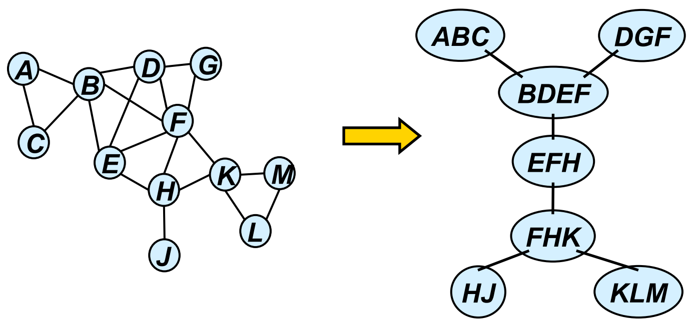


- *Inference may require too much memory*
- *Condition on some of the variables*



Inference vs. Conditioning

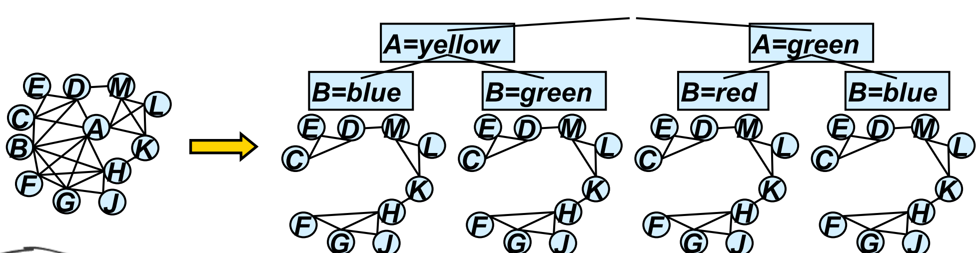
- **By Inference (thinking)**



Exponential in treewidth
Time and space

- ***By Conditioning (guessing)***

•



Exponential in cycle-cutset
Time. linear space

Inference for Optimization: Bucket Elimination

Algorithm BE-mpe (Dechter 1996, Bertele and Briochi, 1977)

$$MPE = \max_{a,e,d,c,b} P(a)P(c \mid a)P(b \mid a)P(d \mid a,b)P(e \mid b,c)$$

$$\max_X \prod$$

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $h^B(a,d,c,e)$

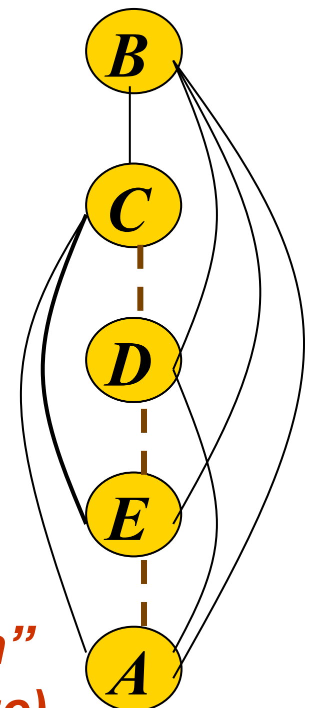
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $h^D(a,e)$

bucket A: $P(a)$ $h^E(a)$

OPT

$W^*=4$
"induced width"
(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_c P(c \mid a') \times h^B(a', d', c, e')$

3. $d' = \arg\max_d h^C(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_a P(a) \cdot h^E(a)$

B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

C: $P(c|a)$ $h^B(a,d,c,e)$

D: $h^C(a,d,e)$

E: $e=0$ $h^D(a,e)$

A: $P(a)$ $h^E(a)$

Return (a', b', c', d', e')

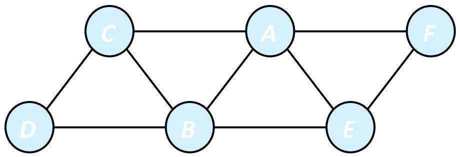
Solution methods

- Solving tree is easy
- Inference: move to trees by clustering
  - (dynamic programming, variable elimination, junction trees)
  - Exploit structure well.
- Search: move to trees by conditioning.
  - Can also exploit structure well.

Outline

- Graphical Models: reasoning principles
- OR Search Trees
- AND/OR Search Spaces for GM
- AND/OR Branch-and-Bound and Best-First Search
- Lower Bounding Heuristics
- Experimental evaluation
- Current work

The Search Space



A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

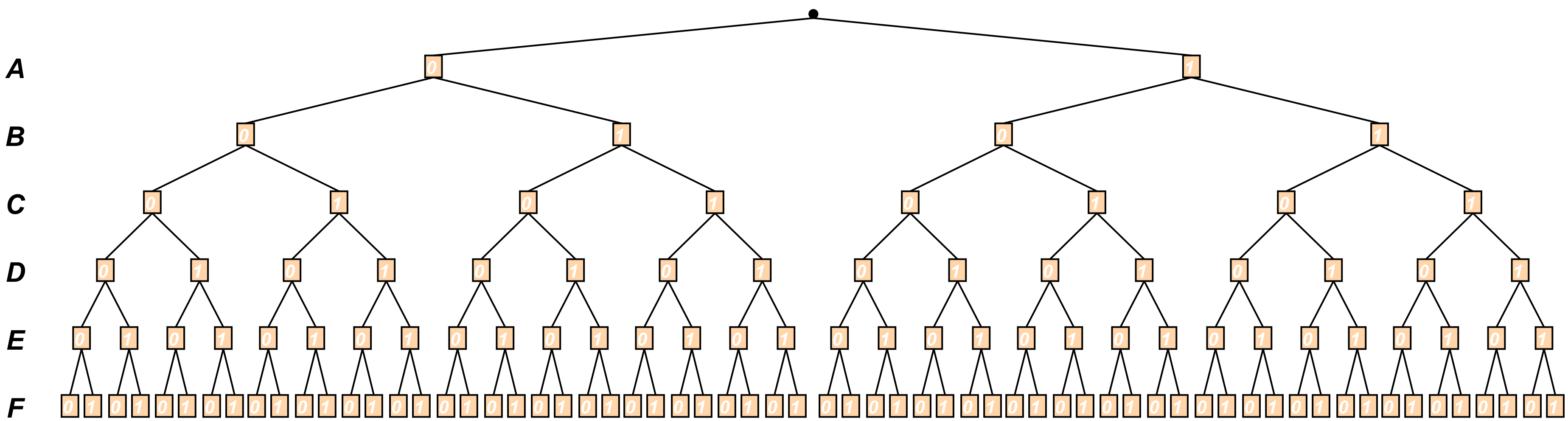
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

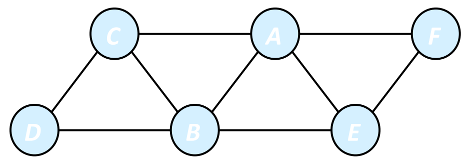
C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$



The Search Space



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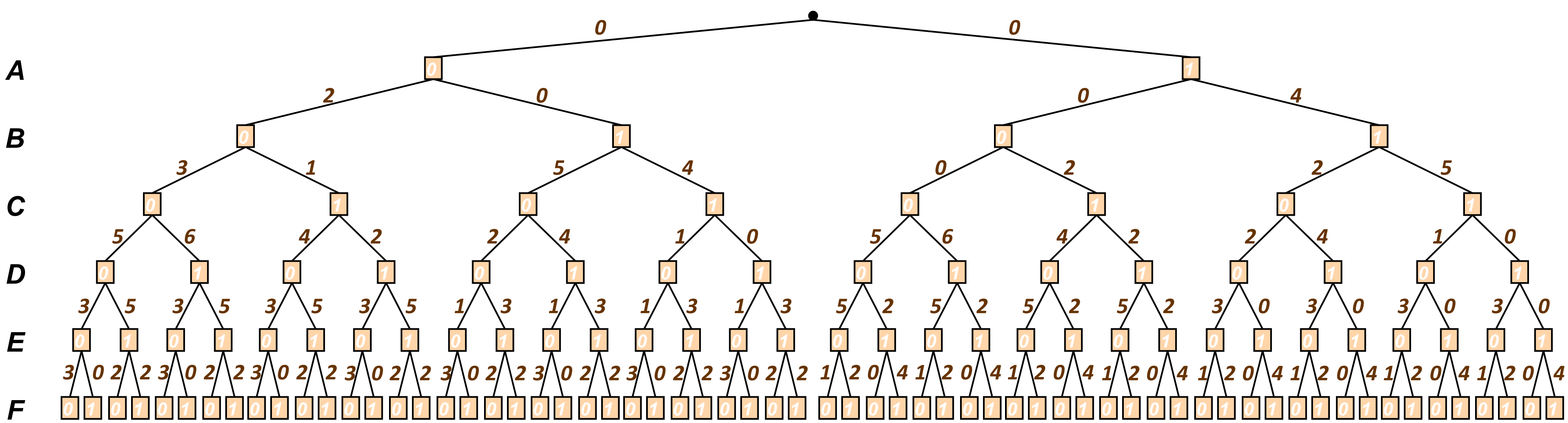
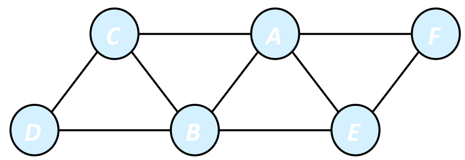


are calculated based on cost components

The Search Space



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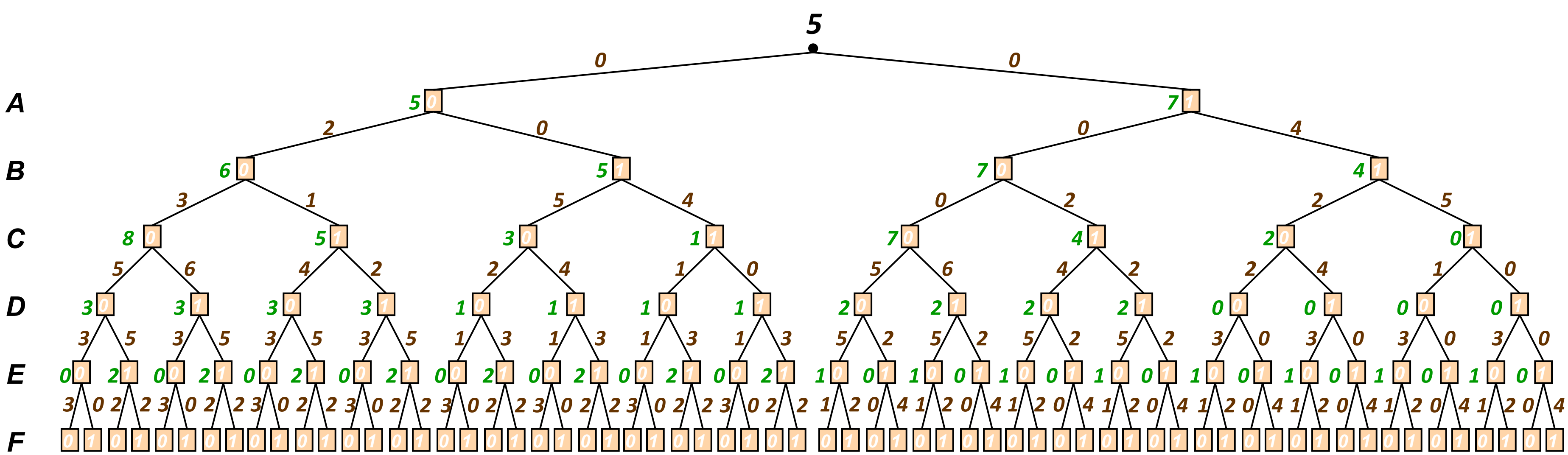
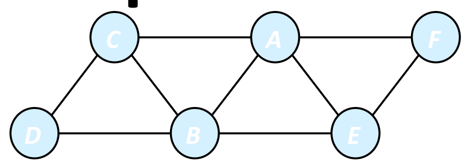


Node value = minimal cost solution below it

An Optimal Solution



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1	1	4

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1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

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0	1	0
1	0	0
1	1	2

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0	0	0
0	1	1
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1	1	4

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B	E	f_7
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C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
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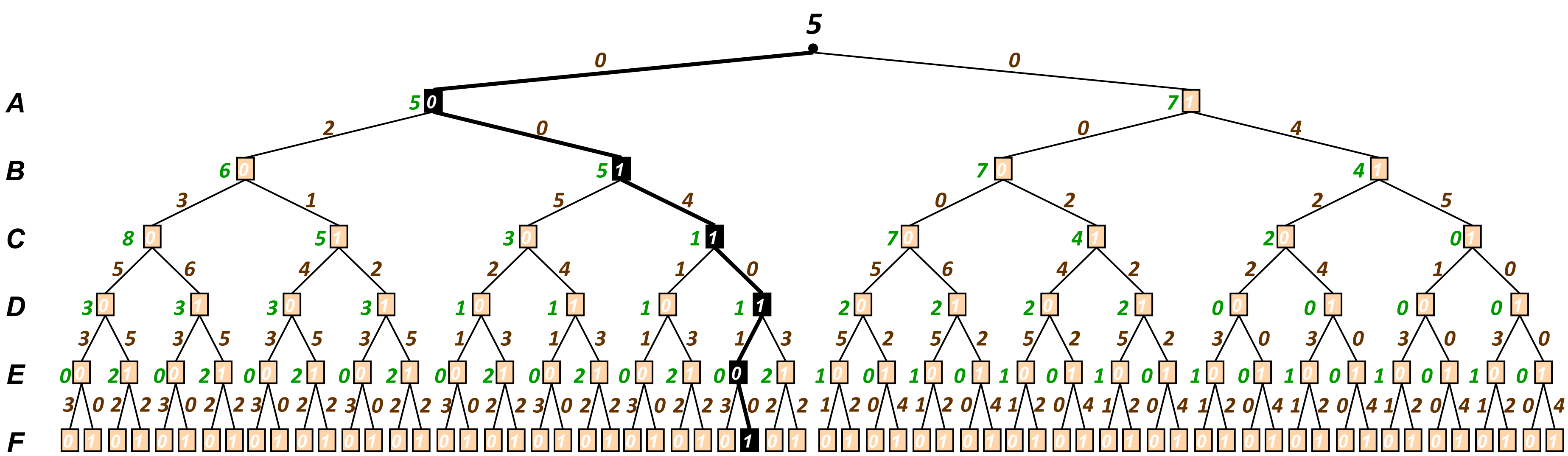
$$f(\mathrm{X}) = \min_X \sum_{i=1}^{9} f_i(\mathrm{X})$$



An optimal assignment is A=0, B=1, C=1, D=1, E=0, F=1 with cost 5

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- Graphical Models: reasoning principles
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- AND/OR Search Spaces for GM
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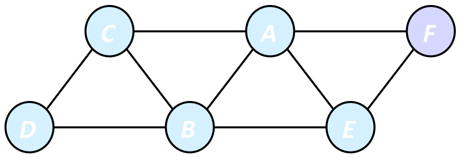
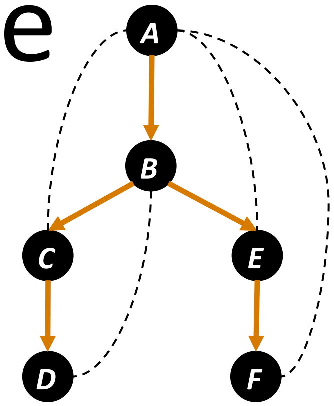
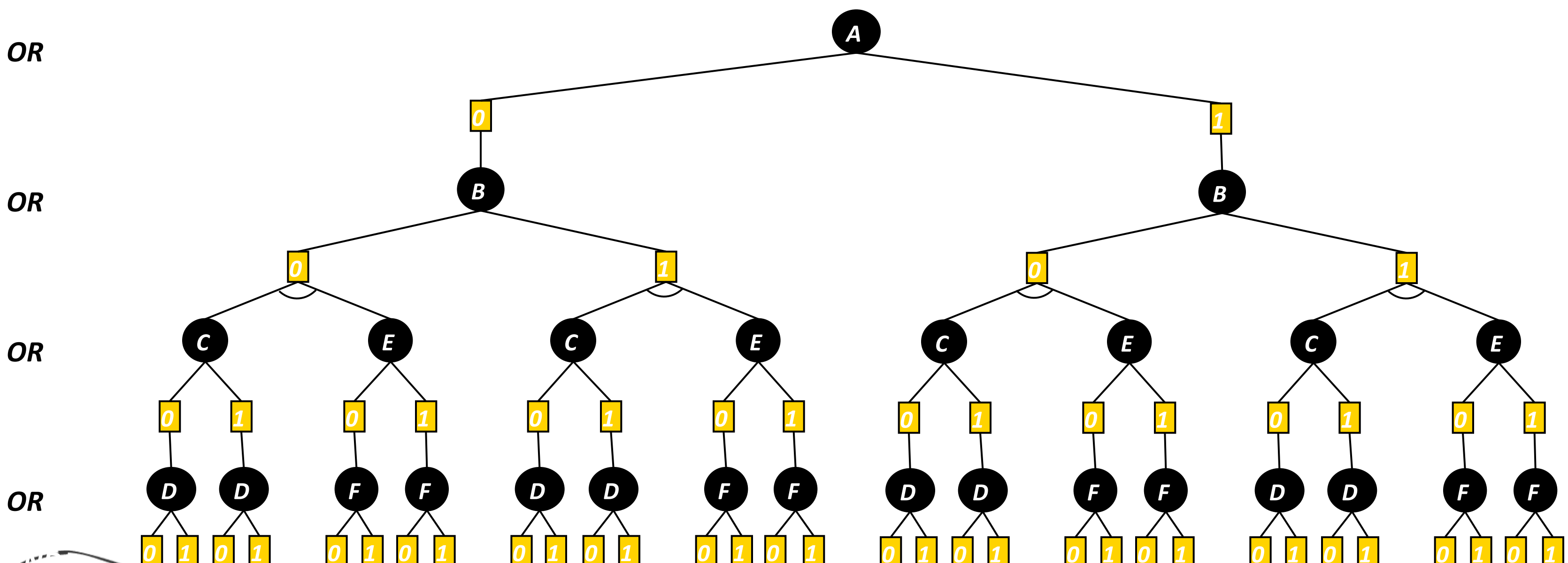
- Current work

The AND/OR Search Tree

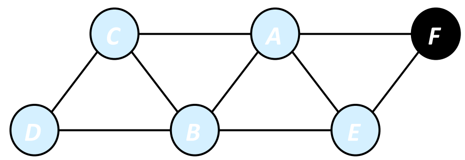


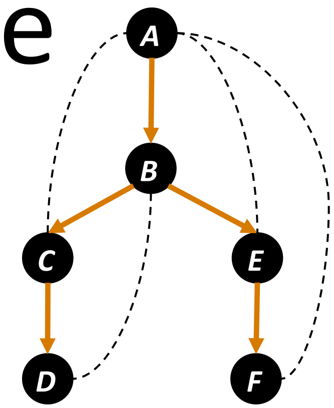


***Pseudo tree** (Freuder & Quinn85)*



The AND/OR Search Tree





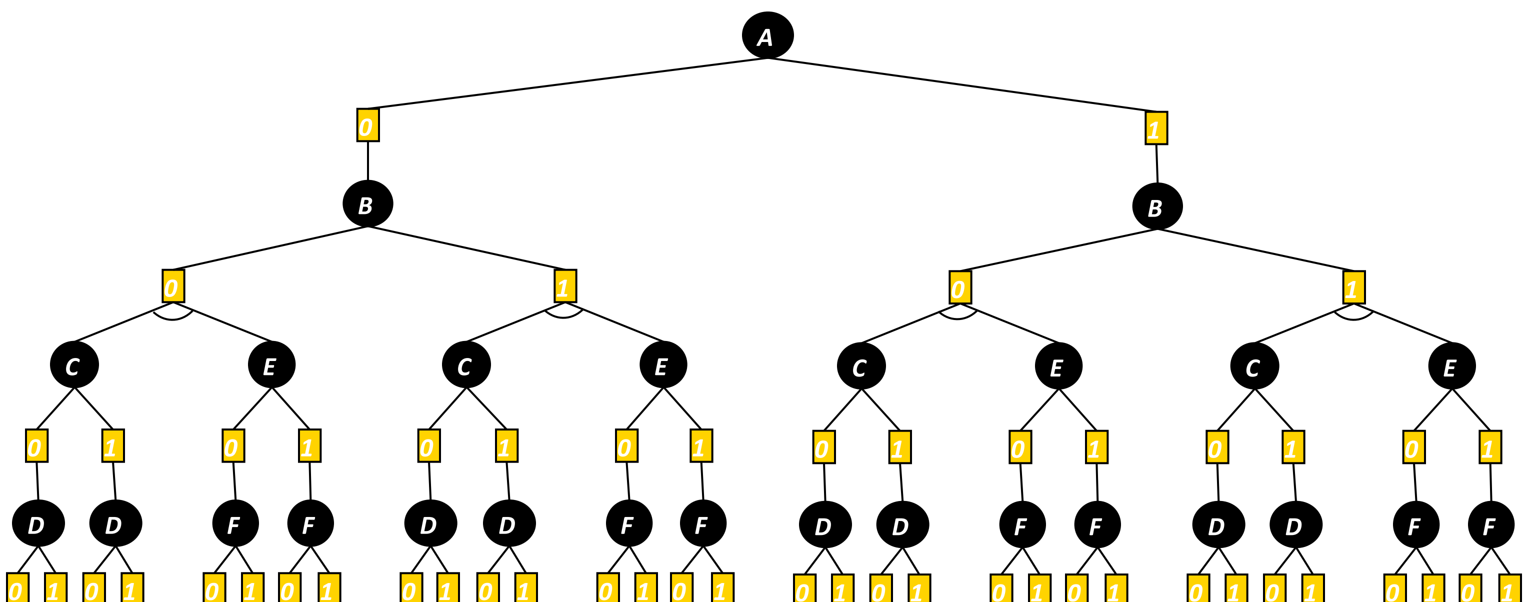
Pseudo tree

OR

OR

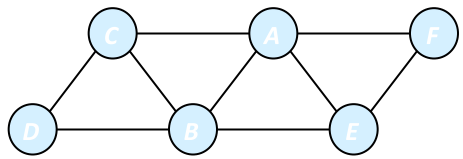
OR

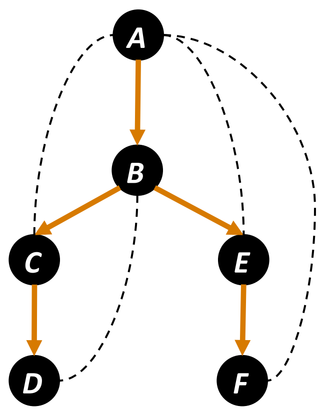
OR



A solution subtree is

Weighted AND/OR Search Tree





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1	1	4

A	C	f_2
0	0	3
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1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

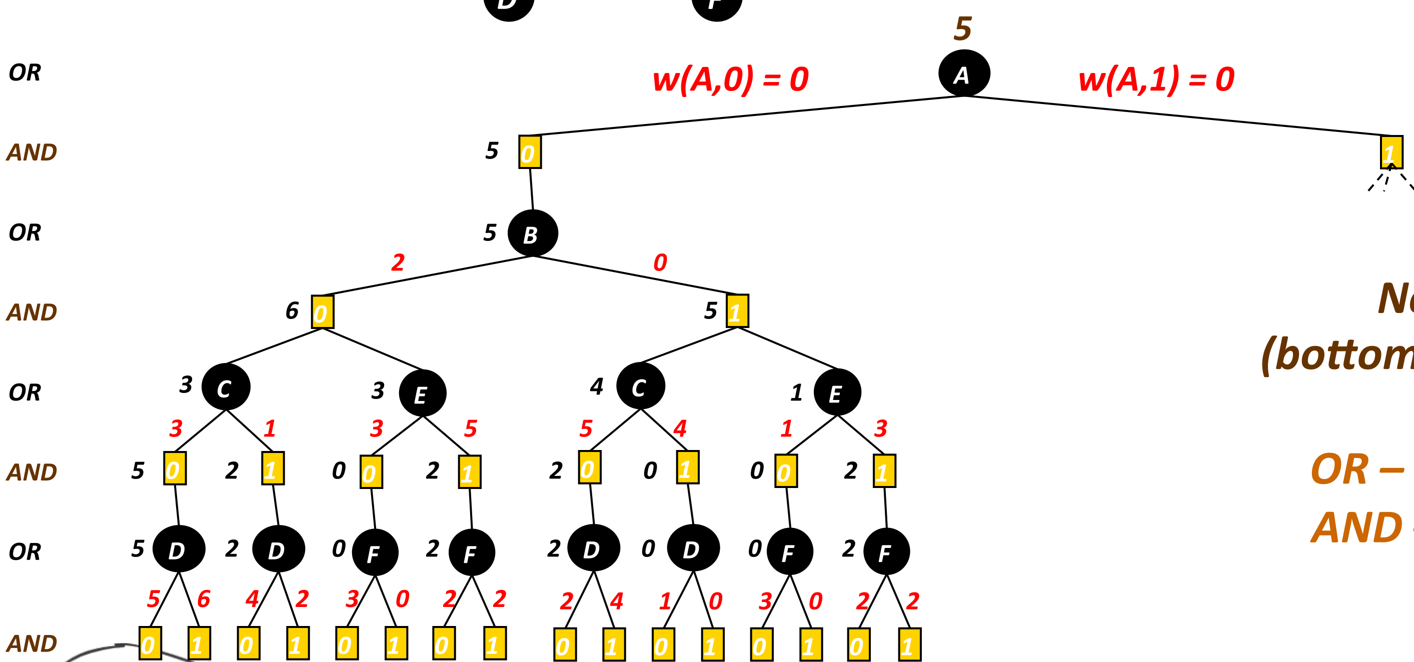
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

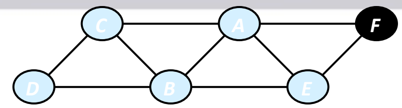
$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$

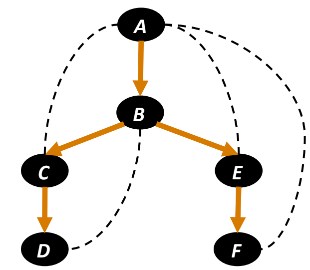


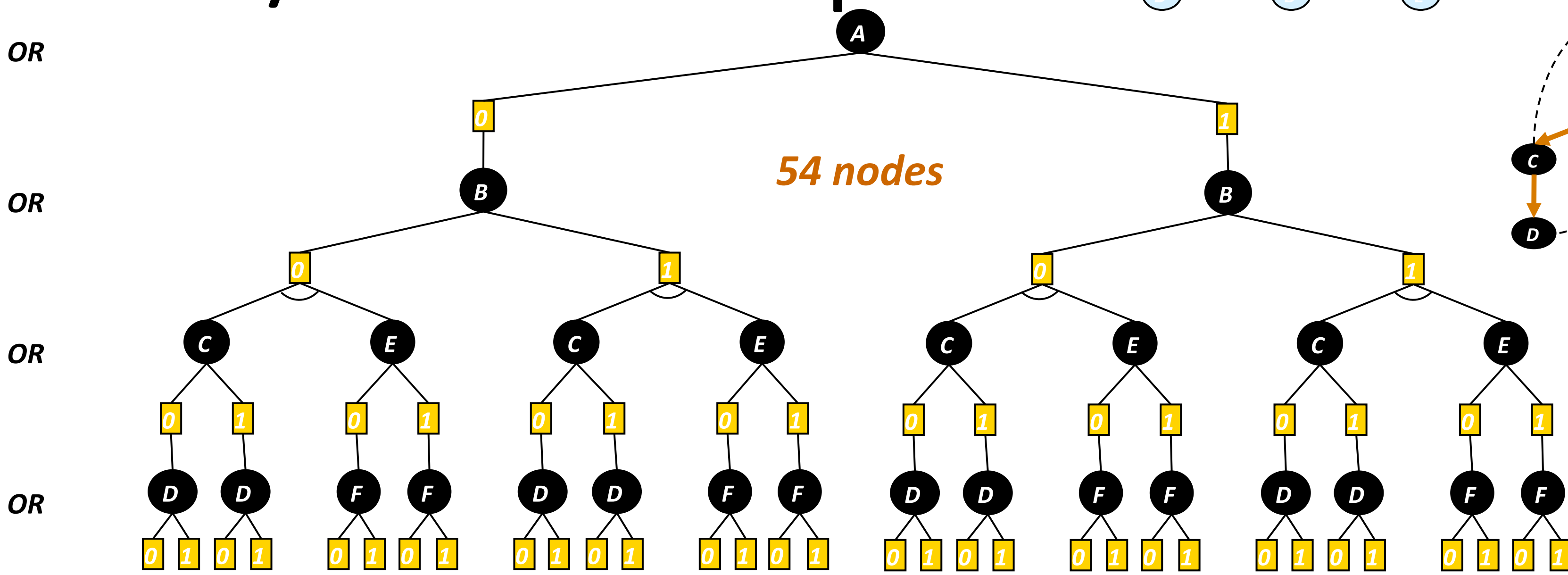
Node Value
(bottom-up evaluation)

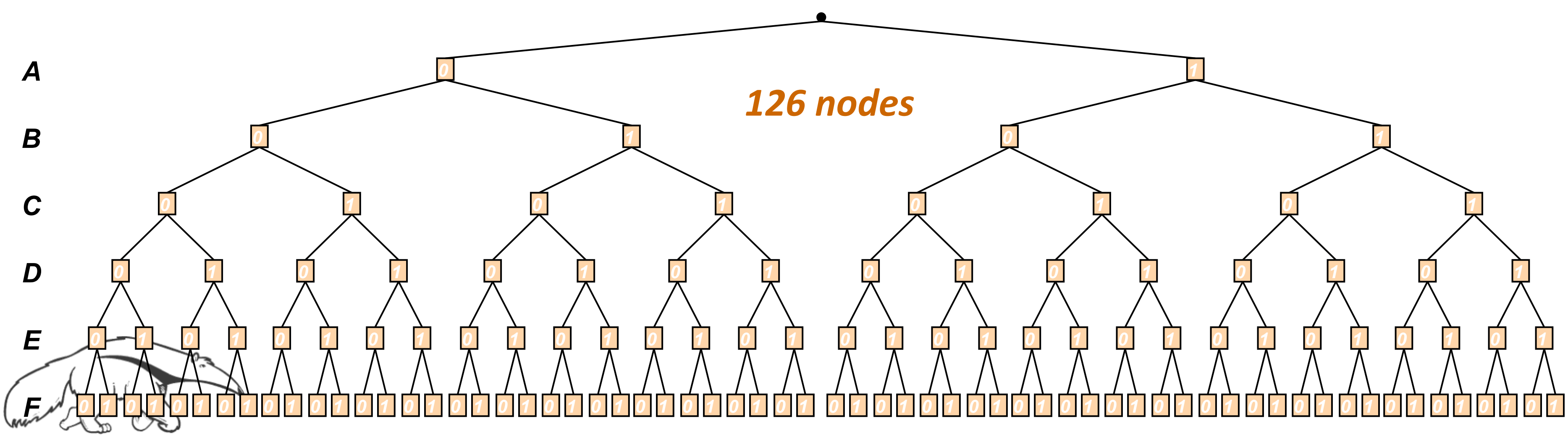
OR – minimization
AND – summation

AND/OR vs. OR Spaces









AND/OR vs. OR Spaces

width	depth	OR space		AND/OR space		
		Time (sec.)	Nodes	Time (sec.)	AND nodes	OR nodes
5	10	3.15	2,097,150	0.03	**10,494**	5,247
4	9	3.13	2,097,150	0.01	**5,102**	2,551
5	10	3.12	2,097,150	0.03	**8,926**	4,463
4	10	3.12	2,097,150	0.02	**7,806**	3,903
5	13	3.11	2,097,150	0.10	**36,510**	18,255

Random graphs with 20 nodes, 20 edges and 2 values per node

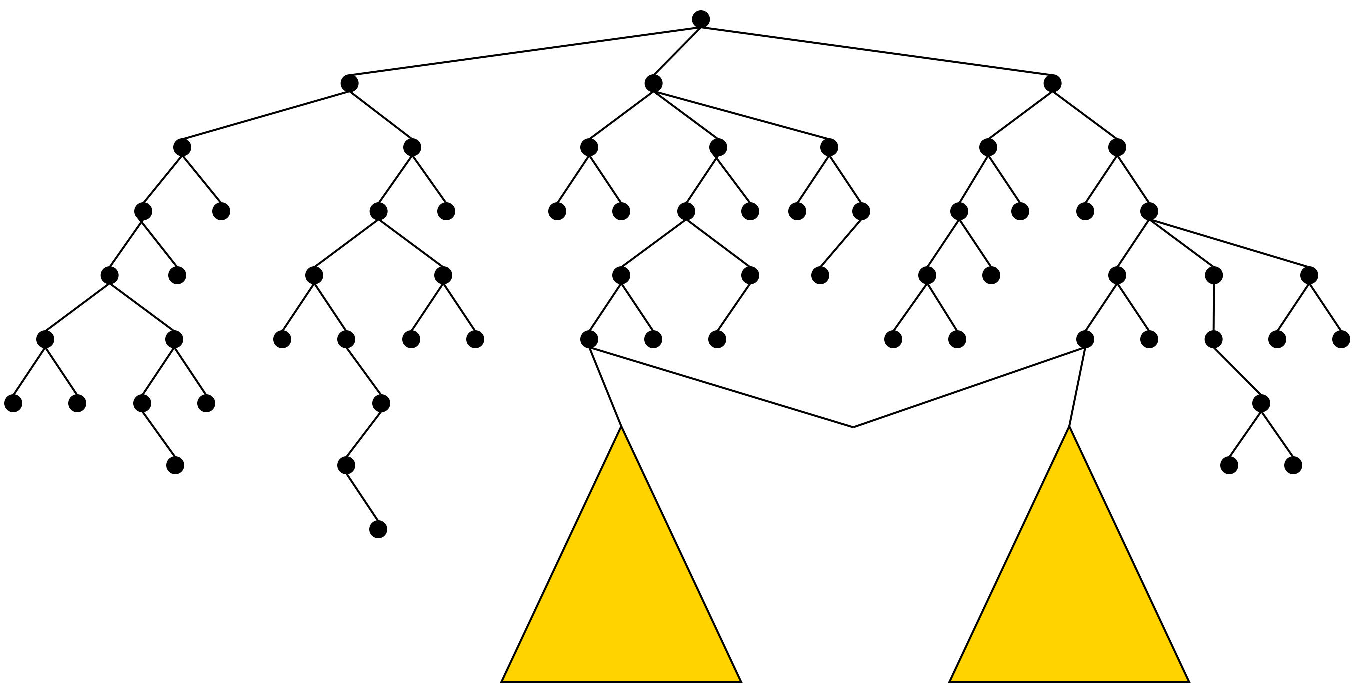
Complexity of AND/OR Tree Search

	AND/OR tree	OR tree
Space	$O(n)$	$O(n)$
Time	$O(n\ k^m)$ $O(n\ k^{w^* \log n})$ (Freuder & Quinn85), (Collin, Dechter & Katz91), (Bayardo & Miranker95), (Darwiche01)	$O(k^n)$

k = *domain size*
m = *depth of pseudo-tree*
n = *number of variables*
*w** = *treewidth*

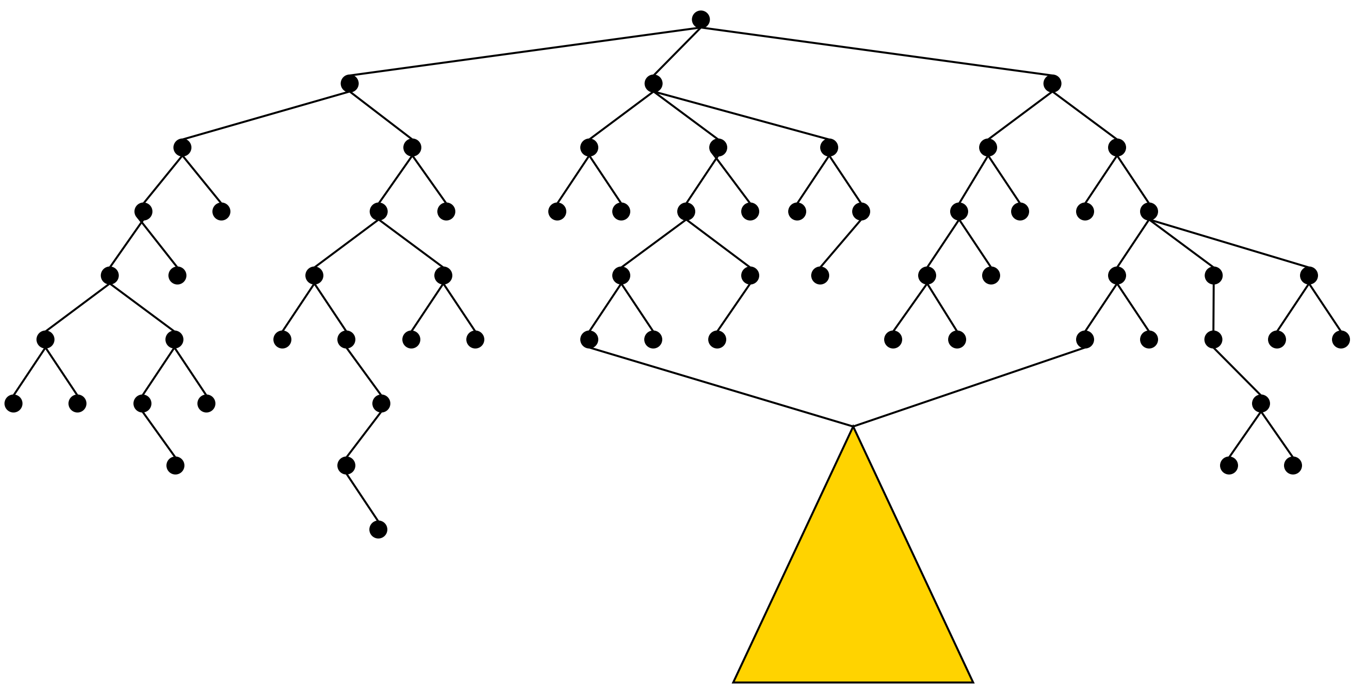
From Search Trees to Search Graphs

- Any two nodes that root dentical sub-trees or sub-graphs can be merged

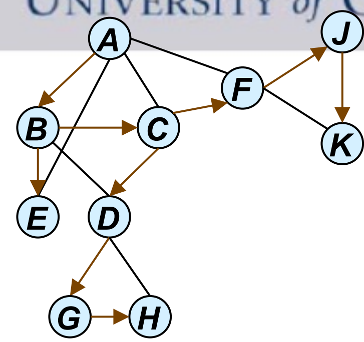


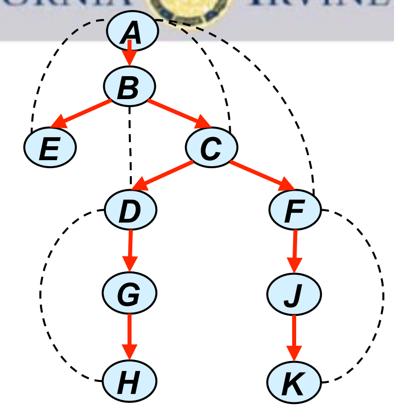
From Search Trees to Search Graphs

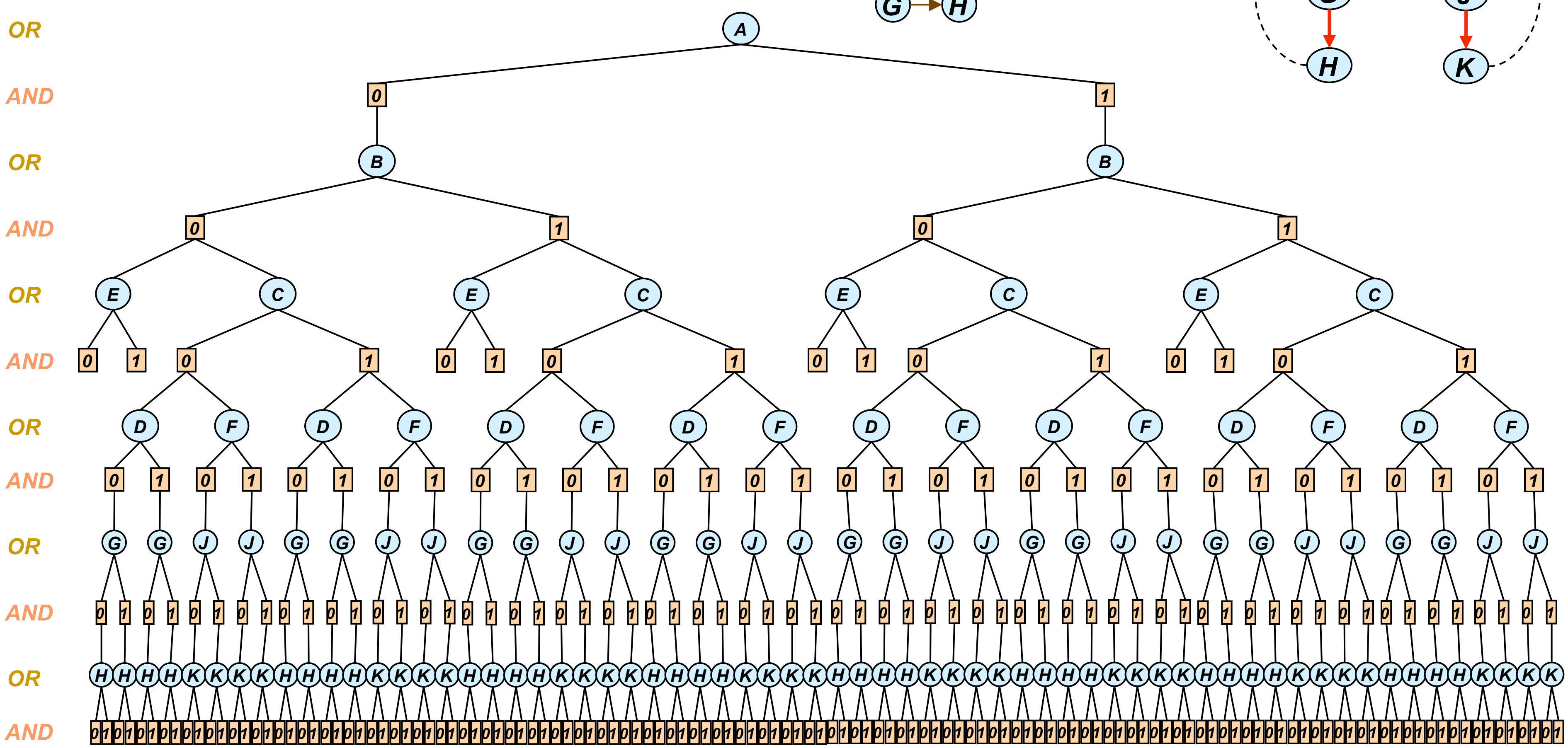
- Any two nodes that root identical sub-trees or sub-graphs can be merged



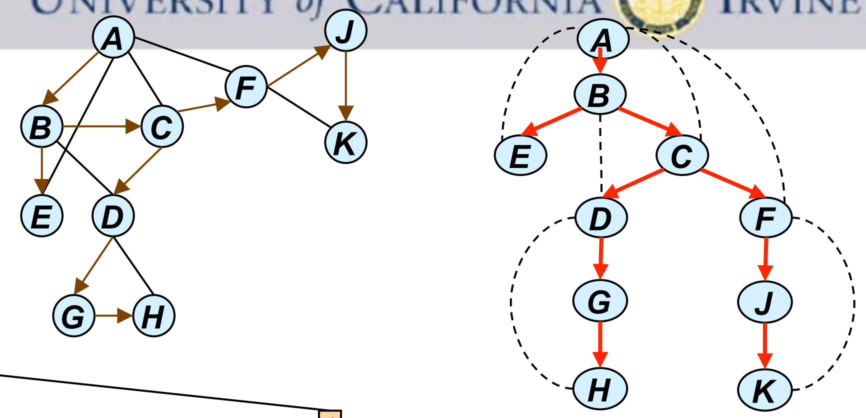
From AND/OR Tree

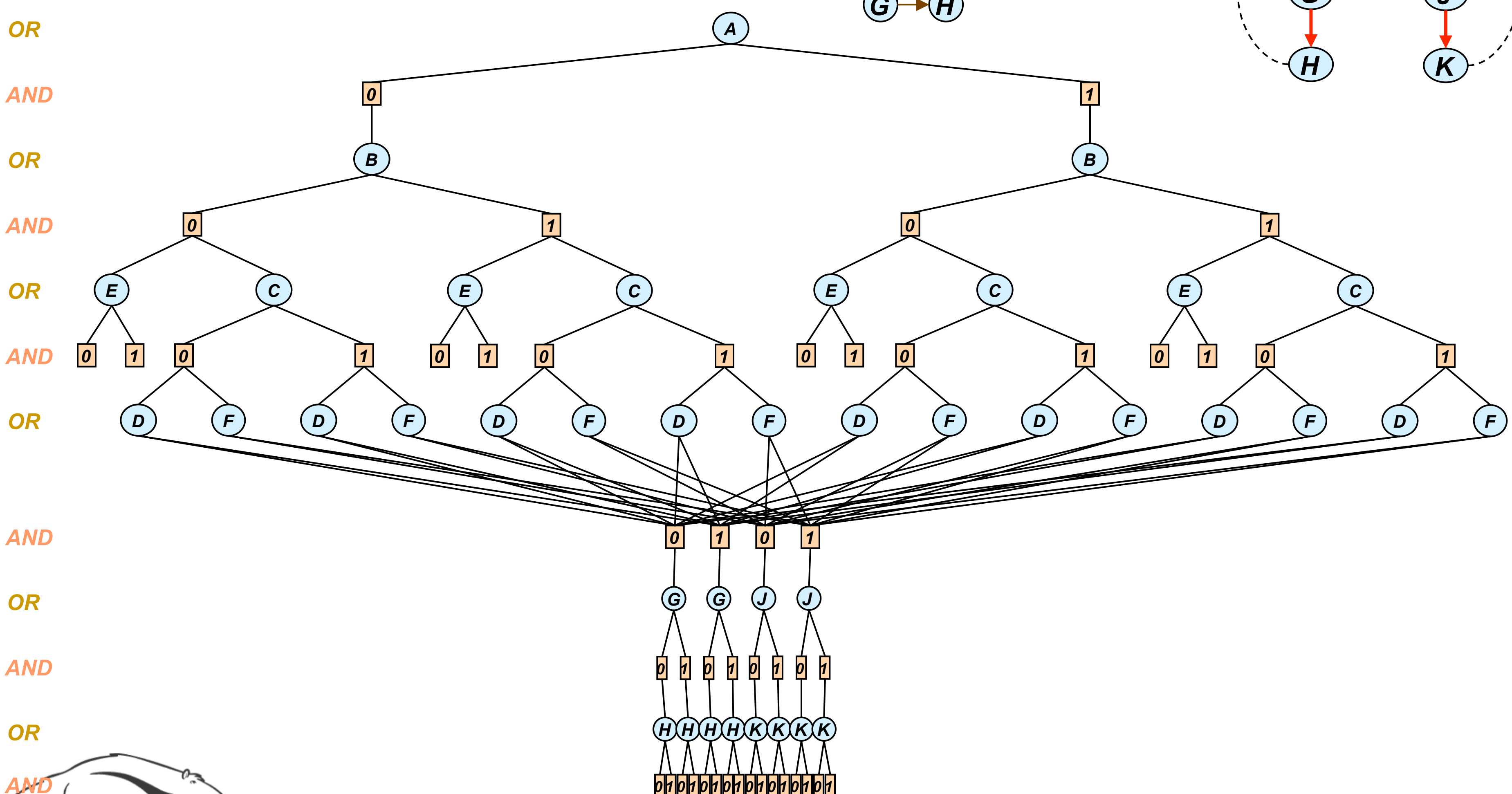






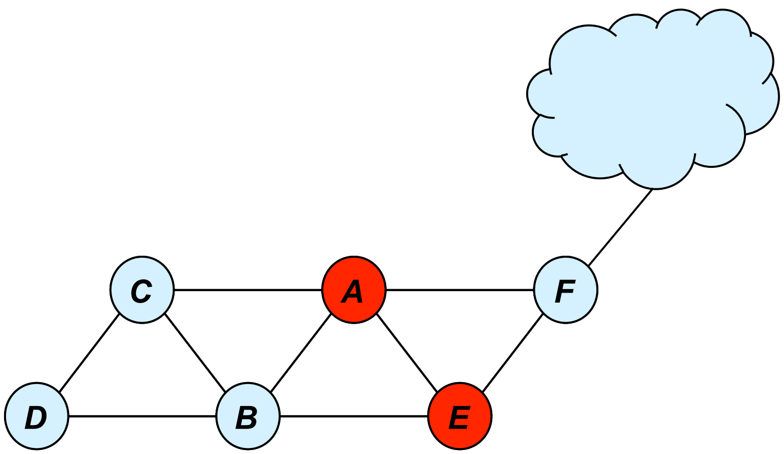
An AND/OR Graph

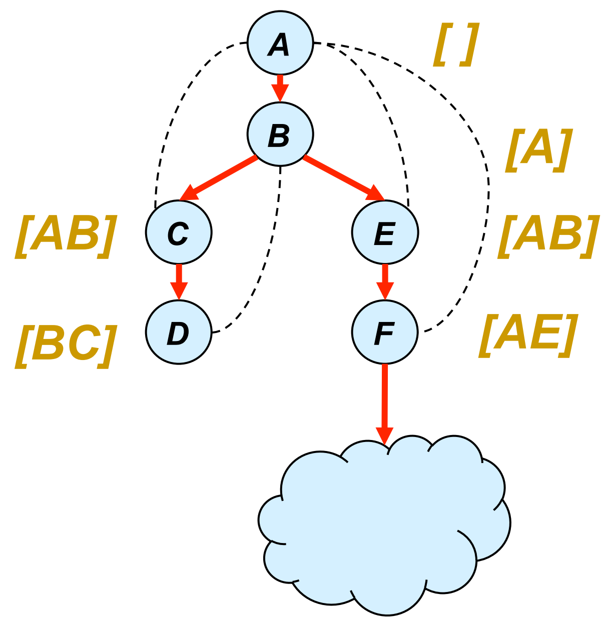




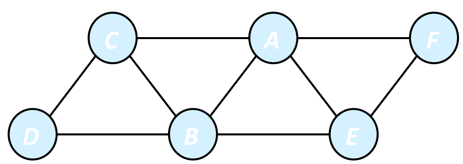
Merging based on context

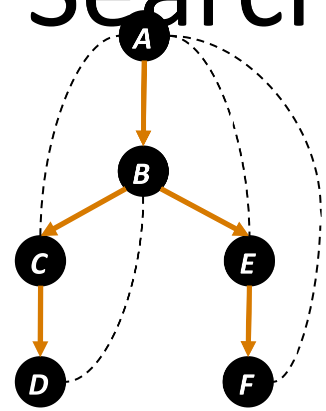
context (X) = ancestors of X connected to → *X* / → *descendants of X*





AND/OR Search Graph





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

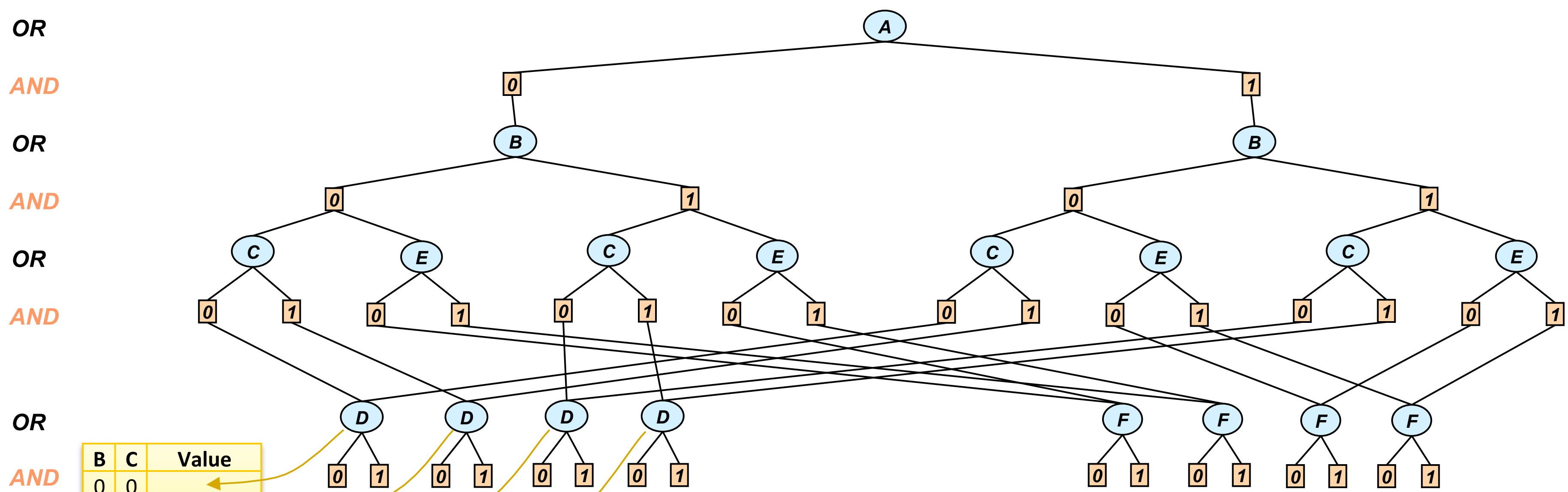
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

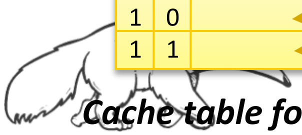
C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$



B	C	Value
0	0	
0	1	
1	0	
1	1	

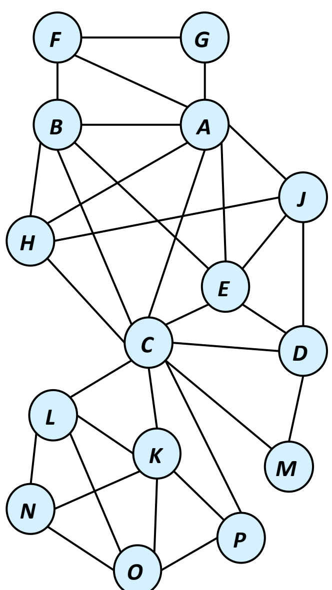


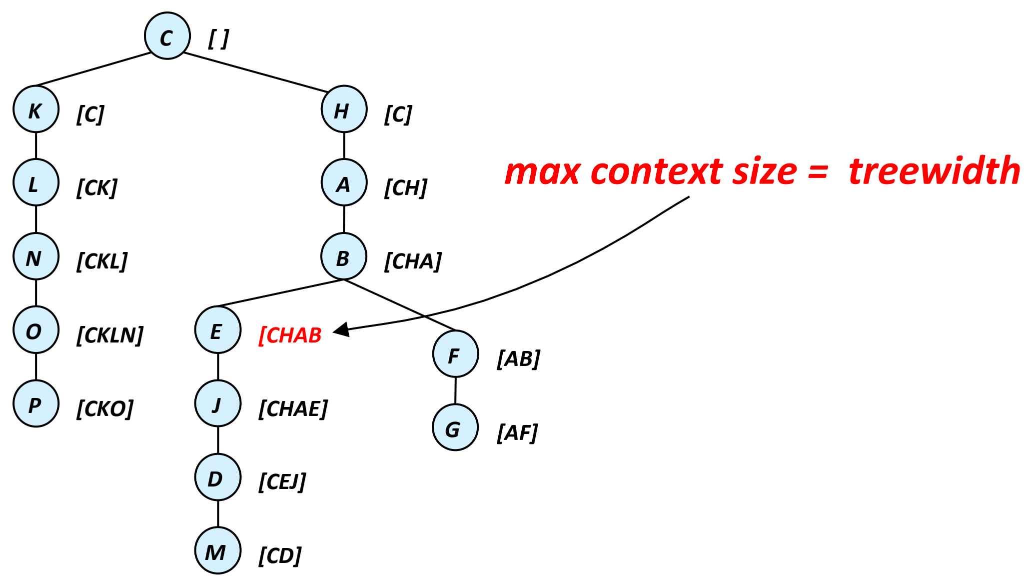
Cache table for D

context minimal graph

How Big Is The Context?

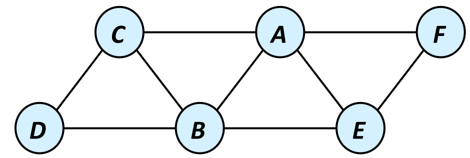
Theorem: The maximum context size for a pseudo tree is equal to the treewidth of the graph along the pseudo tree.

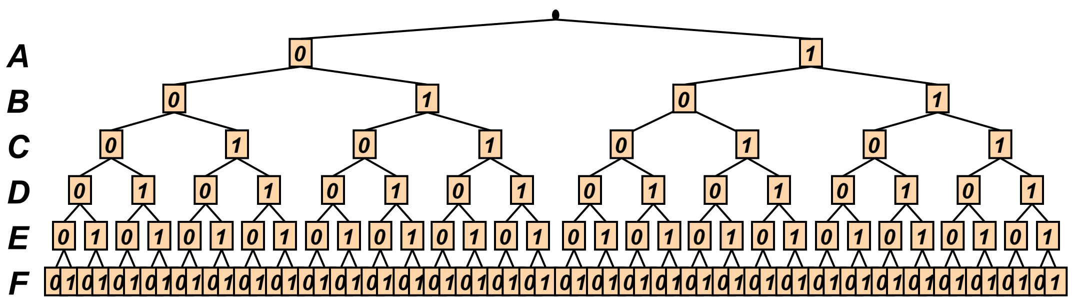




(C K H A B E J L N O D P M F G)

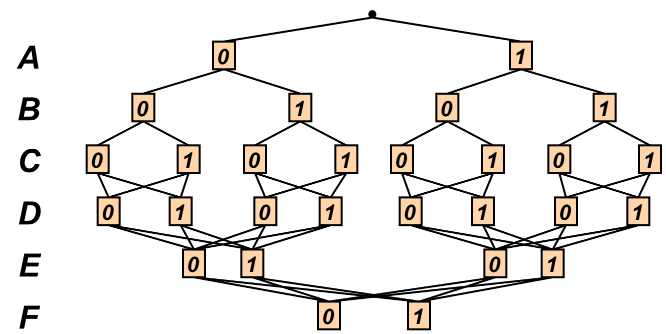
All Four Search Spaces





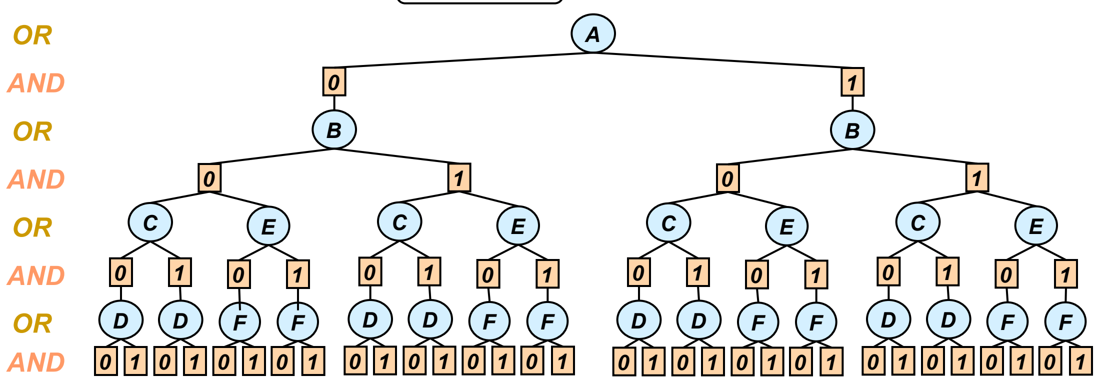
Full OR search tree

126 nodes



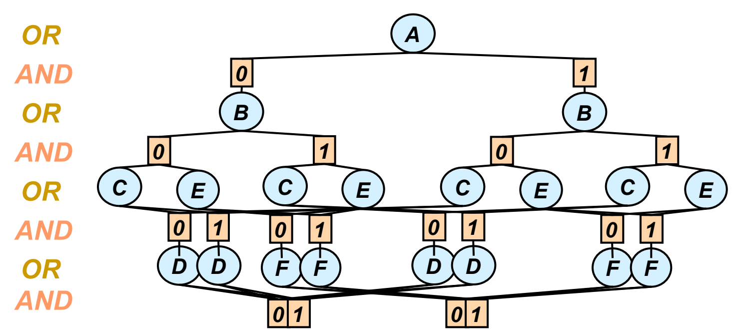
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes



Context minimal AND/OR search graph

18 AND nodes

Any query is best computed Over the c-minimal AO space

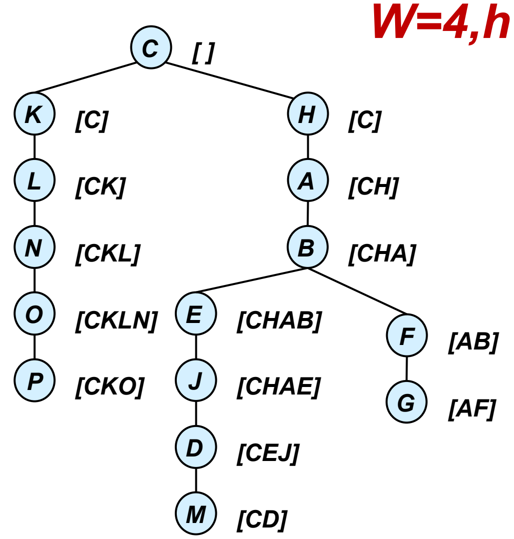
Complexity of AND/OR Graph Search

	AND/OR graph	**OR graph**
Space	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$
Time	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$

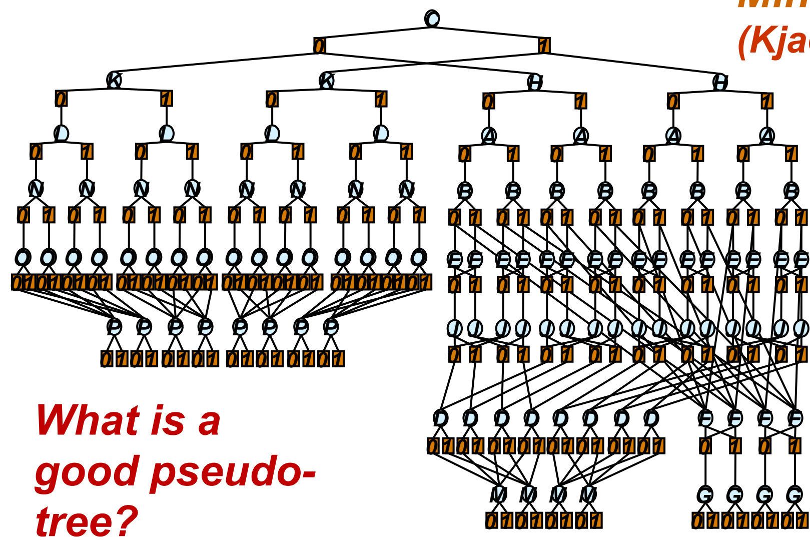
***k* = domain size**
***n* = number of variables**
w*= treewidth
pw*= pathwidth

The impact of the pseudo-tree

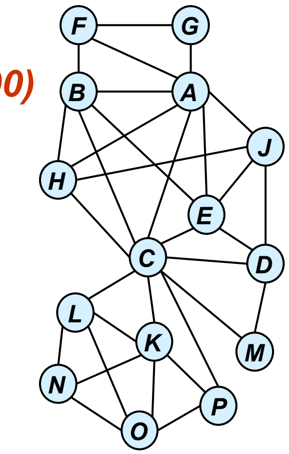
W=4,h=8



(C K H A B E J L N O D P M F G)



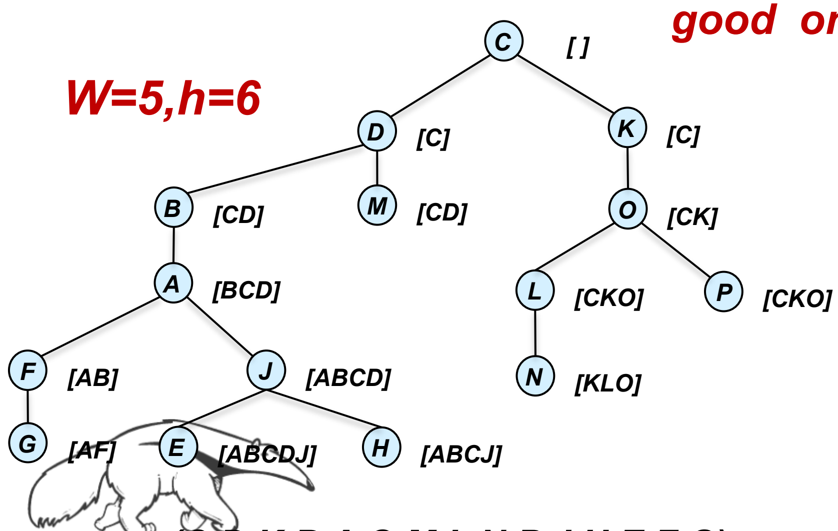
Min-Fill (Kjaerulff90)



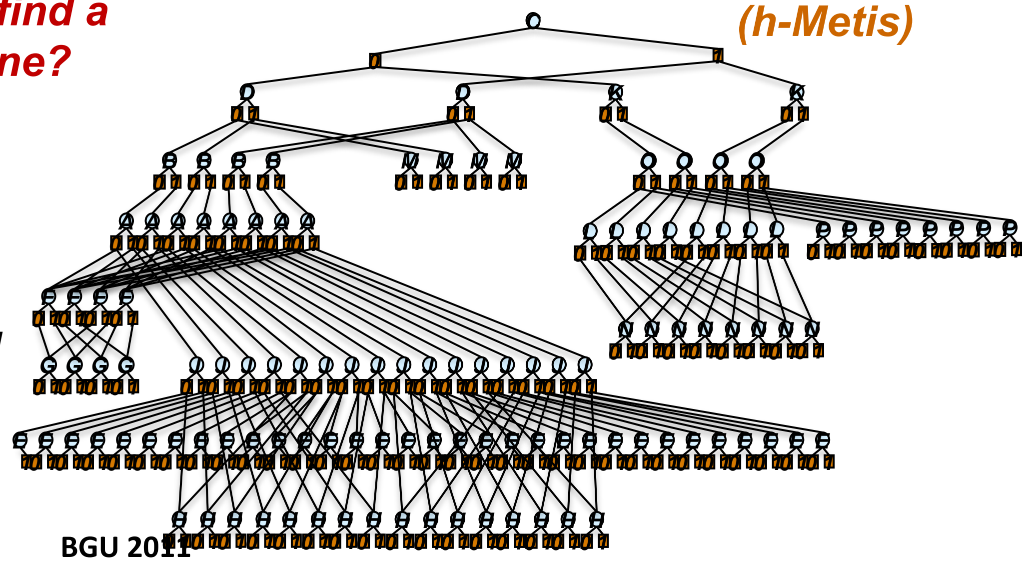
What is a good pseudo-tree?
How to find a good one?

Hypergraph Partitioning (h-Metis)

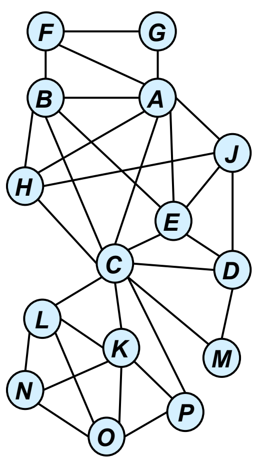
W=5,h=6

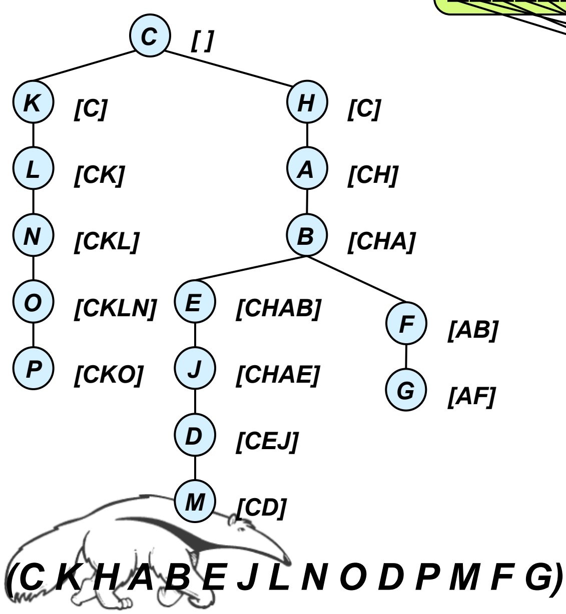


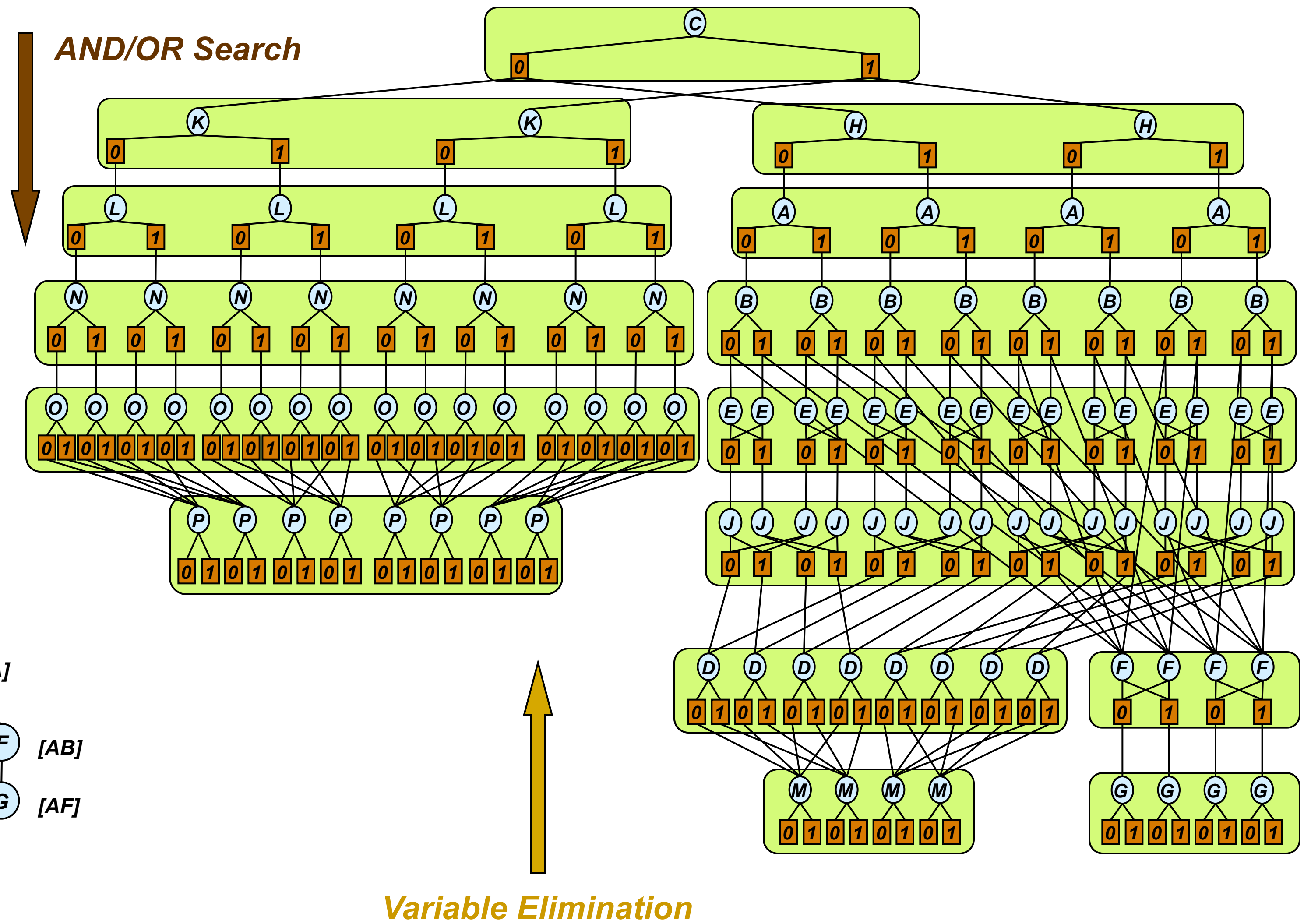
(C D K B A O M L N P J H E F G)



AND/OR Context Minimal Graph







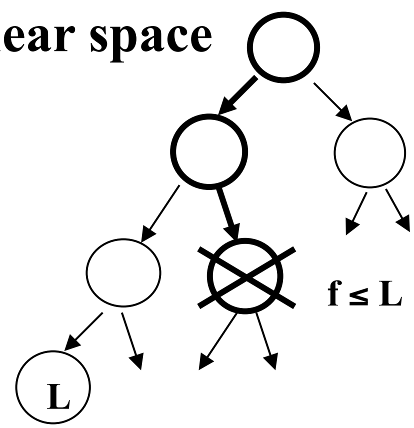
Outline

- Graphical Models: reasoning principles
- OR Search Trees
- AND/OR Search Spaces for GM
- AND/OR Branch-and-Bound and Best-First Search
- Lower Bounding Heuristics
- Experimental evaluation
- Current work

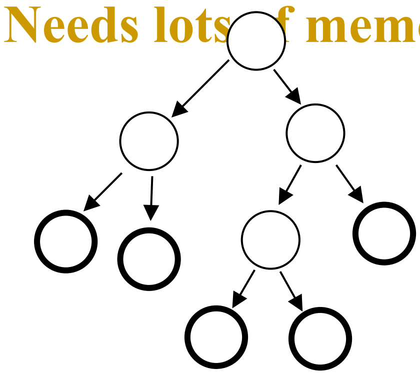
Basic Heuristic Search Schemes

Heuristic function f(x) computes a lower bound on the best extension of x and can be used to guide a heuristic search algorithm. We focus on

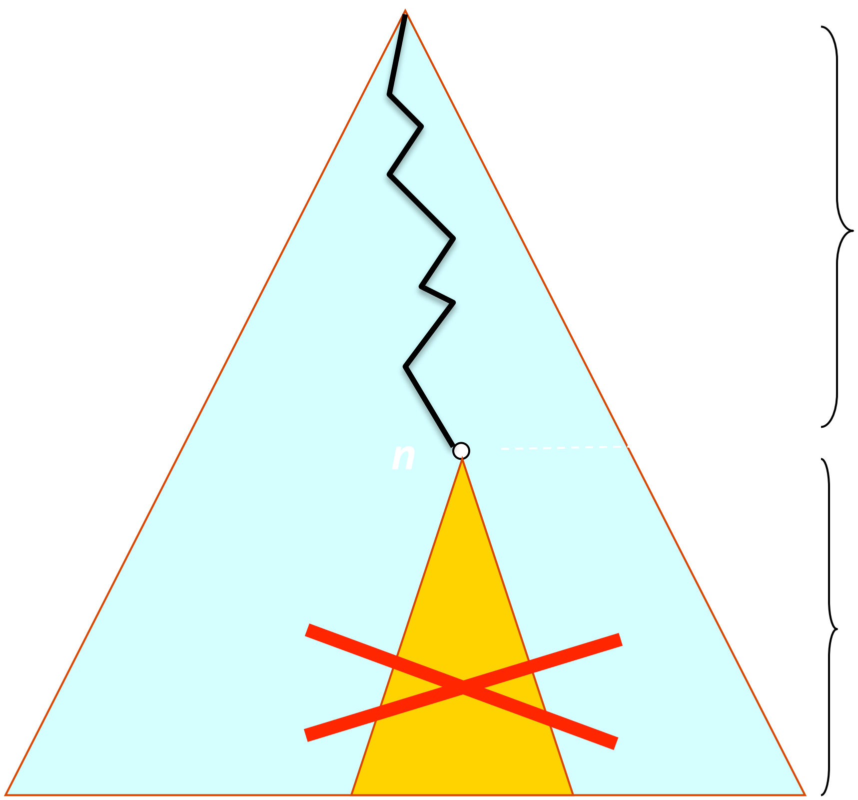
1.Branch and Bound
Use heuristic function $f(x^p)$ to prune the depth-first search tree.
Linear space



2.Best-First Search
Always expand the node with the highest heuristic value $f(x^p)$.
Needs lots of memory



Branch-and-Bound Search



g(n)=cost of the search path to n

H(n) = estimates the optimal cost below n

OR Search Tree

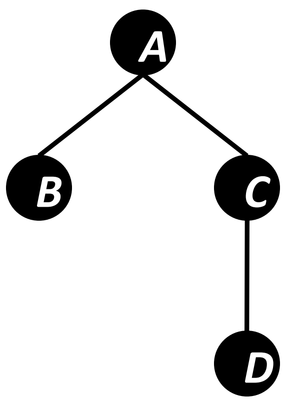
Upper Bound UB

Lower Bound LB(n)

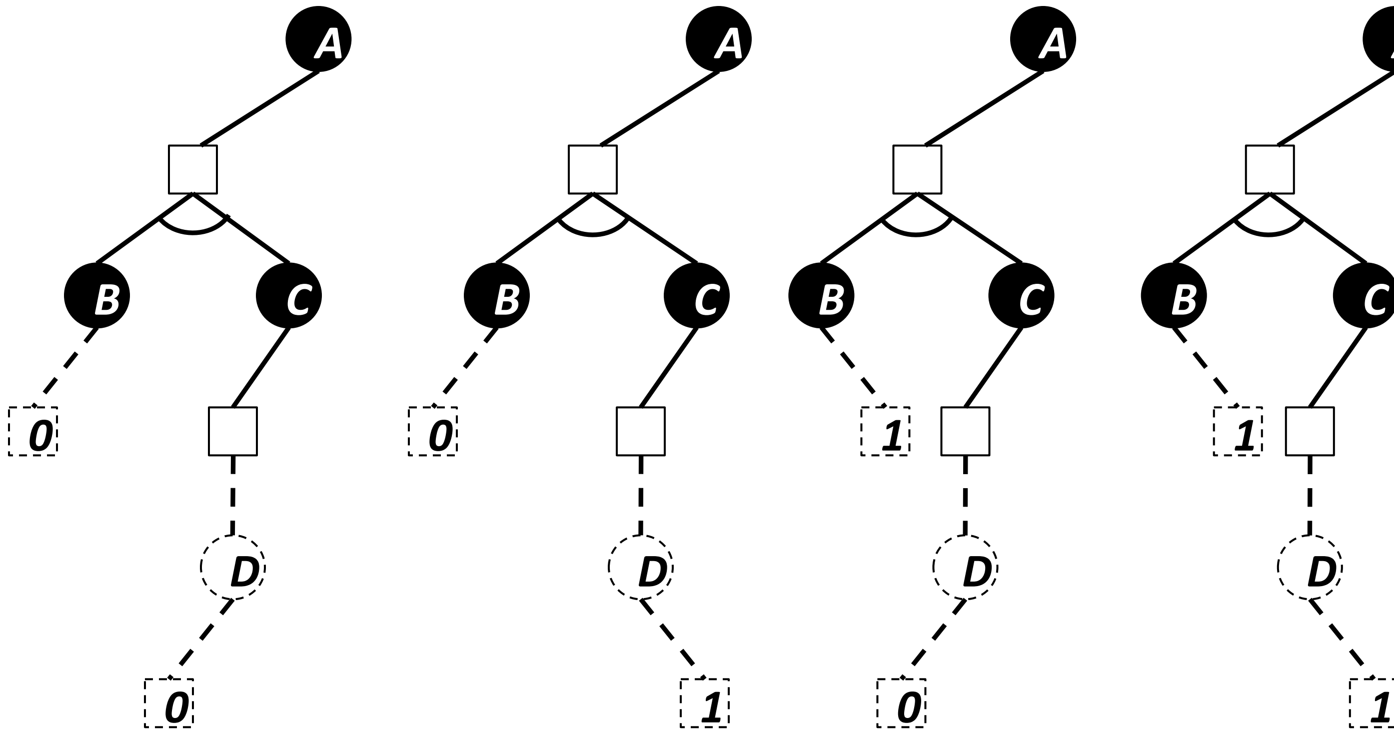
Prune if LB(n) ≥ UB

(Lawler & Wood66)

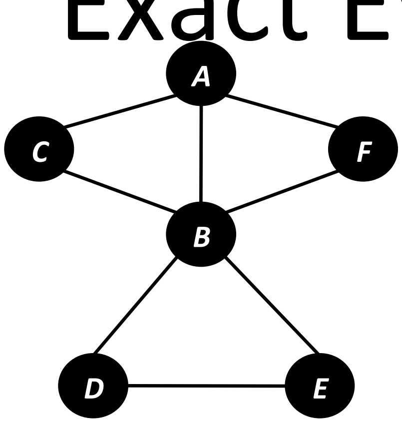
Partial Solution Tree



Pseudo tree



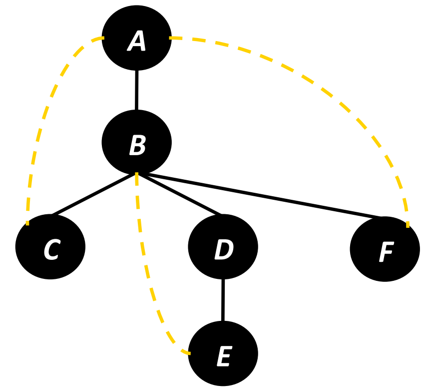
Exact Evaluation Function

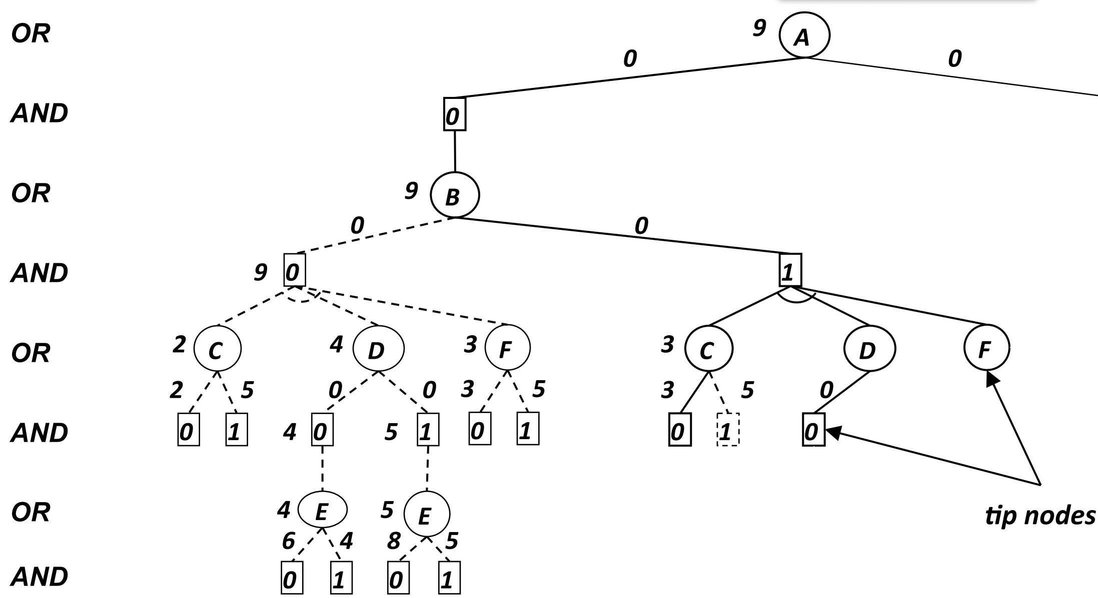


A	B	C	$f_1(ABC)$
0	0	0	2
0	0	1	5
0	1	0	3
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	2

A	B	F	$f_2(ABF)$
0	0	0	3
0	0	1	5
0	1	0	1
0	1	1	4
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

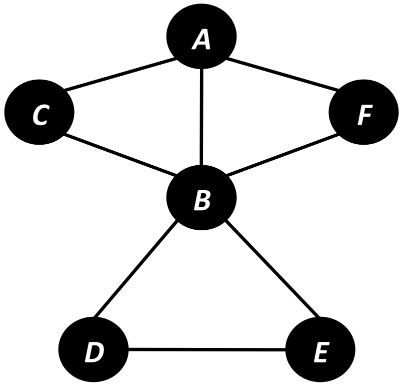
B	D	E	$f_3(BDE)$
0	0	0	6
0	0	1	4
0	1	0	8
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	4





$f^*(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + v(D,0) + v(F)$

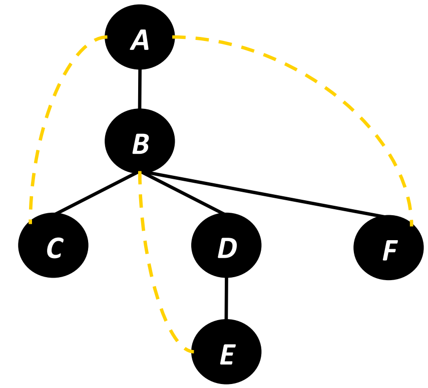
Heuristic Evaluation Function

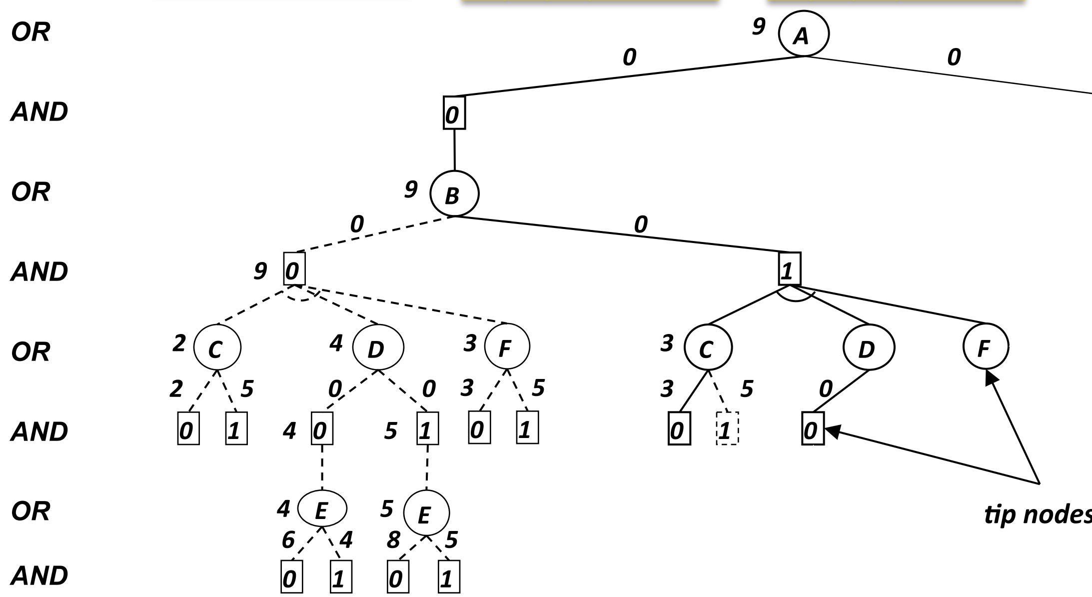


A	B	C	$f_1(ABC)$
0	0	0	2
0	0	1	5
0	1	0	3
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	2

A	B	F	$f_2(ABF)$
0	0	0	3
0	0	1	5
0	1	0	1
0	1	1	4
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

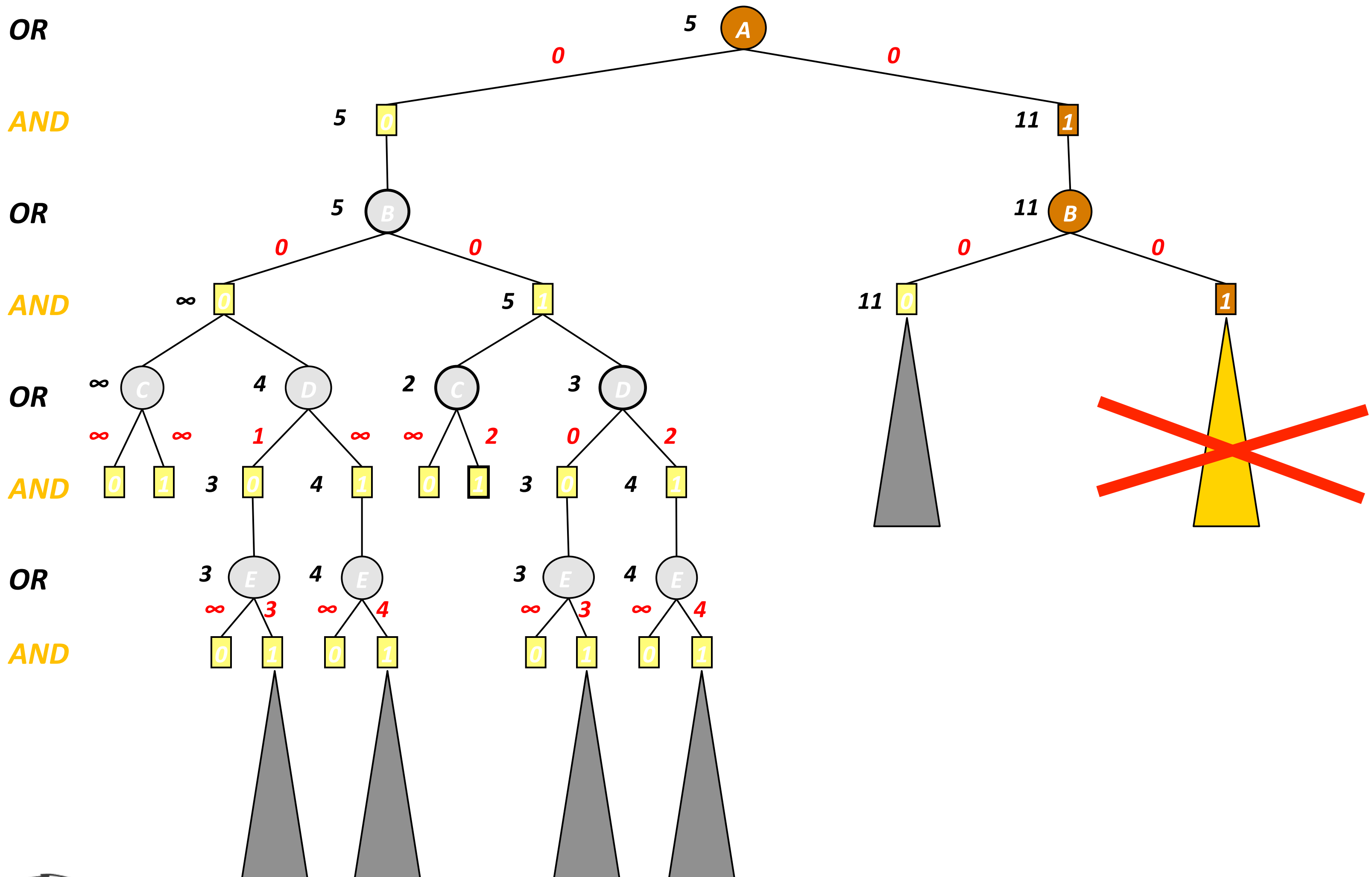
B	D	E	$f_3(BDE)$
0	0	0	6
0	0	1	4
0	1	0	8
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	4





$$f(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + h(D,0) + h(F) = 12 \le f^*(T')$$

AND/OR Branch-and-Bound Search



Best-First Search Principle

- Best-first search expands first the node with the best heuristic evaluation function among all nodes encountered so far
- It never expands nodes whose evaluation function is beyond the optimal one, unlike depth-first search algorithms (Dechter & Pearl85)
- Superior among memory intensive algorithms employing the

Best-First AND/OR Search

- Maintains the set of best partial solution trees
- **EXPAND** (top-down)
  - Traces down marked connectors from root (best partial solution tree)
  - Expands a tip node by generating its successors **n’**
  - Associate each successor with heuristic estimate **h(n’)**
    - Initialize **v(n’) = h(n’)**
- **REVISE** (bottom-up)
  - Updates node values **v(n)**
    - OR nodes: **minimization**
    - AND nodes: **summation**
  - Marks the most promising solution tree from the root
  - Label the nodes as SOLVED:
    - OR is SOLVED if marked child is SOLVED
    - AND is SOLVED if all children are SOLVED
- Terminate when root node is SOLVED

[specializes Nilsson’s AO* to graphical models (Nilsson80)]

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How to Generate Heuristics *(Pearl86)*

- The principle of relaxed models
  - Linear relaxation for integer programs
  - Mini-Bucket Elimination for graphical models
  - Bounded directional consistency ideas
  - Pattern databases

Inference for Optimization: Bucket Elimination

Algorithm BE-mpe (Dechter 1996, Bertele and Briochi, 1977)

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)P(d|a,b)P(e|b,c)$$

$$\max_X \prod$$

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $h^B(a,d,c,e)$

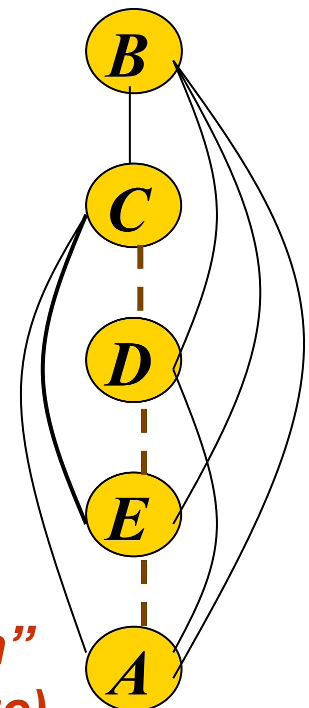
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $h^D(a,e)$

bucket A: $P(a)$ $h^E(a)$

$P(a|e=0)$

$W^*=4$
"induced width"
(max clique size)



Mini-bucket approximation: MPE task

Split a bucket into mini-buckets =>bound complexity

bucket (X) =

$$\{ h_1, \ldots, h_r, h_{r+1}, \ldots, h_n \}$$

$$h^X = \max_X \prod_{i=1}^{n} h_i$$

$$\{ h_1, \ldots, h_r \} \qquad \{ h_{r+1}, \ldots, h_n \}$$

$$g^X = \left(\max_X \prod_{i=1}^{r} h_i\right) \cdot \left(\max_X \prod_{i=r+1}^{n} h_i\right)$$

$$h^X \le g^X$$

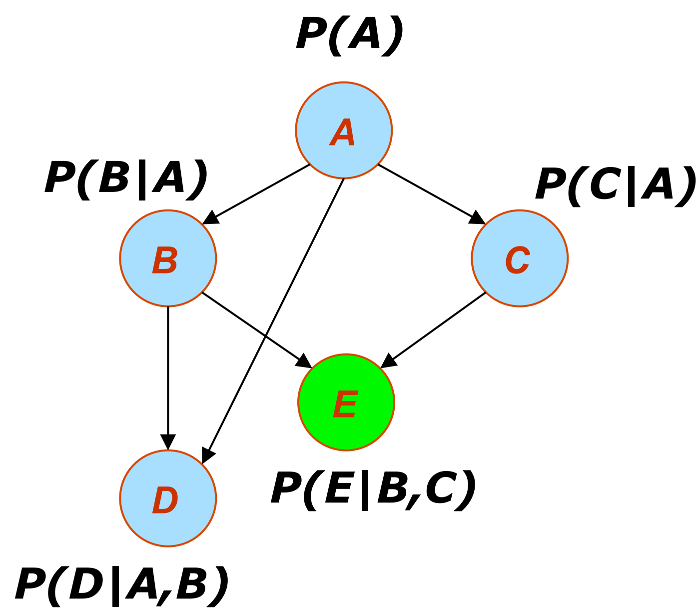
Exponential complexity decrease : $O(e^n) \rightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Elimination

Node duplication, renaming

	$max_B\Pi$	$max_B\Pi$
Bucket B	**P(E|B,C)**	**P(B|A) P(D|A,B)**
Bucket C	**P(C|A)** h^B **(C,E)**	
Bucket D		h^B **(A,D)**
Bucket E	**E = 0** h^C **(A,E)**	
Bucket A	**P(A)** h^E **(A)** h^D **(A)**	

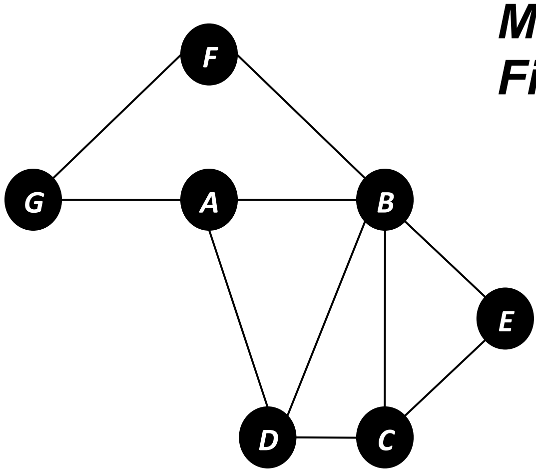
W=2

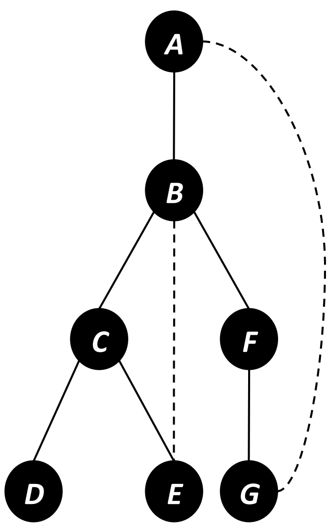


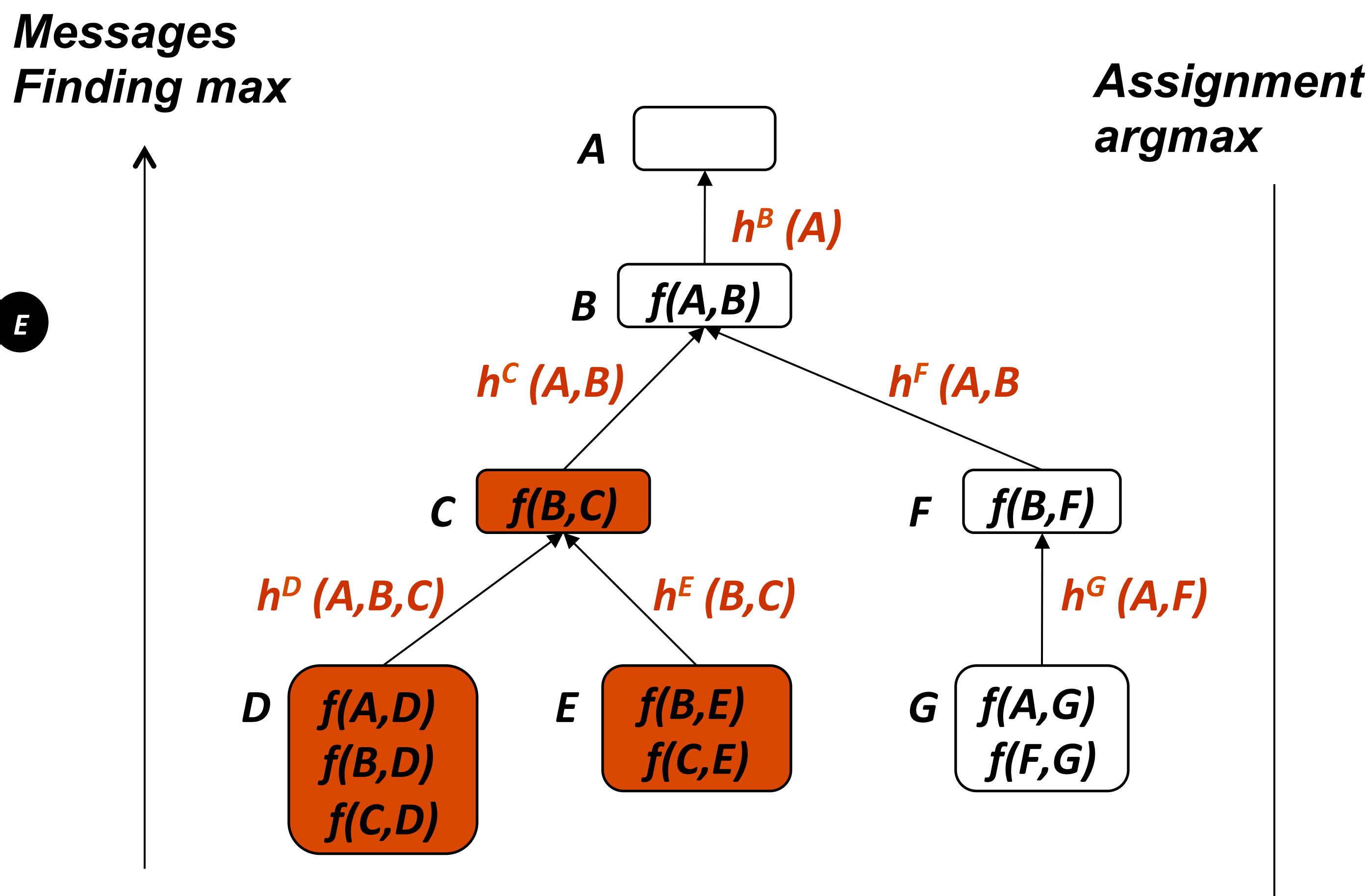
MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

Bucket Elimination

$$\min_{a,b,c,d,e,f,g} f(a,b) + f(a,d) + a(b,c) + f(a,d) + f(b,d) + f(c,d) + f(b,e) + f(c,e) + f(b,f) + f(a,g) + f(f,g) =$$



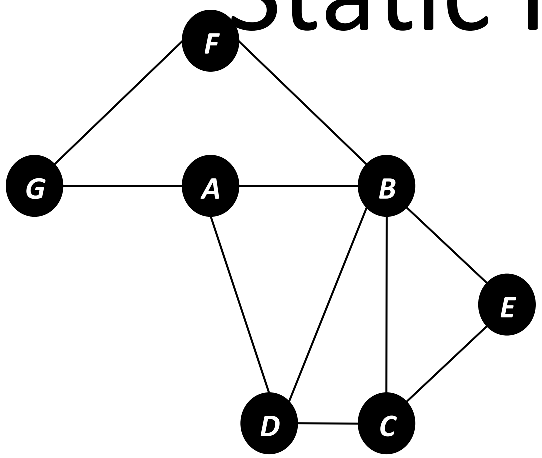


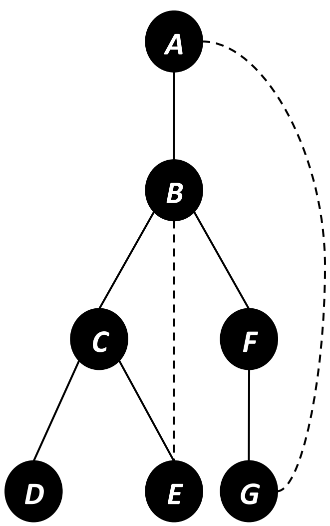


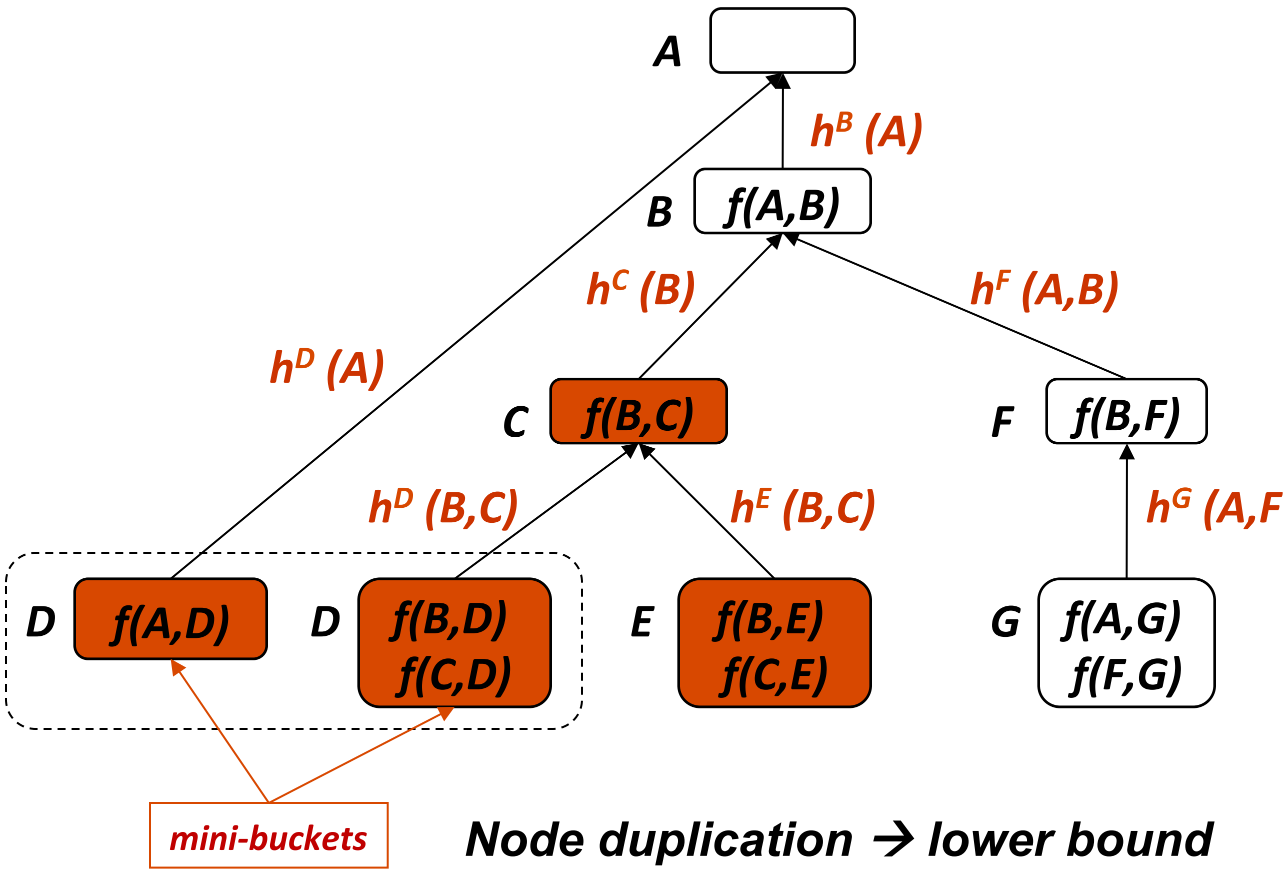
$$h^*(a, b, c) = h^D(a, b, c) + h^E(b, c)$$

Ordering: (A, B, C, D, E, F, G)

Static Mini-Bucket Heuristics







Node duplication → lower bound

$$h(a, b, c) = h^D(a) + h^D(b, c) + h^E(b, c)$$
$$\leq h^*(a, b, c)$$

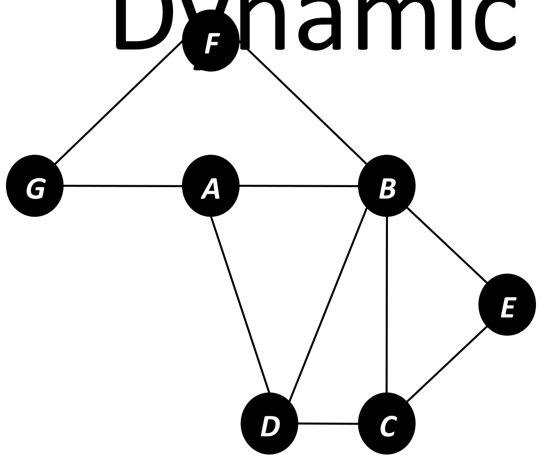
Ordering: (A, B, C, D, E, F, G)

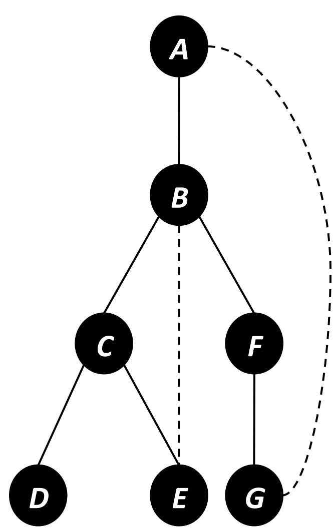
Mini-Bucket Elimination

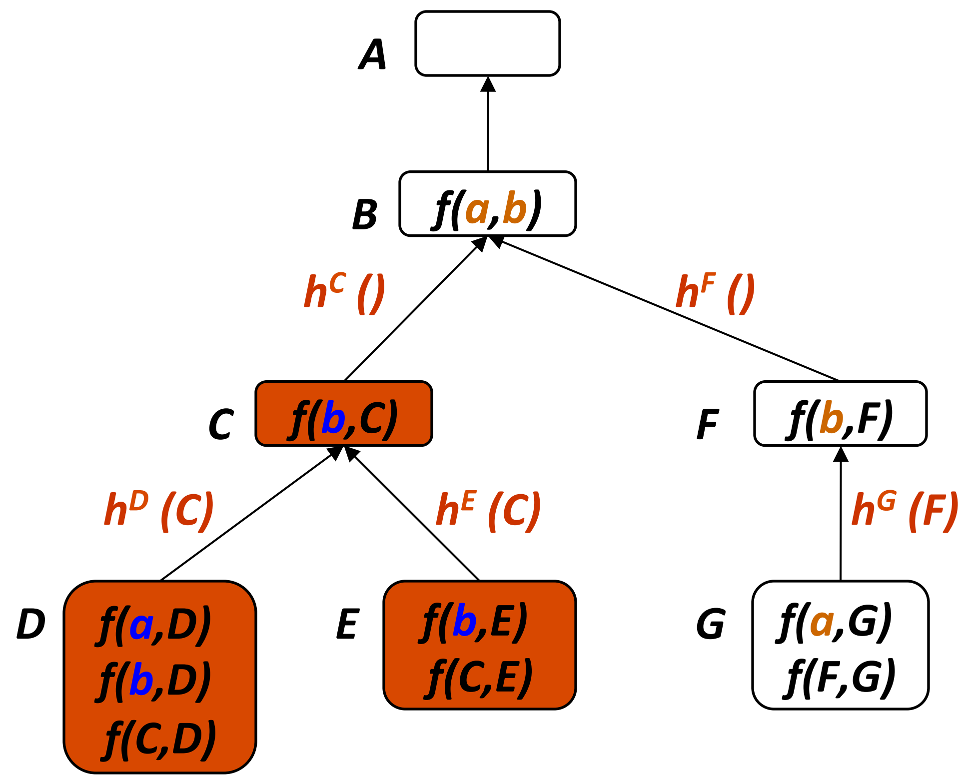
- Approximation of the Bucket Elimination algorithm by partitioning large buckets into "mini-buckets" which are processed separately (Dechter & Rish97)

- Properties
  - Parameterized by i-bound (controls complexity)
  - Computes a lower bound on the exact solution
  - Approximation improves with the i-bound

Dynamic Mini-Bucket Heuristics







$$h(a, b, c) = h^D(c) + h^E(c)$$
$$= h^*(a, b, c)$$

Ordering: (A, B, C, D, E, F, G)

Outline

- Graphical Models: reasoning principles
- OR Search Trees
- AND/OR Search Spaces for GM
- AND/OR Branch-and-Bound and Best-First Search
- Lower Bounding Heuristics
- Experimental evaluation
- Current work

What are the empirical questions?

- What is the
  - Impact of AND/OR decomposition?
  - Impact of caching (Tree search vs graph search)?
  - The heuristic strength vs. search tradeoff.
  - impact of depth-first BB vs Best-first
  - The (w,h) of the pseudo-tree

Experiments

- Optimization task
  - Most Probable Explanation in belief networks
  - Optimal wcsp
- Algorithms
  - AOBB-C+SMB(i): AOBB w/ full caching and mini-buckets
  - AOBF-C+SMB(i): AOBF w/ full caching and mini-buckets
  - Samlam (Recursive Conditioning) (Darwiche01)
  - Superlink (genetic linkage analysis) (Fishelson&Geiger02)
  - ILOG CPLEX 11.0
- Benchmarks
  - ISCAS'89 circuits
  - Grid networks
  - Genetic linkage analysis
  - Mastermind games

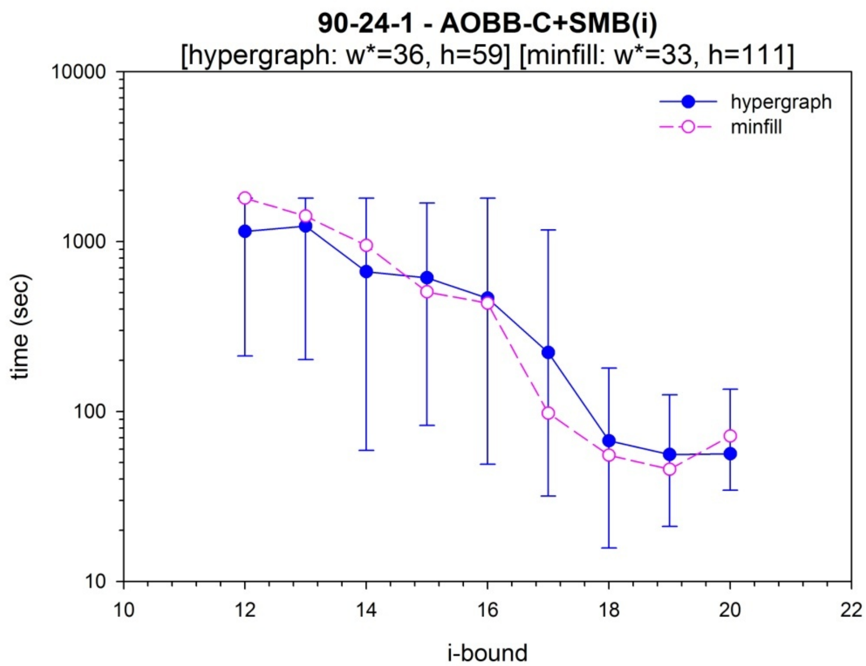
Grid Networks (BN)

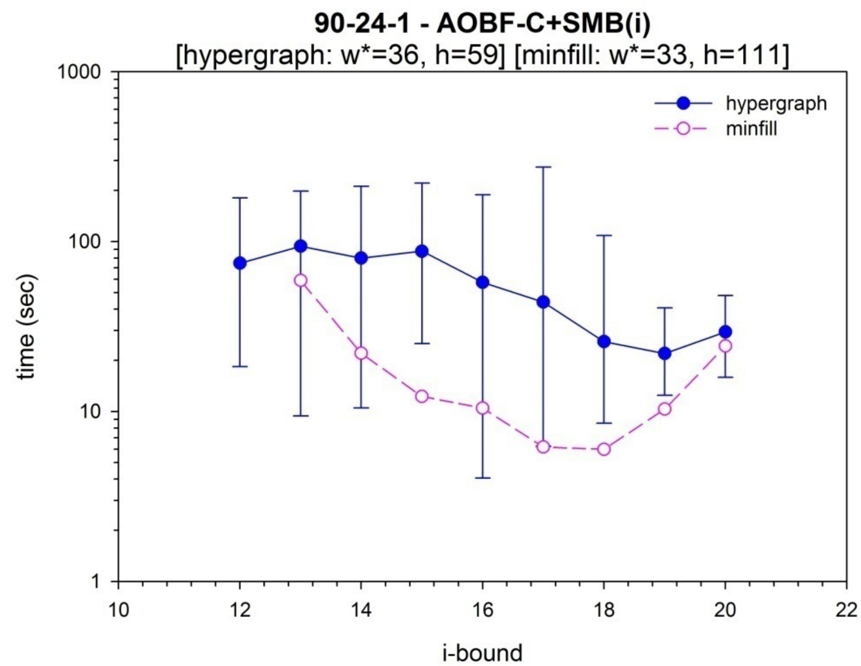
(Sang et al.05)

grid (w*, h) (n, e)	**Samlam**	**MBE(i)** **BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=12		**MBE(i)** **BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=14		**MBE(i)** **BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=16		**MBE(i)** **BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=18	
		time	nodes	time	nodes	time	nodes	time	nodes
		0.28		0.64		1.69		4.60	
90-24-1		-	-	-	-	-	-	-	-
(33, 111)	out	-	-	2338.67	24,117,151	1548.09	18,238,983	138.67	1,413,764
(576, 20)		-	-	1273.09	9,047,518	596.27	4,923,760	70.42	473,675
		out		21.94	75,637	10.59	33,770	**6.06**	5,144
		0.63		1.25		3.72		11.66	
90-34-1		-	-	-	-	-	-	-	-
(45, 153)	out	-	-	-	-	-	-	-	-
(1154, 80)		-	-	-	-	-	-	-	-
		out		out		**243.63**	596,978	270.88	667,013
		0.78		1.67		4.20		12.36	
90-38-1		-	-	-	-	-	-	-	-
(47, 163)	out	2032.33	6,835,745	-	-	807.38	2,850,393	568.69	2,079,146
(1444, 120)		969.02	2,623,971	1753.10	3,794,053	203.67	614,868	165.45	488,873
		101.69	174,786	103.80	146,237	54.00	95,511	**53.44**	78,431

Min-fill pseudo tree. Time limit 1 hour.

Impact of the Pseudo Trees and heuristic power





Runtime distribution for AOBB and AOBF with different i-bound for
The mini-bucket heuristics for pseudo-trees created by hypergraph and min-fill
over 20 independent runs for 90-24-1 grid instance.

Genetic Linkage Analysis

(Fishelson & Geiger02)

pedigree (w*, h) (n, d)	Samlam Superlink	MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF-C+SMB(i) i=12		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF-C+SMB(i) i=14		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF-C+SMB(i) i=16		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF-C+SMB(i) i=18	
		time	nodes	time	nodes	time	nodes	time	nodes
		0.42		0.83		1.78		5.75	
ped30		-	-	-	-	-	-	-	-
(23, 118)	out	-	-	-	-	-	-	214.10	1,379,131
(1016, 5)	13095.83	10212.70	93,233,570	8858.22	82,552,957	-	-	34.19	193,436
		out		out		out		**30.39**	72,798
		0.58		2.31		7.84		33.44	
ped33		-	-	-	-	-	-	-	-
(37, 165)	out	2804.61	34,229,495	737.96	9,114,411	3896.98	50,072,988	159.50	1,647,488
(581, 5)	-	1426.99	11,349,475	307.39	2,504,020	1823.43	14,925,943	86.17	453,987
		out		140.61	407,387	out		**74.86**	134,068
		4.20		31.33		96.28		out	
ped42		-	-	-	-	-	-		
(25, 76)	out	-	-	-	-	-	-		
(448, 5)	561.31	-	-	-	-	2364.67	22,595,247		
		out		out		**133.19**	93,831		

Min-fill pseudo tree. Time limit 3 hours.

ISCAS'89 Benchmark (WCSP)

iscas (w*, h) (n, d)	MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF+SMB(i) i=8		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF+SMB(i) i=10		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF+SMB(i) i=12		MBE(i) BB-C+SMB(i) AOBB+SMB(i) AOBB-C+SMB(i) AOBF+SMB(i) i=14		toolbar toolbar-BTD	
	time	nodes	time	nodes	time	nodes	time	nodes	time	nodes
	0.08		0.08		0.14		0.28			
c499	-	-	-	-	1.53	4,495	6.20	35,314	-	-
(23, 55)	96.46	1,265,425	39.65	526,517	1.42	18,851	37.26	486,656	100.96	1,203,734
(499, 2)	19.28	99,906	7.36	40,285	0.47	2,401	5.83	34,708		
	3.91	14,049	2.45	8,816	**0.34**	1,032	2.52	8,755		
	0.16		0.19		0.38		0.94			
s1196	-	-	-	-	-	-	-	-	-	-
(54, 97)	-	-	1347.95	12,392,442	-	-	1949.37	15,775,180	376.35	1,276,514
(562, 2)	3347.38	13,554,137	503.30	2,425,152	2299.72	11,488,366	734.66	3,524,780		
	22.67	72,075	**2.89**	9,336	13.02	40,210	7.27	21,989		
	0.16		0.22		0.38		0.92			
s1238	-	-	-	-	-	-	-	-	-	-
(59, 94)	-	-	-	-	1722.53	18,302,873	1394.86	14,213,319	-	-
(541, 2)	1897.37	8,386,634	1682.99	7,431,223	281.05	1,350,933	248.27	1,220,658		
	34.09	137,960	29.41	111,205	12.31	53,095	**6.64**	26,101		

Min-fill pseudo tree. Time limit 1 hour.

UAI 2010 evaluation, 2008, 2006

- Toulbar2: INRA
  Summary: Toulbar2 is an open source exact anytime Weighted CSP solver using Branch and Bound and soft local consistency
  Team members: S. de Givry, D. Allouche, A. Favier, T. Schiex
  Additional contributors: M. Sanchez, S. Bouveret, H. Fargier, F. Heras, P. Jegou, J. Larrosa, K. L. Leung, S. N'diaye, E. Rollon, C. Terrioux, G. Verfaillie, M. Zytnicki
  Contact person: Thomas Schiex, Thomas.Schiex@toulouse.inra.fr
  Detailed description
- Daoopt: UCI Irvine
  Summary: "daoopt" and "daoopt.anytime" are based on AND/OR branch and bound graph search, with mini bucket heuristics and LDS (Limited Discrepancy Search) initialization.
  Team members: Lars Otten, Rina Dechter
  Additional Contributor: Radu Marinescu
  Contact person: Lars Otten, lotten@ics.uci.edu
  Detailed description
  Web-site: http://graphmod.ics.uci.edu

3rd in all 3 categorie
After Toolbar, Joris

What did we learn

- Use the highest heuristic memory allows
- Generate heuristic in pre-processing
- Use AOBB with caching (best-first run out of memory). We also have adaptive-caching
- Try to get the best induced-width pseudo-tree
- Use some good initial upper-bound from local search

Outline

- Graphical Models: reasoning principles
- OR Search Trees
- AND/OR Search Spaces for GM
- AND/OR Branch-and-Bound and Best-First Search
- Lower Bounding Heuristics
- Experimental evaluation
- Current work

Recent related highlights

- Improving Heuristics due to “moment matching” or soft arc-consistency. (Flerova, Ihler,Dechter, Otten, 2011)
- Anytime behavior of AOBB (Otten and Dechter, SOCS 2011)
- Improving treewidth (Kask et. Al. 2011)
- Improving mini-bucket parttiotning (Rollon 2010)

Bounding algorithms

non-iterative message-passing schemes

e.g. ***MBE*** *[Dechter, Rish 2003]*

iterative schemes using re-parametrization

e.g. ***MPLP*** *[Globerson et al. 2007],*

Max-sum diffusion *[Kovalevsky et al. 1975]*

Soft arc-consistency

[Schiex 2000, Bistarelli et al. 2000]

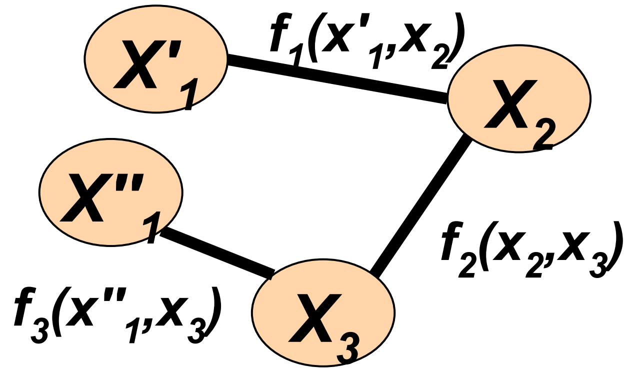
↓ ↓

Mini-Bucket with moment-matching

Intuition behind MBE-MM

How to make max-marginals equal?

Do cost shifting!

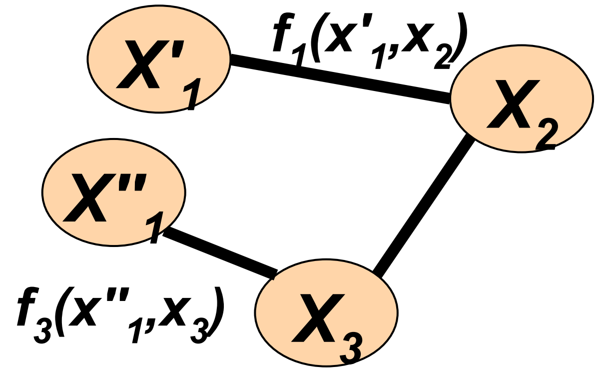


$$C_1(x_2, x_3) = \max_{x_1} f_1(x_1, x_2) \cdot f_3(x_1, x_3)$$

$$C_1(x_2, x_3) = \max_{x_1} f_1(x_1, x_2) g(x_1) \cdot f_3(x_1, x_3) / g(x_1)$$

Intuition behind MBE-MM

$$x_1^{*\prime} = \underset{x_1'}{argmax} \max_{x_2}[f_1(x_1', x_2) \cdot g(x_1')]$$



$$x_1^{*\prime\prime} = \underset{x_1''}{argmax} \max_{x_3}[f_3(x_1'', x_3)/g(x_1'')]$$

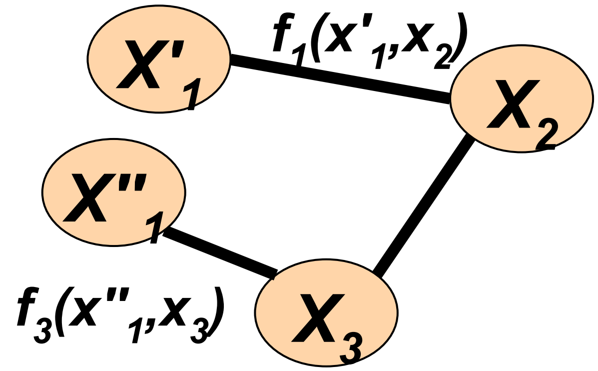
$$x_1^{*} = x_1^{*\prime} = x_1^{*\prime\prime}$$

$$\max_{x_2} f_1(x_1, x_2) g(x_1) = \max_{x_3} f_3(x_1, x_3)/g(x_1)$$

Intuition behind MBE-MM

$$F'_1(x_1, x_2) = f_1(x_1, x_2)\sqrt{\frac{\max\limits_{x_3} f_3(x_1, x_3)}{\max\limits_{x_2} f_1(x_1, x_2)}}$$

$$F''_1(x_1, x_3) = f_3(x_1, x_3)\sqrt{\frac{\max\limits_{x_2} f_1(x_1, x_2)}{\max\limits_{x_3} f_3(x_1, x_3)}}$$



Linear relaxation-based schemes (MPLP class, Globerson and Jakkola)

Find: $\boldsymbol{x} = (x_1, \ldots, x_n)$ to all the variables which maximizes the sum of the factors:

$$\mathrm{MAP}(\boldsymbol{\theta}) = \max_{\boldsymbol{x}} \sum_{i \in V} \theta_i(x_i) + \sum_{f \in F} \theta_f(\boldsymbol{x}_f). \tag{1.1}$$

Best upper bound by Equivalence preserving transformations:

$$\min_{\boldsymbol{\delta}} L(\boldsymbol{\delta}), \tag{1.2}$$

$$L(\boldsymbol{\delta}) = \sum_{i \in V} \max_{x_i} \Big(\theta_i(x_i) + \sum_{f: i \in f} \delta_{fi}(x_i) \Big) + \sum_{f \in F} \max_{\boldsymbol{x}_f} \Big(\theta_f(\boldsymbol{x}_f) - \sum_{i \in f} \delta_{fi}(x_i) \Big).$$

$\delta_{fi}(x_i)$ ***Is the cost shifted from f to value x_i of X_i.***

There are several variations of scheme computing the optimizing shifts based on partial gradient descent, which differ by what is being kept constant. The 1.2 tas Dual of a linear relaxation of the original problem.

Grids

The dash '-' indicates that the was no solution within 24 hours

Time in seconds for grid instances

All grids are binary

Instances (n,w,h)	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=10 time	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=5 time	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=3 time
50-12-5	0	2	9
144,15,48	0	0	1
	0	1	0
50-14-5	1	64	211
196,18,64	0	6	25
	0	6	24
50-15-5	1	42	431
225,19,76	0	22	24
	0	29	24
50-16-5	97	6759	14047
256,21,79	1	257	11918
	1	209	11872
50-17-5	18	2674	19951
289,22,84	0	70	2293
	0	44	2302
50-18-5	1131	—	—
324,24,84	10	47196	—
	10	19697	—
50-19-5	4664	—	—
361,25,93	9	8808	—
	11	5375	864000
50-20-5	3589	—	—
400,27,97	11	28985	—
	26	6529	—
75-16-5	7	245	2457
256,21,73	0	32	511
	1	37	521
75-17-5	8	279	568
289,22,78	1	186	292
	1	133	301

The dash '-' indicates that the was no solution within 24 hours

Time in seconds for grid instances

All grids are binary

Instances (n,w,h)	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=10 time	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=5 time	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=3 time
90-20-5 400,27,99	14 1 0 9	1199 585 389	7281 5575 5317
90-21-5 441,28,106	15 0 1	1585 861 593	7722 9064 9054
90-22-5 484,30,109	50 6 1	2327 1172 604	27283 17130 17060
90-23-5 529,31,116	564 10 17	70188 29635 25511	— — —
90-24-5 576,33,110	2393 73 68	— — 27818	— — —
90-25-5 625,34,132	2496 223 277	— — —	— — —
90-26-5 676,36,136	386 21 16	36469 7077 4000	— 70798 70290
90-30-5 900,42,151	— 25439 21895	— — —	— — —

Runtime for pedigree instances for AOBB

Instances (n,k,w,h)	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=10	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=8	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=6	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=4
pedigree1 298,4,15,48	0 1 4	1 0 7	0 0 11	0 0 42
pedigree13 88,3,32,102	— 57583 70328	— — —	— — —	— — —
pedigree19 693,5,26,95	— — —	— — —	— — —	— — —
pedigree20 387,5,22,60	44 25 87	137 112 262	167 378 582	4460 10805 11203
pedigree23 309,5,25,51	4 0 9	13 2 24	22 3 20	45 31 89
pedigree25 993,5,25,69	58 0 1	145 4 3	1303 36 48	— 13321 4670
pedigree30 1015,5,21,108	109 21 34	246 36 99	1690 442 508	13198 — —
pedigree31 1006,5,30,85	— — —	— — —	— — —	— — —

The dash '-' indicates that the was no solution within 24 hours

Runtime Runtime for pedigree instances for AOBB

Instances (n,k,w,h)	AOBB-MBE(z) AOBB-MBE-MM(z) AOBB-MPLP(z) z-bound=10	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=8	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=6	AOBB-MBE(z) AOBB-MBE-MM(z) OBB-MPLP(z) z-bound=4
pedigree33 581,4,28,98	89 5 8	177 7 8	201 260 287	24142 1958 2076
pedigree37 726,5,21,56	6 1 0	13 0 1	33 8 6	298 174 145
pedigree39 953,5,21,76	136 15 17	732 29 26	2871 294 377	30724 8315 9100
pedigree50 478,6,17,47	6 17 46	16 11 886	39 47 12146	25440 — —
pedigree51 871,5,39,98	— — —	— — —	— — —	— — —
pedigree7 867,4,32,90	— 1987 4975	— 5993 13211	— 46261 54000	— — —
pedigree9 935,7,27,100	46434 1206 2161	— 7086 9397	— — —	— — —

The dash '-' indicates that the was no solution within 24 hours

Recent related highlights

- Improving Heuristics due to “moment matching” or soft arc-consistency. (Flerova, Ihler,Dechter, Otten, 2011)
- Anytime behavior of AOBB (Otten and Dechter, SOCS 2011)
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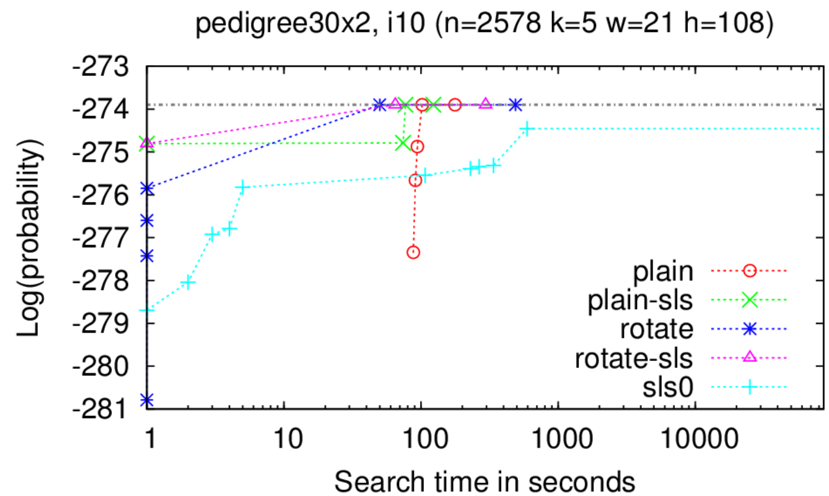
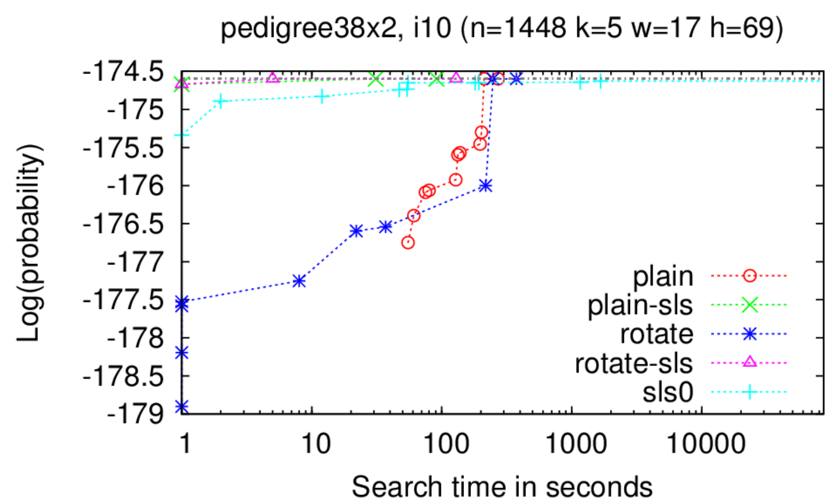
- Improving mini-bucket parttiotning (Rollon 2010)

Select Anytime Results

- AOBB+SLS receives initial boost

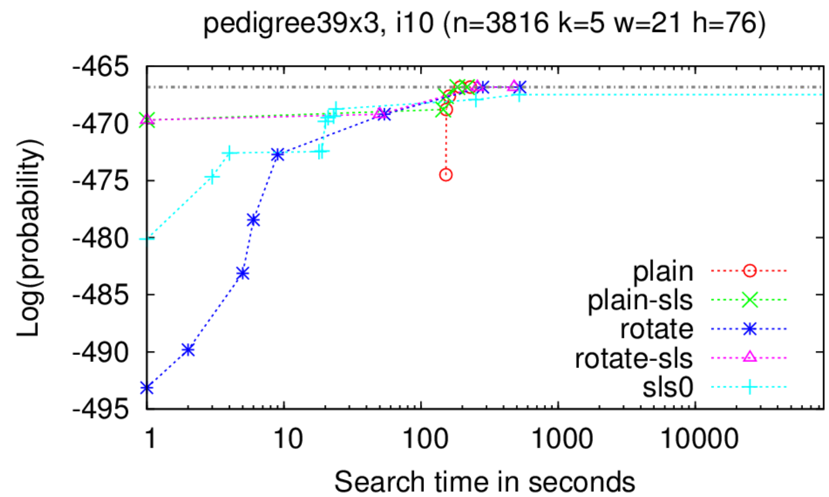


- AOBB+SLS gets to optimality faster

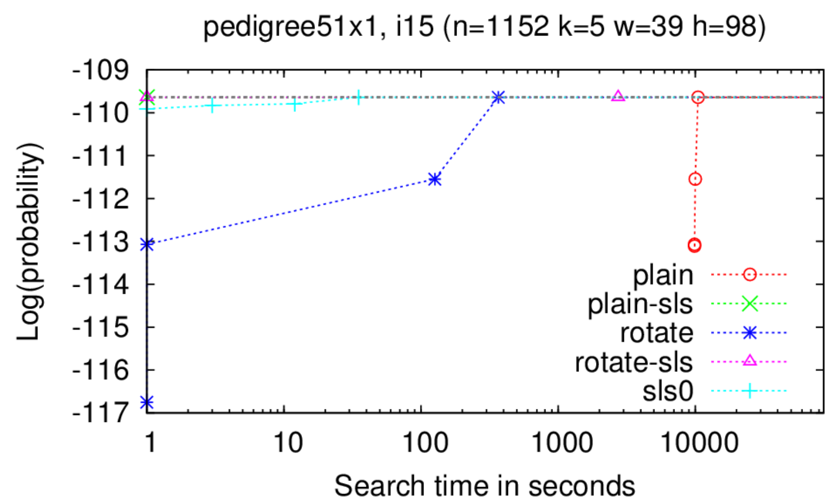


Select Detailed Results

- AOBB+SLS finds optimal solution, SLS doesn't

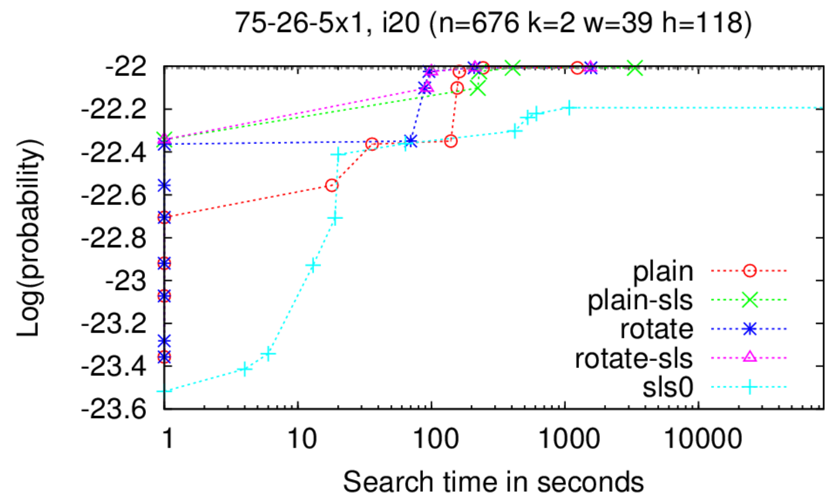


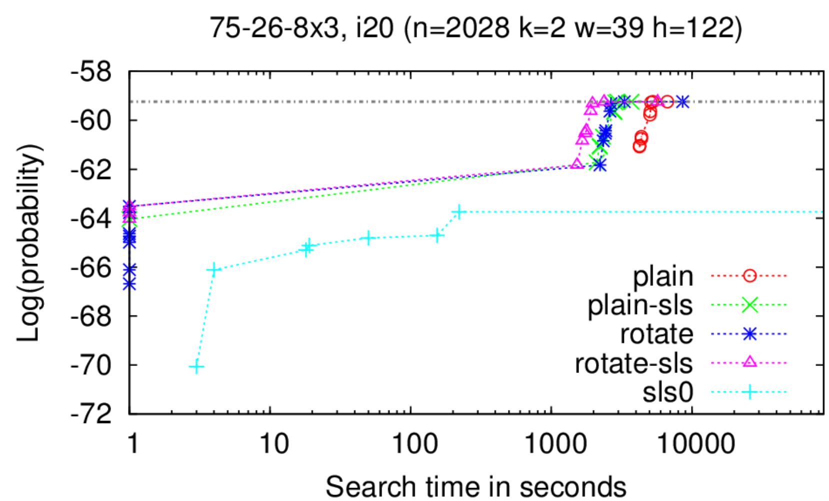
- AOBB+SLS finds optimum right away



Select Detailed Results

- All AOBB variants outperform SLS, reach optimality
- Rotating AOBB receives boost from initial SLS





Software

- AND/OR search algorithms
- Bucket-tree elimination
- Generalized belief propagation
- Samplesearch sampling

are available at:

- http://graphmod.ics.uci.edu/group/Software

Thank you!