Weighted AND/OR Graphs/Diagrams for Probabilistic and constraints Databases.

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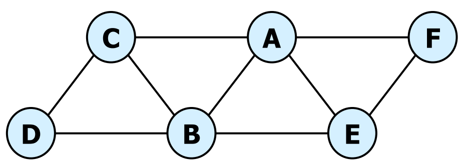
A Constraint Network and its Search Graphs



A	B	C	R_{ABC}
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

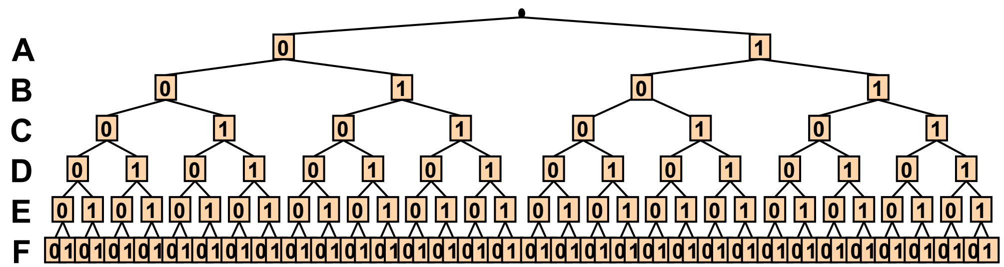
B	C	D	R_{BCD}
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

A	B	E	R_{ABE}
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

A	E	F	R_{AEF}
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

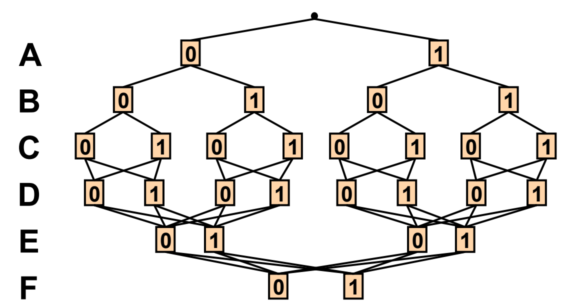
Contest-Minimal AND/OR Graph

Functional information
Is not needed



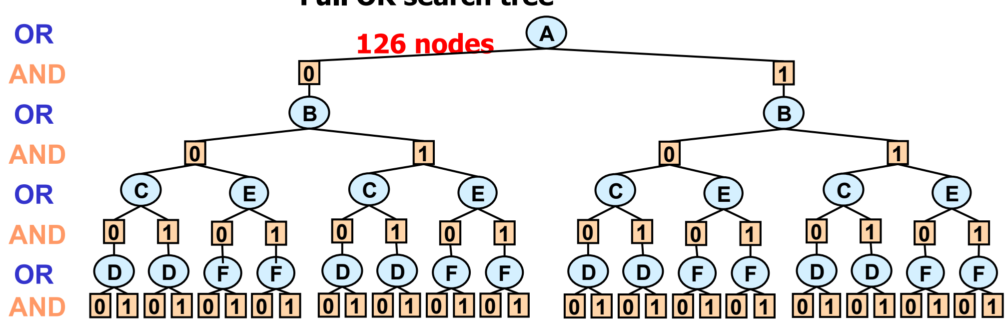
Full OR search tree

126 nodes



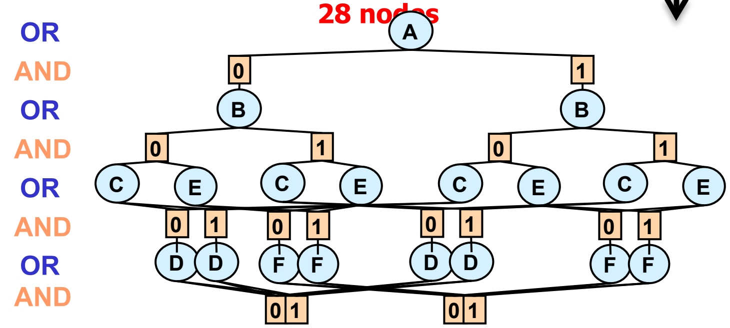
Context minimal OR search graph

28 nodes



Full AND/OR search tree

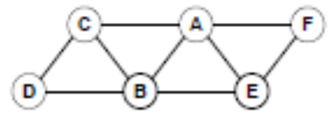
54 AND nodes



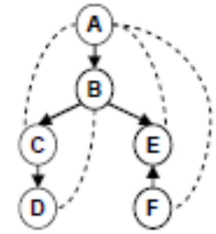
Context minimal AND/OR search graph

18 AND nodes

AND/OR Search Tree for Constraint Networks



(a) Constraint graph



(b) Pseudo tree

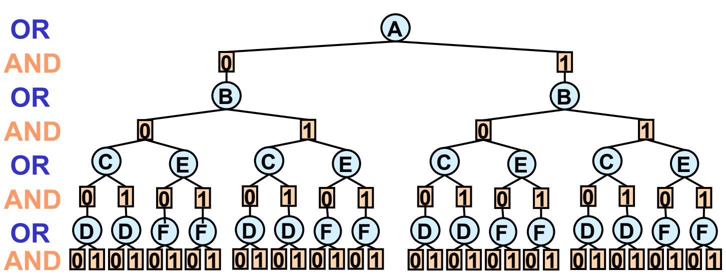
A	B	C	R_{ABC}
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

B	C	D	R_{BCD}
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

A	B	E	R_{ABE}
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

A	E	F	R_{AEF}
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

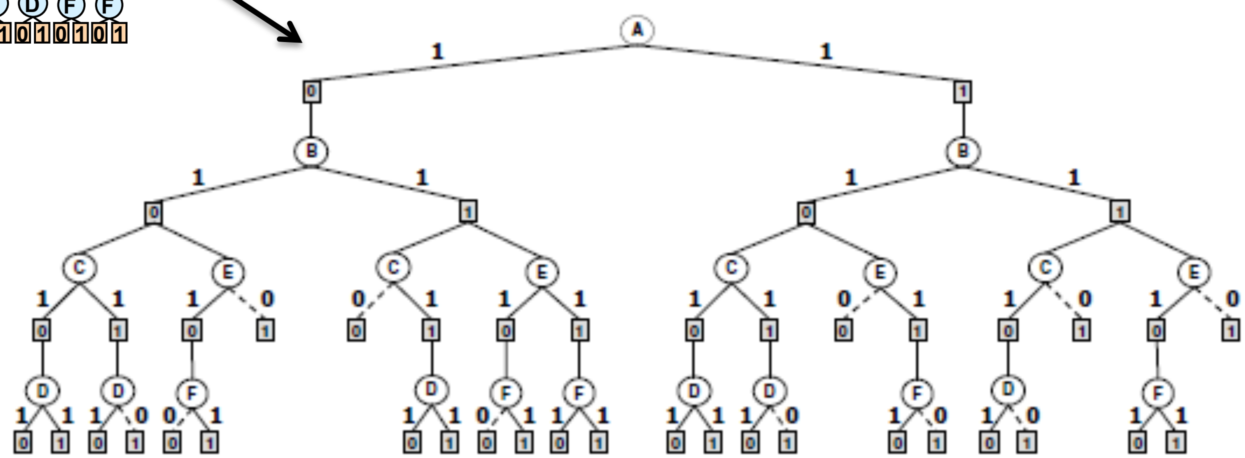
(c) Relations



Full AND/OR search tree

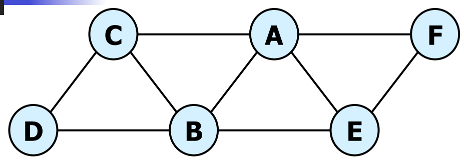
54 AND nodes

Taking the constraints into account



(d) AND/OR tree

Or Search Tree and Graph for Constraint Networks



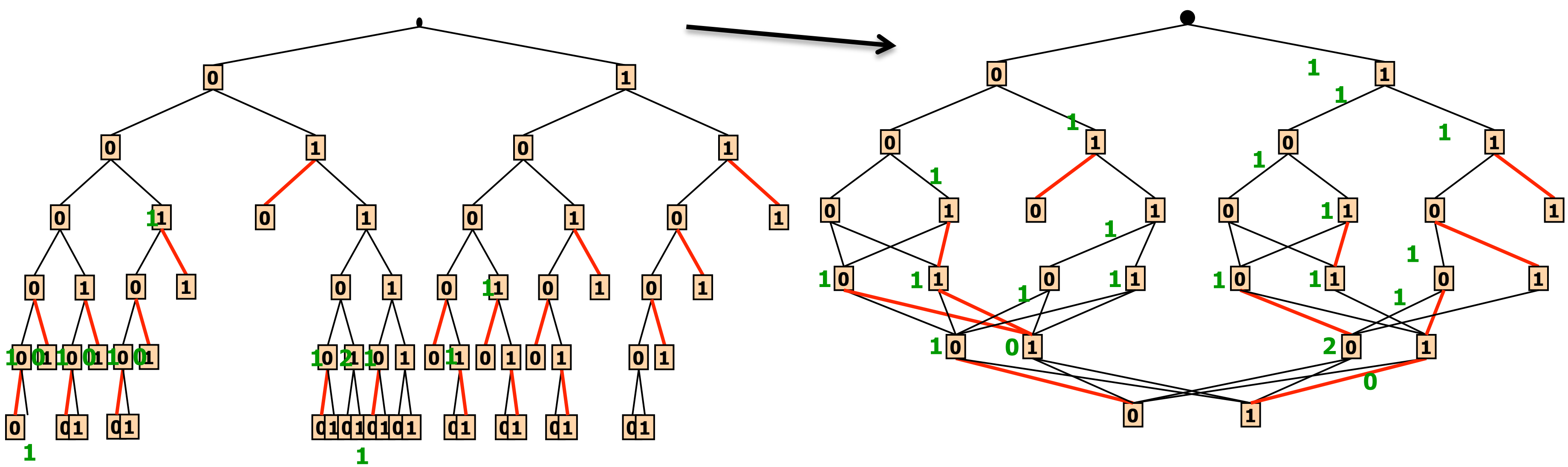
A	B	C	R_{ABC}
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

B	C	D	R_{BCD}
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

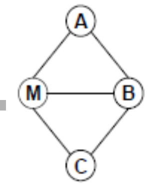
A	B	E	R_{ABE}
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

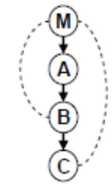
A	E	F	R_{AEF}
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Or search tree
To Or search graph



Weighted AND/OR Search Tree and Context Minimal Graph for Cost Networks





M	A	B	f(M,A,B)
0	0	0	12
0	0	1	5
0	1	0	18
0	1	1	2
1	0	0	4
1	0	1	10
1	1	0	6
1	1	1	4

M	B	C	g(M,B,C)
0	0	0	3
0	0	1	5
0	1	0	14
0	1	1	12
1	0	0	9
1	0	1	15
1	1	0	7
1	1	1	6

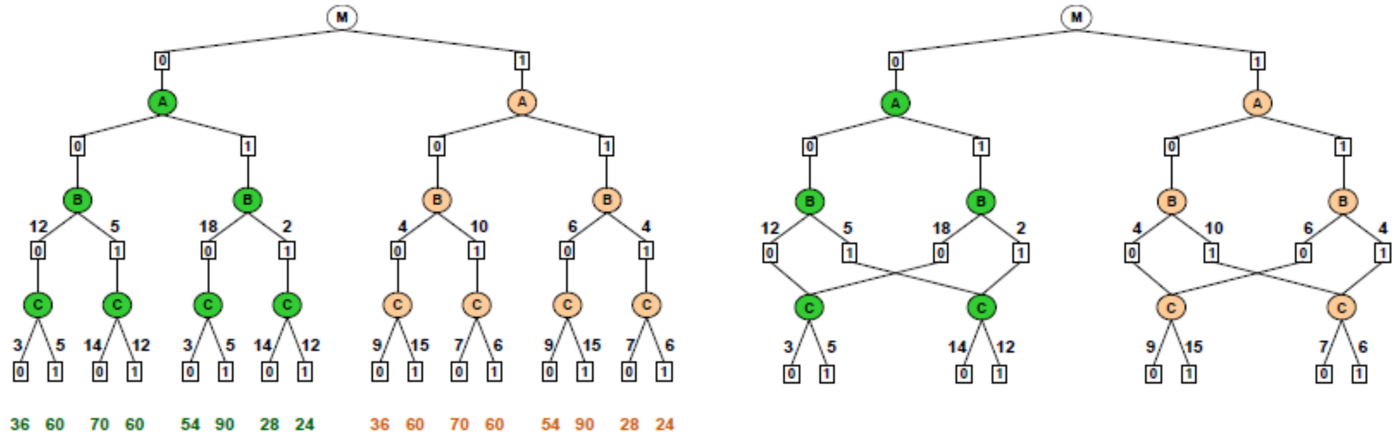


Figure 20: AND/OR search tree and context minimal graph

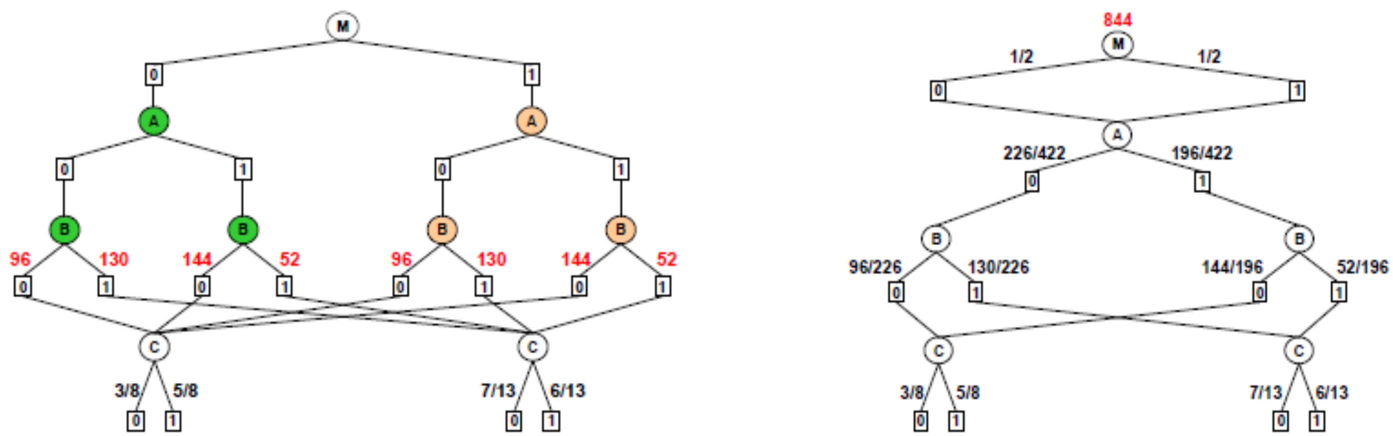


Figure 22: AOMDD for the weighted graph

Weighted AND/OR Tree for Bayesian Network

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

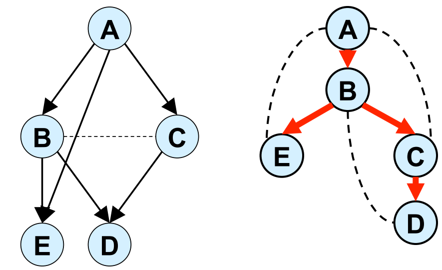
A	B=0	B=1
0	.4	.6
1	.1	.9

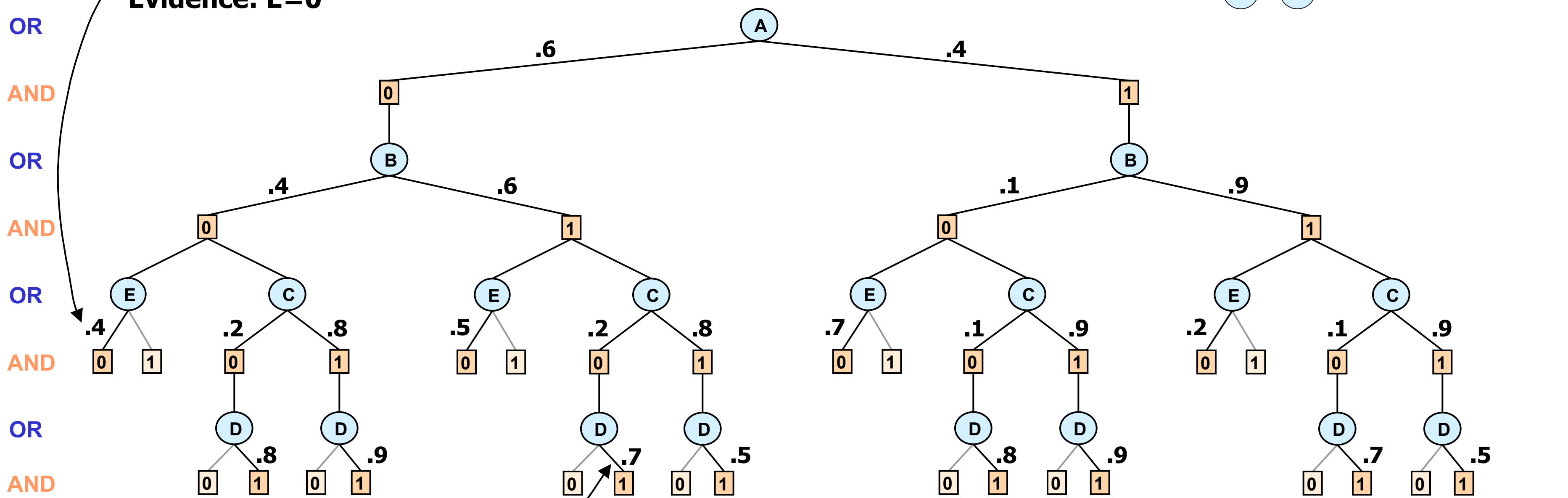
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





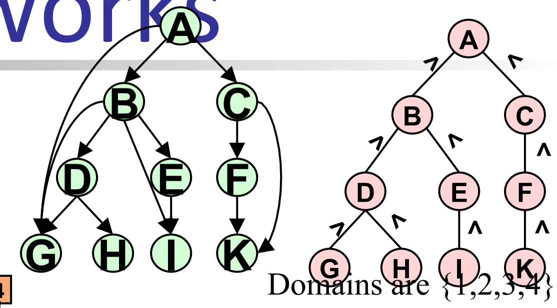
$P(D \mid B,C)$

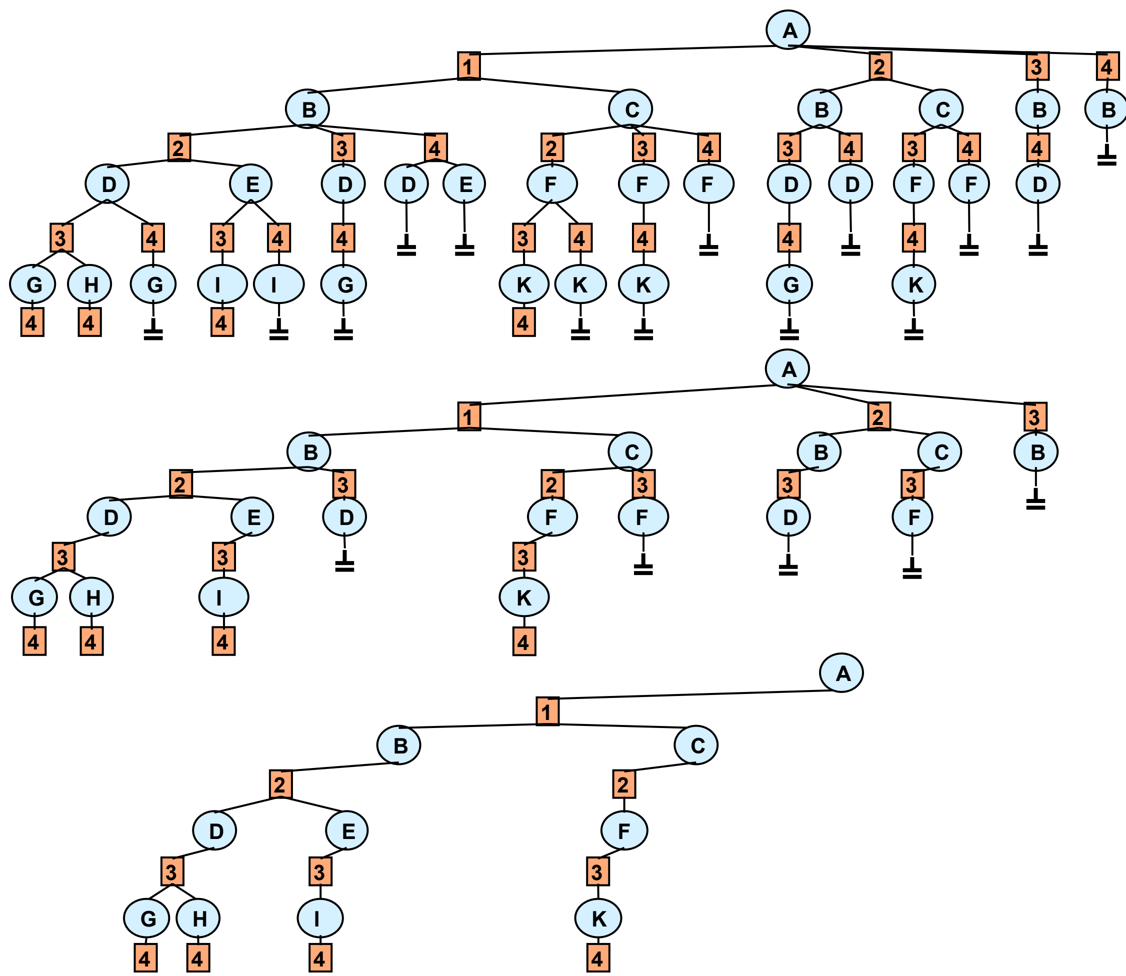
B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

(D=1,E=0)

Representing Mixed Networks





From local
To global

Or making
Global into
local

Outline

- Motivation
- Background in Graphical models
- AND/OR search trees and Graphs
- Minimal AND/OR graphs
- From AND/OR search graphs to AOMDDs
- Compilation of AOMDDs
- AOMDDs and earlier BDDs

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Constraint Networks

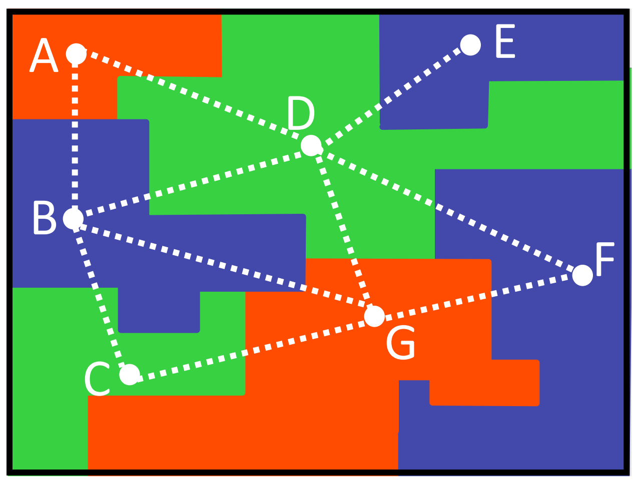
Map coloring

Variables: countries (A B C etc.)

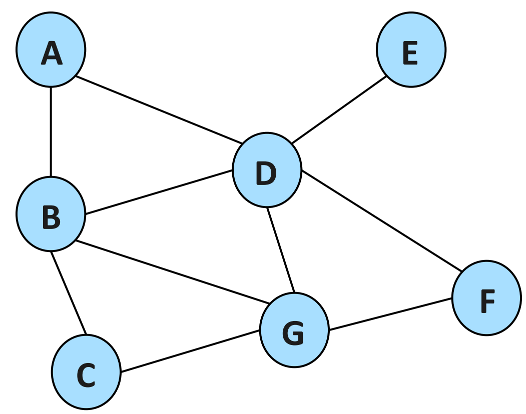
Values: colors (red green blue)

Constraints: **A ≠ B, A ≠ D, D ≠ E**, …

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red

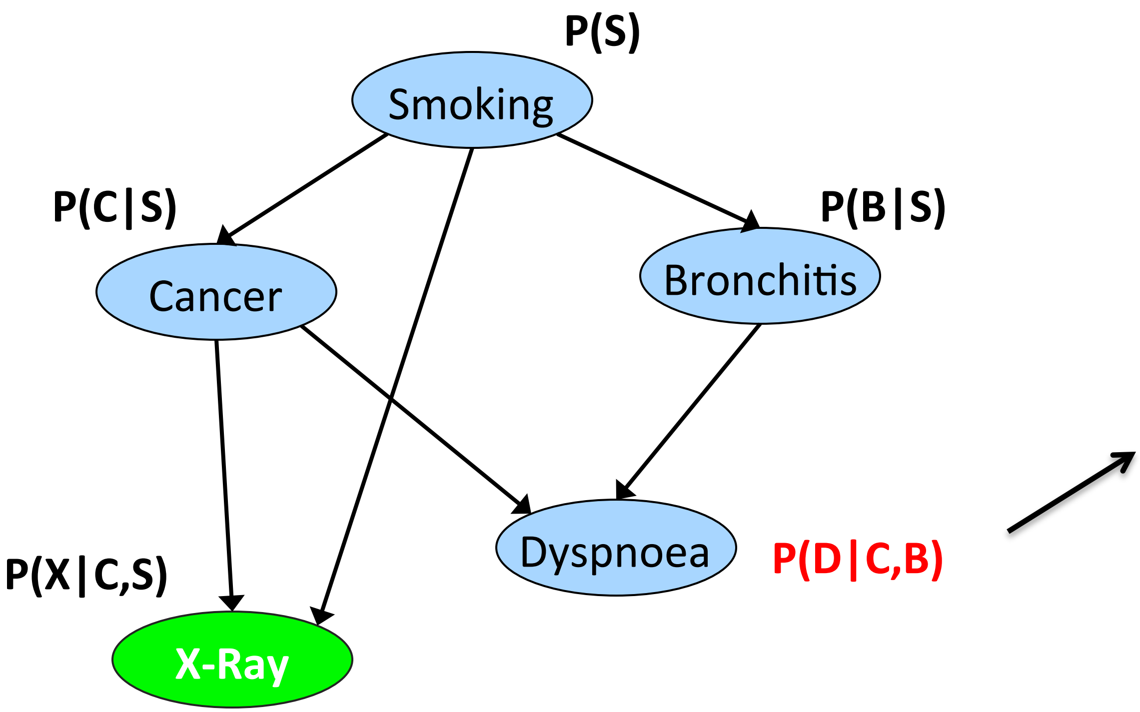


Constraint graph



Bayesian Networks

BN = (X,D,G,P)



P(D|C,B)

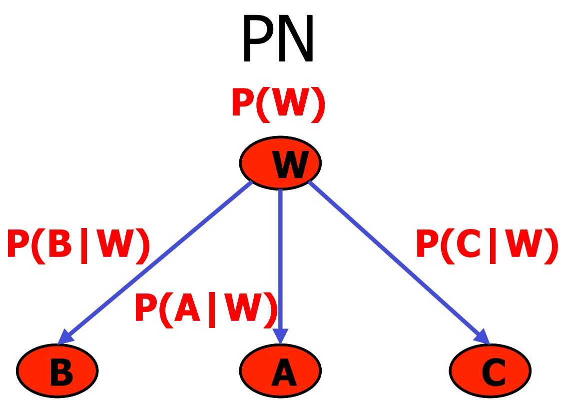
C	B	D=0	D=1
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

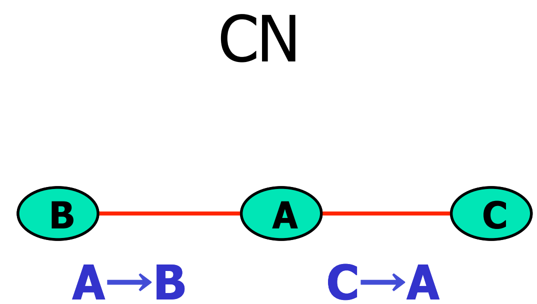
P(S,C,B,X,D) = P(S)· P(C|S)· P(B|S)· P(X|C,S)· P(D|C,B)

MPE = Find a maximum probability assignment, given evidence

MPE = find argmax P(S)· P(C|S)· P(B|S)· P(X|C,S)· P(D|C,B)

Mixed Probabilistic and Deterministic networks





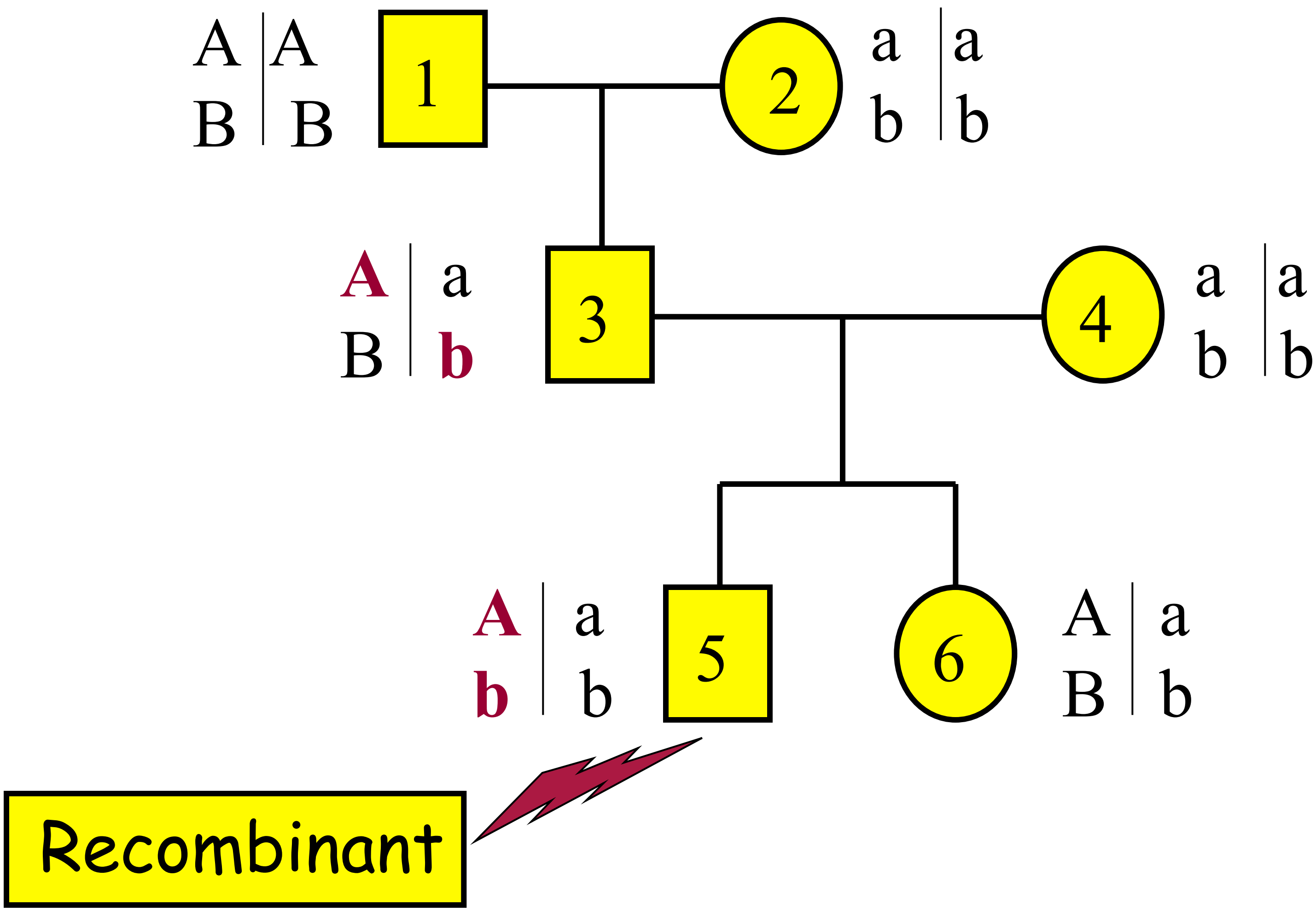
Semantics?

Algorithms?

Query:
Is it likely that Chris goes to the party if Becky does not but the weather is bad?

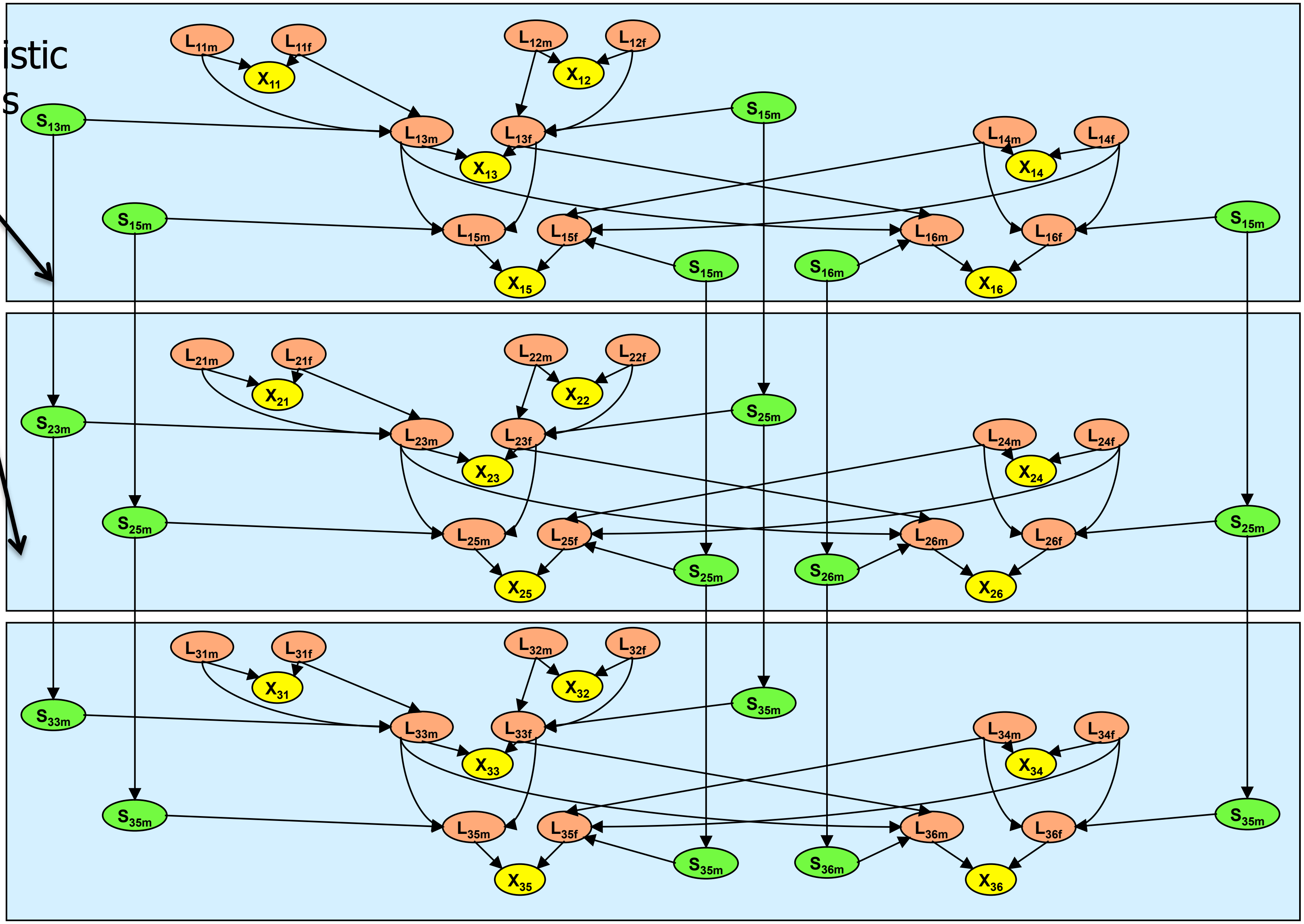
$$P(C, \neg B \mid w = bad, A \rightarrow B, C \rightarrow A)$$

Two Loci Inheritance Pedigree



Networks with Determinism

Probabilistic functions



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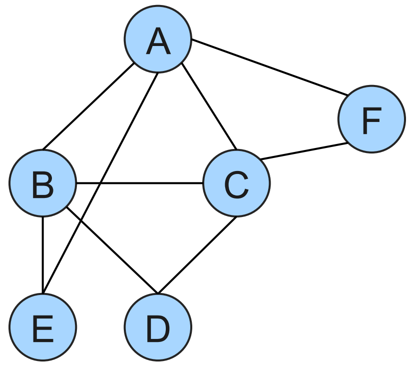
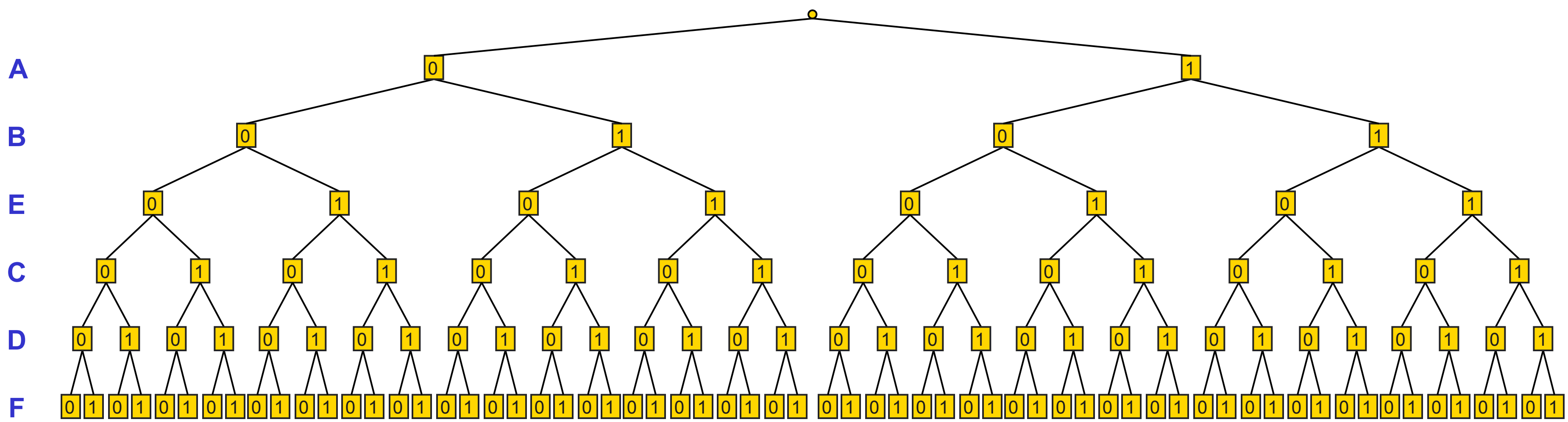
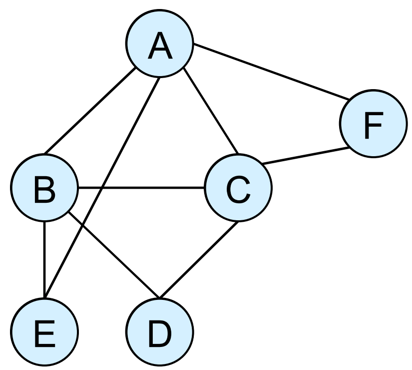
Classic OR Search Space



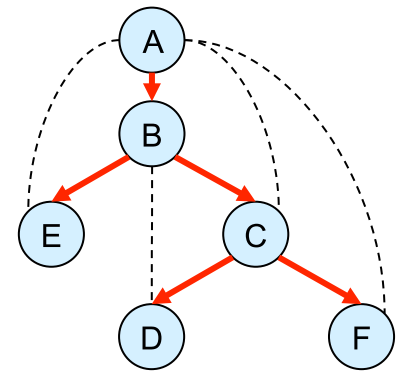
Ordering: A B E C D F



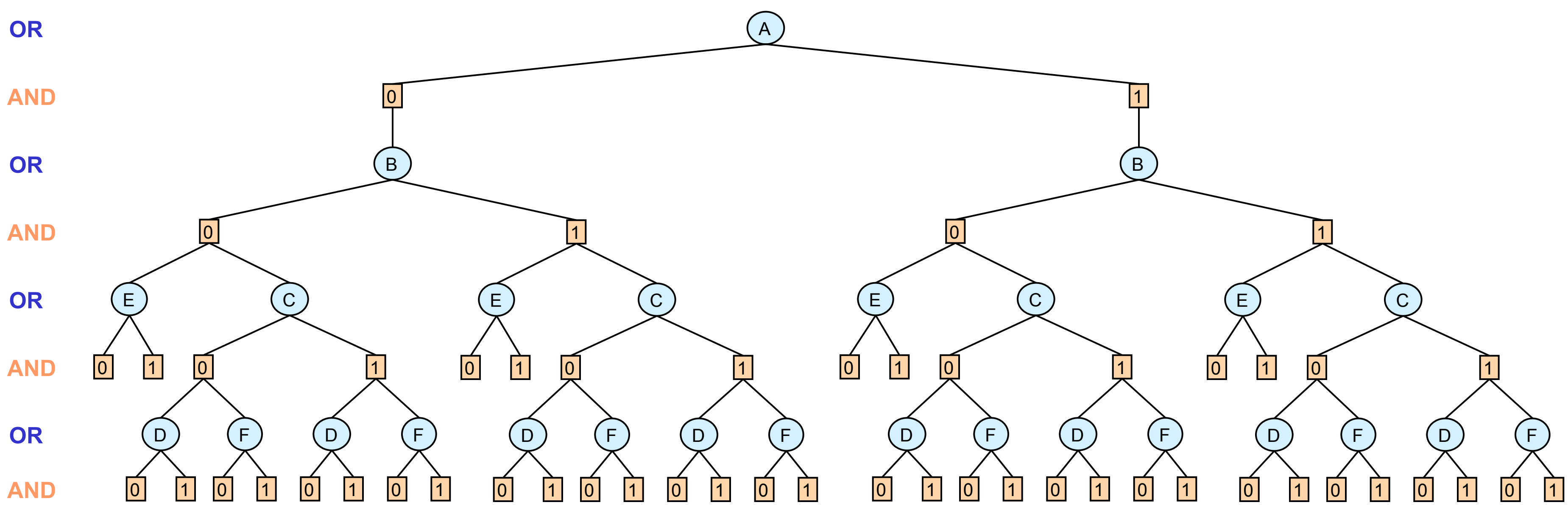
AND/OR Search Space



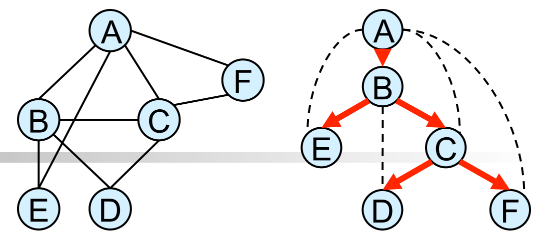
Primal graph

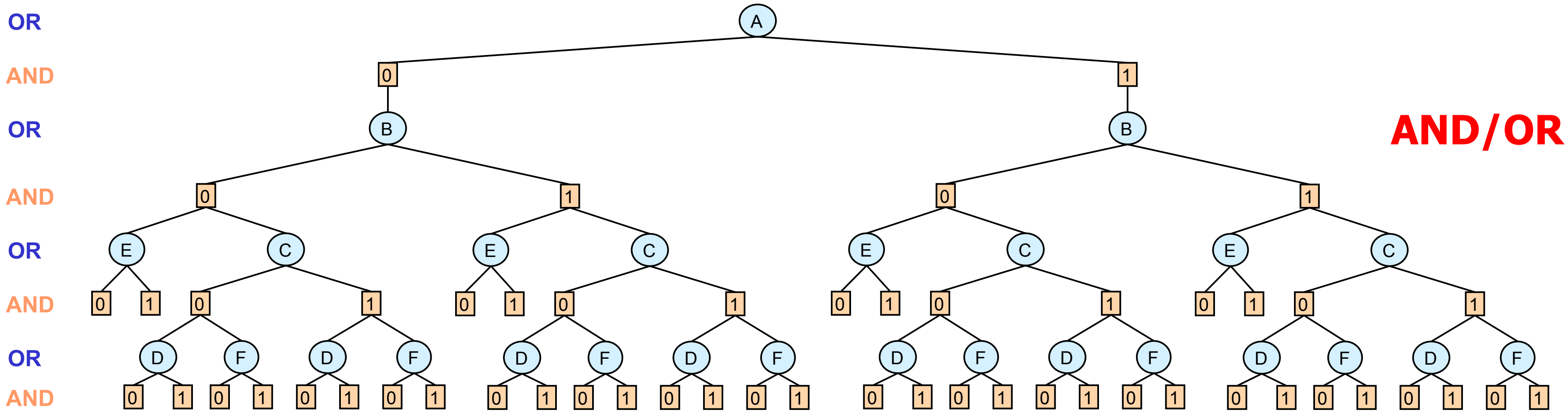


DFS tree

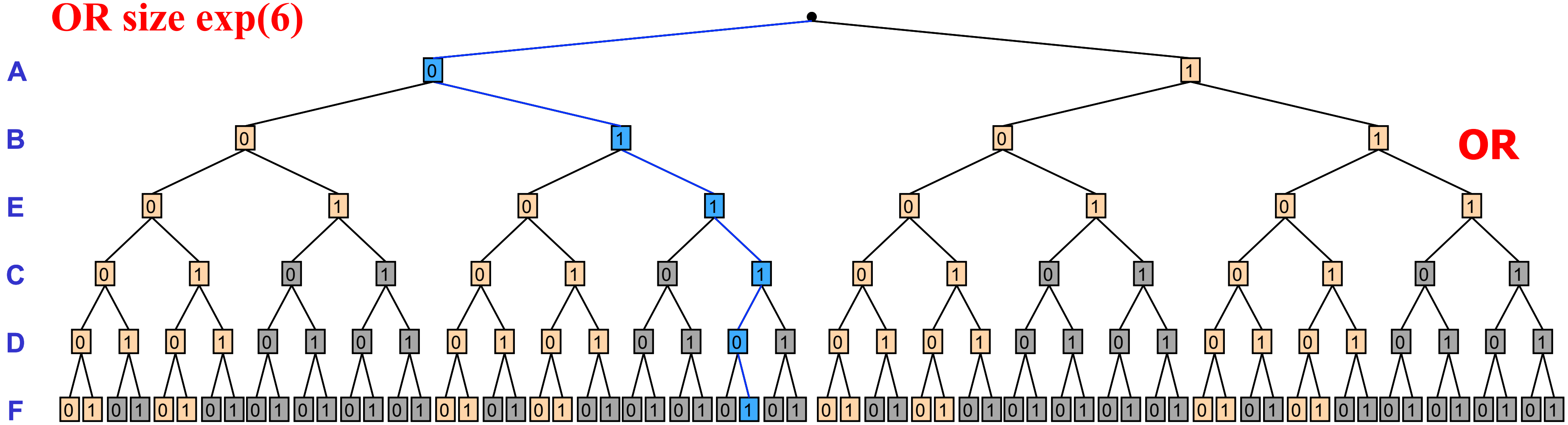


AND/OR vs. OR



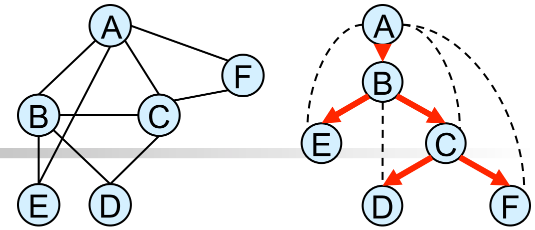


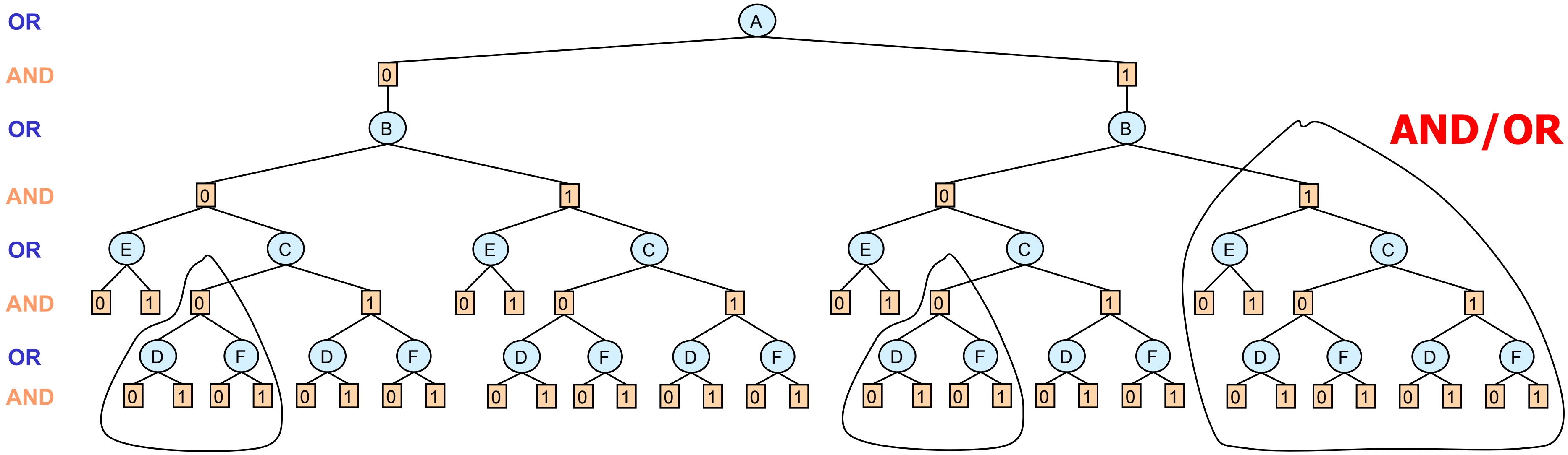
AND/OR size: exp(4),
OR size exp(6)

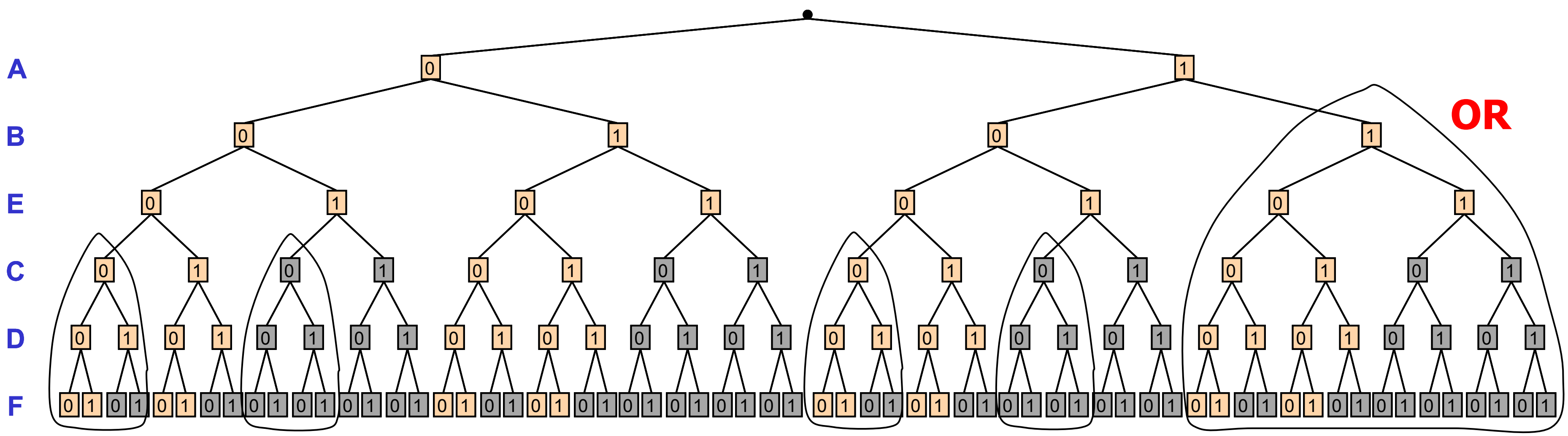


AND/OR vs. OR with Constraints

No-goods
(A=1,B=1)
(B=0,C=0)







AND/OR vs. OR with Constraints

No-goods
(A=1,B=1)
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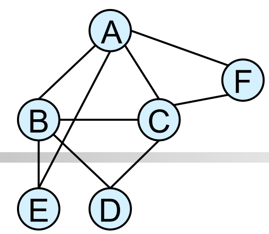
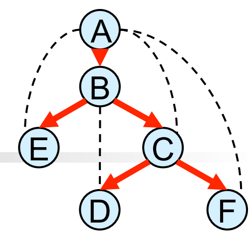
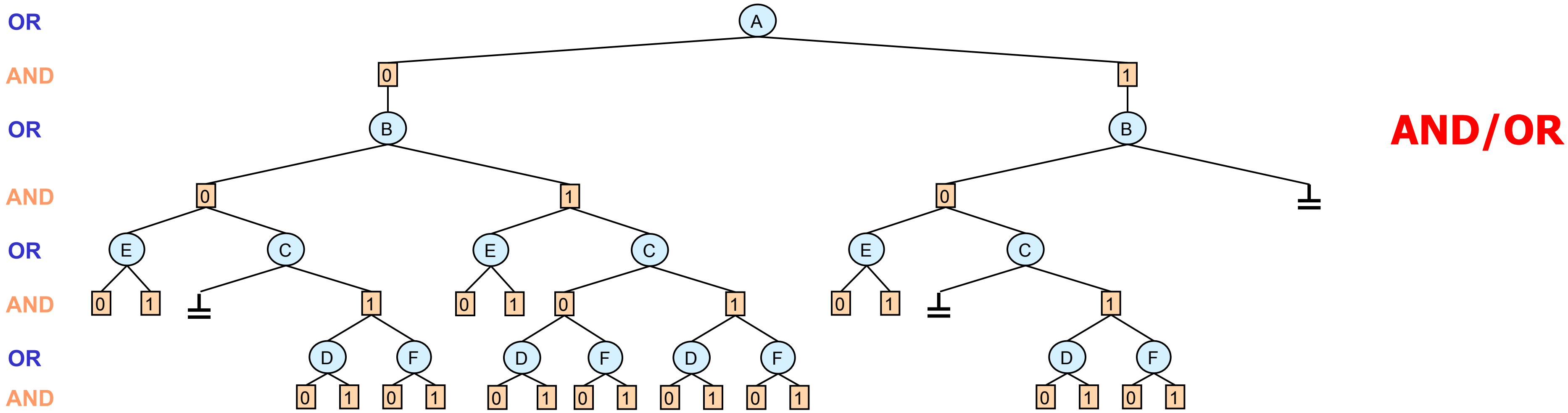
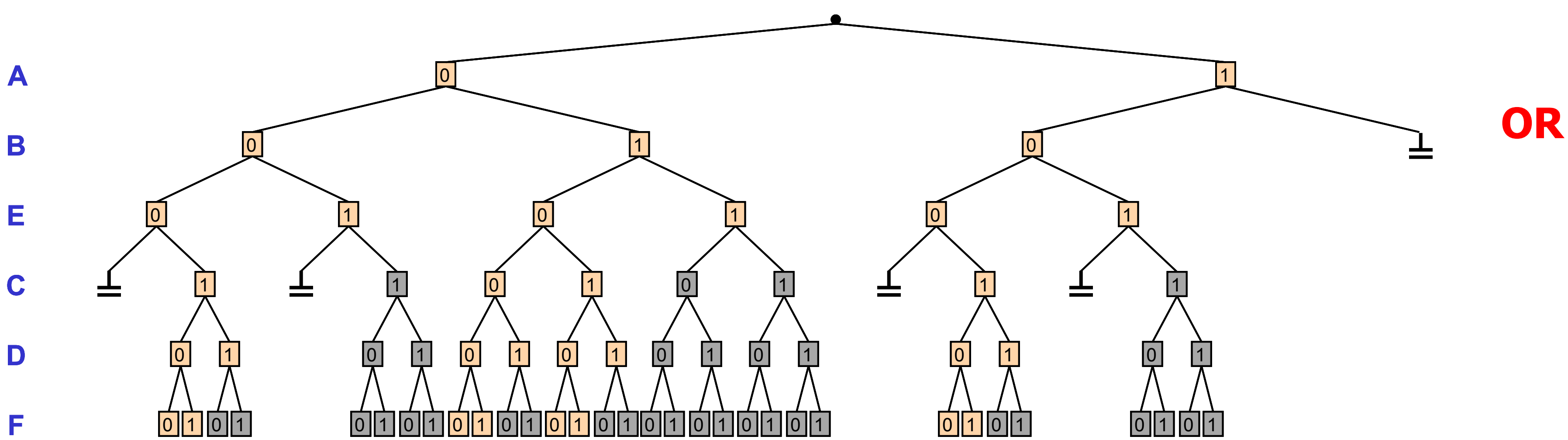
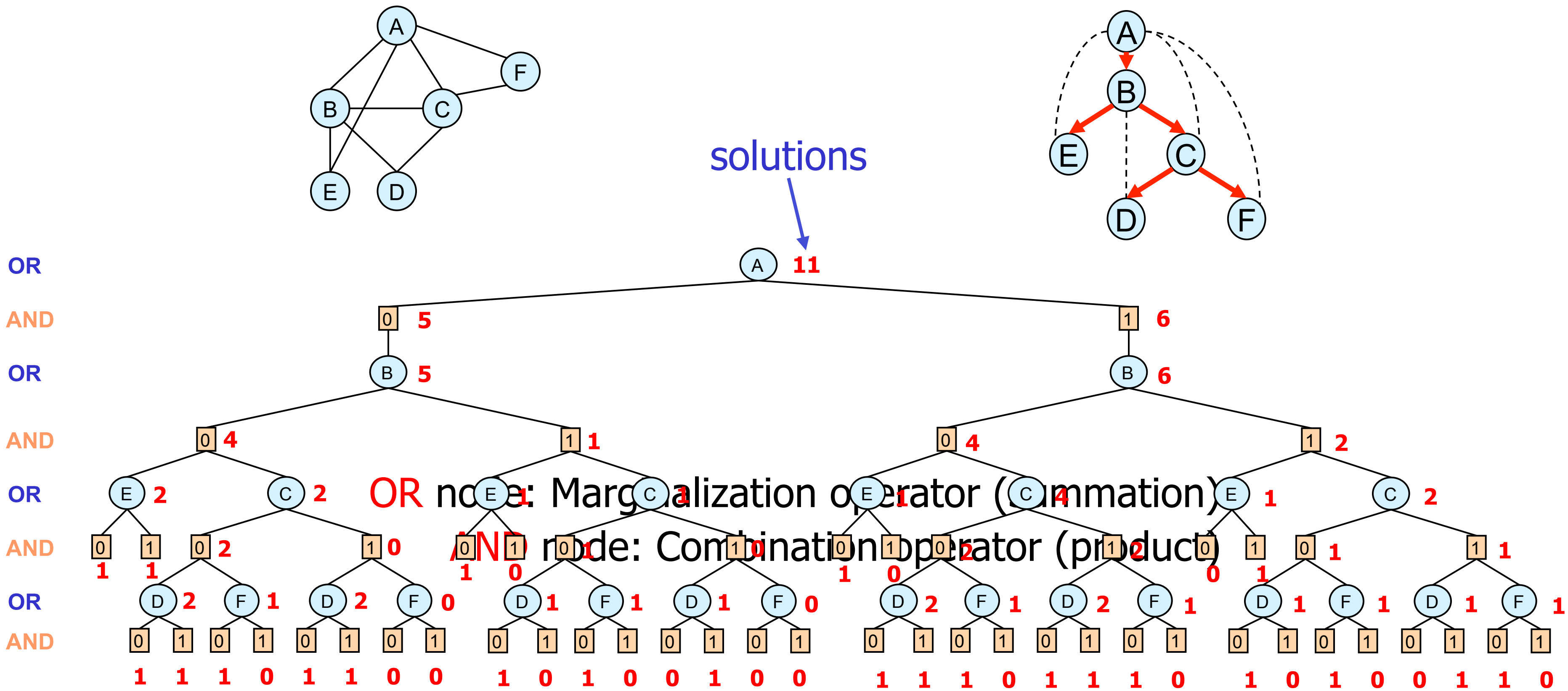








Counting Solutions by DFS traversal (Sum-Product Networks)

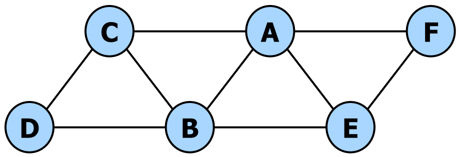


OR node: Marginalization operator (summation)

AND node: Combination operator (product)

Value of node = number of solutions below it

Cost Networks



A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

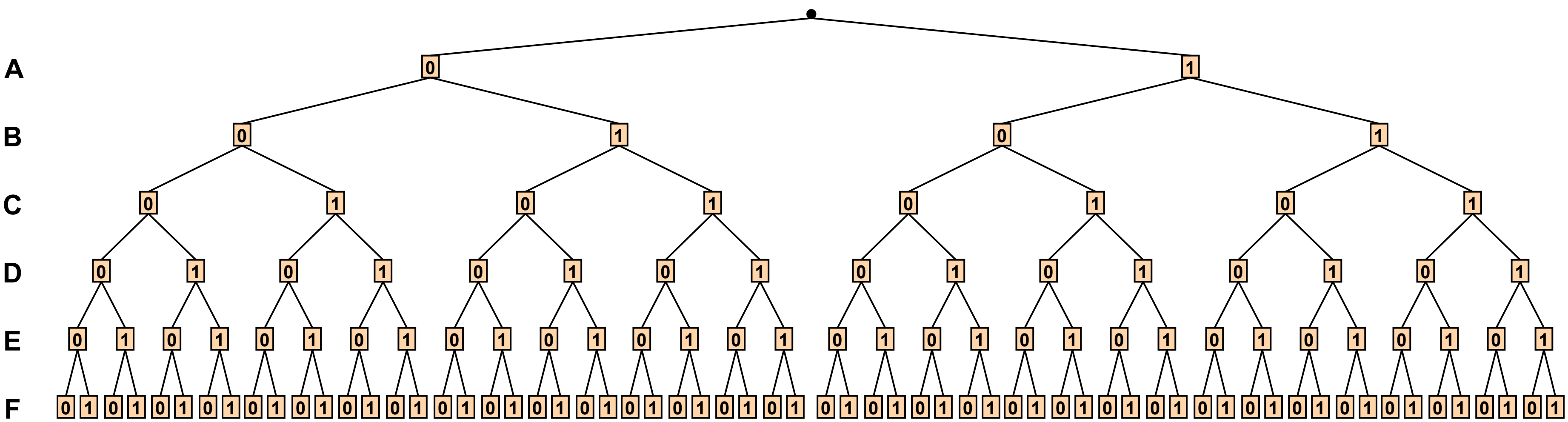
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

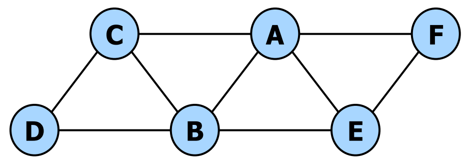
C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

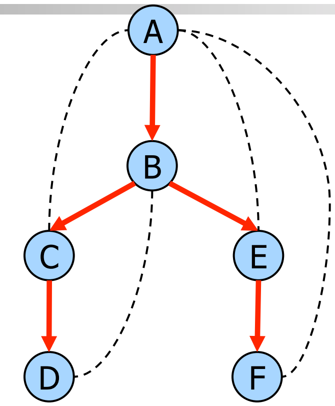
E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \sum_{i=1}^{9} f_i(X)$$

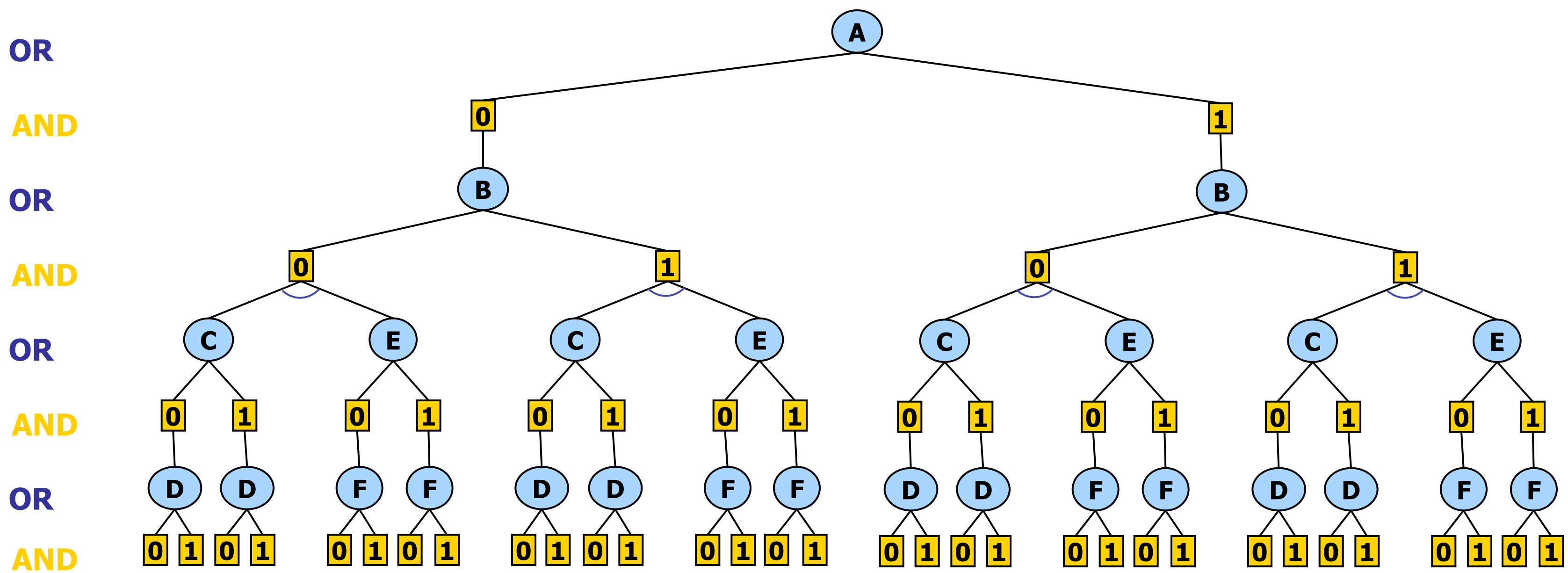


AND/OR Search Tree for a Cost Network



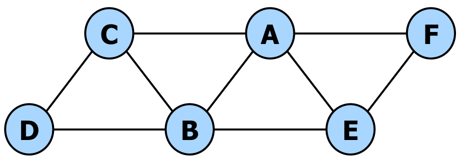


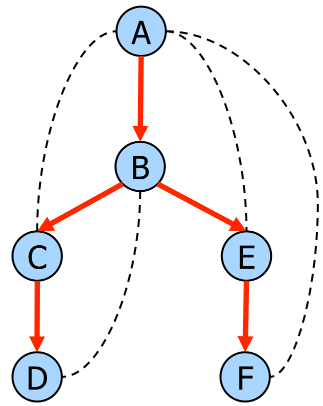
Pseudo tree



A solution subtree is (A=0, B=1, C=0, D=0, E=1, F=1)

Weighted AND/OR Search Tree for a Cost Network





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

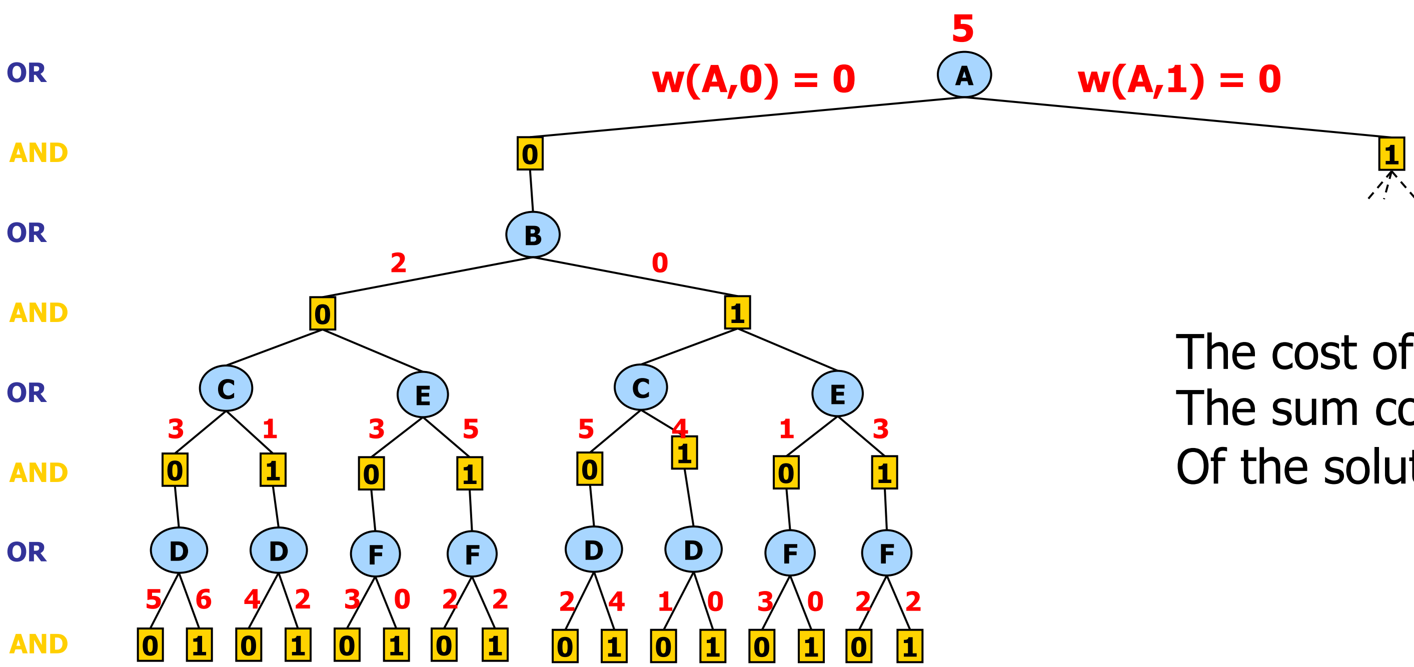
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

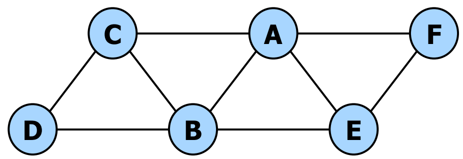
E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

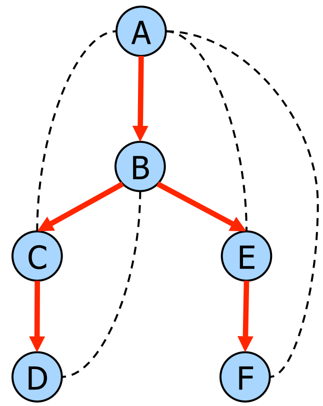
$$f(X) = \sum_{i=1}^{9} f_i(X)$$



The cost of a solution is
The sum cost of weights
Of the solution tree

Optimizing over Weighted AND/OR Tree for a Cost Network





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

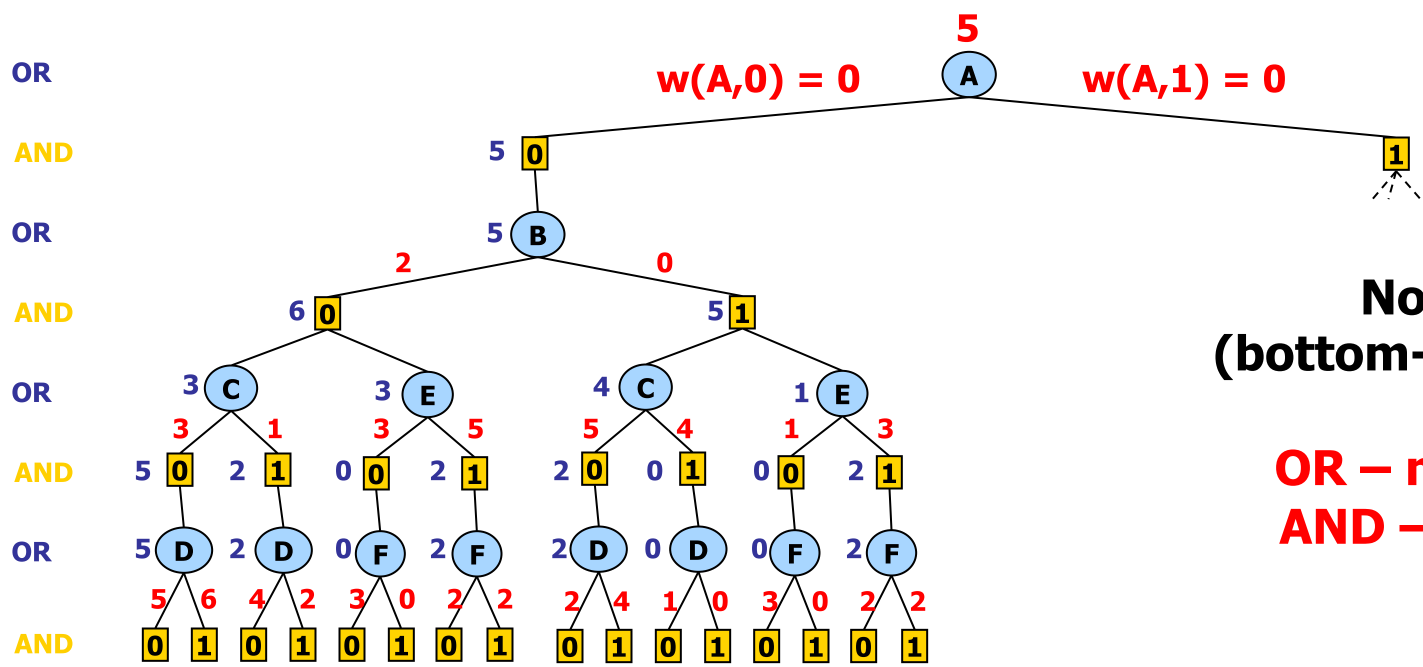
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \sum_{i=1}^{9} f_i(X)$$



Node Value
(bottom-up evaluation)

OR – minimization
AND – summation

Weighted AND/OR Tree for Bayesian Network

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

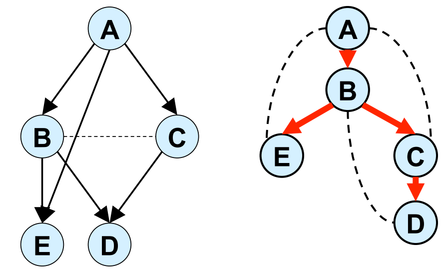
A	B=0	B=1
0	.4	.6
1	.1	.9

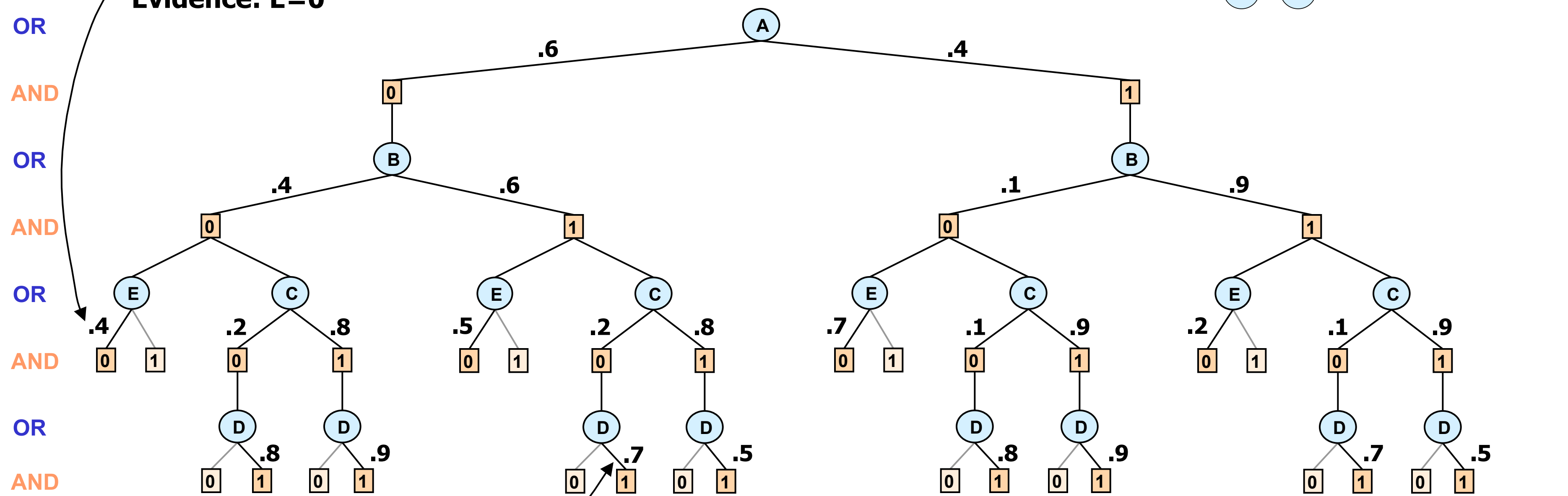
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

Weighted AND/OR Tree for Bayesian Network (Sum-Product Networks)

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

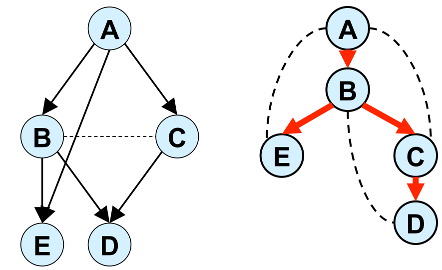
A	B=0	B=1
0	.4	.6
1	.1	.9

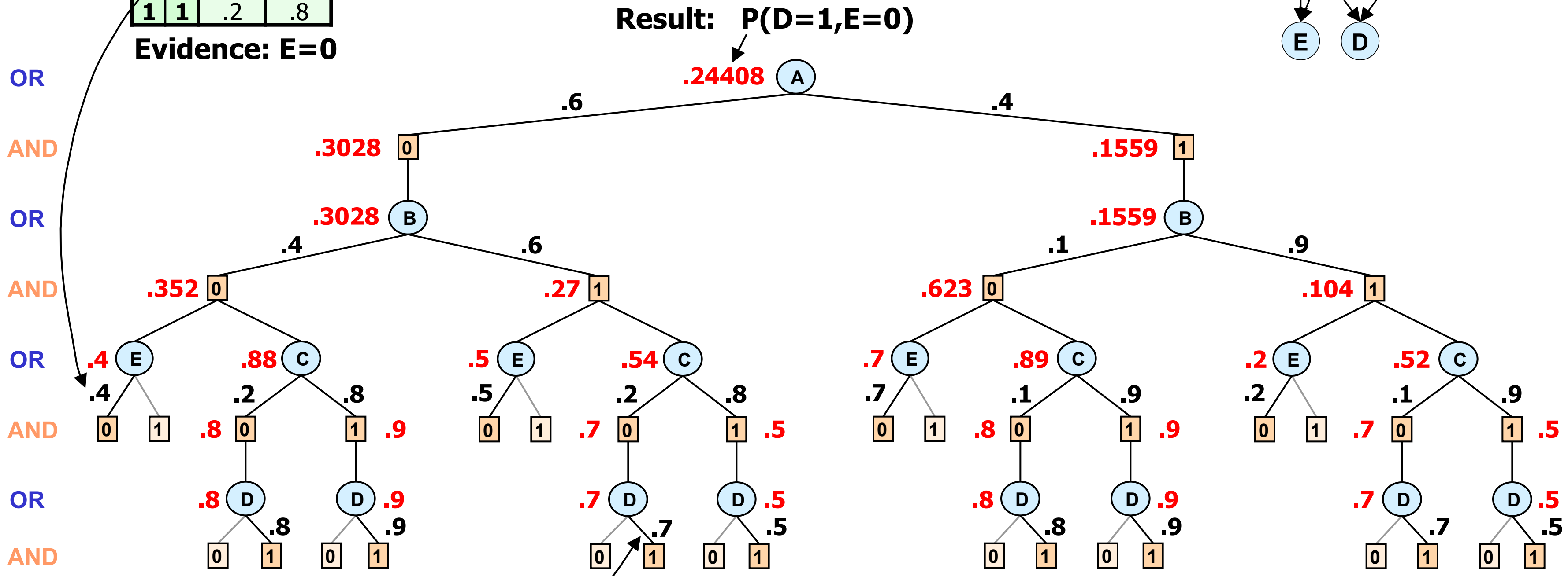
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

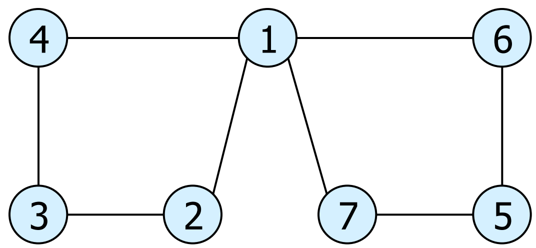
OR node: Marginalization operator (summation)

AND node: Combination operator (product)

Value of node = updated belief for sub-problem below

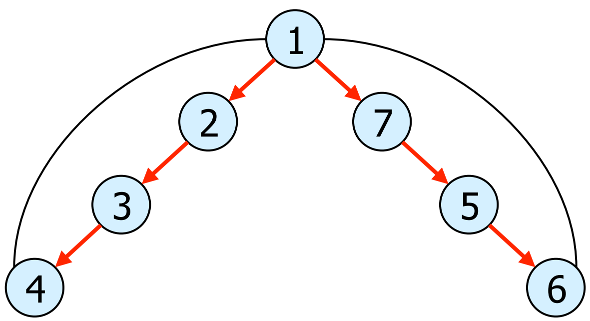
Pseudo-Trees

(Freuder 85, Bayardo 95, Bodlaender and Gilbert, 91)

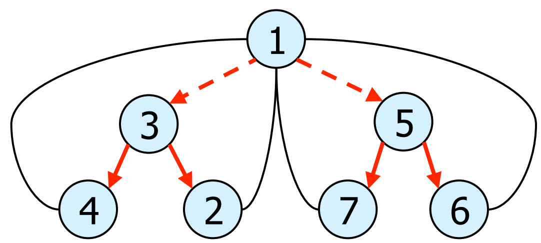


(a) Graph

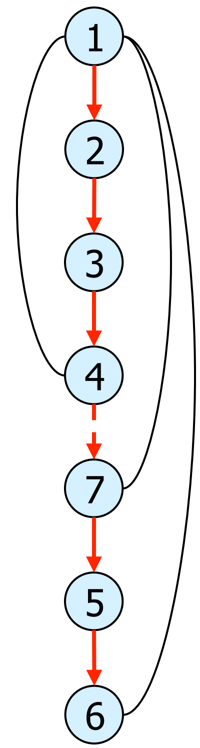
$$h \le w^* \log n$$



(b) DFS tree depth=3



(c) pseudo- tree depth=2



(d) Chain depth=6

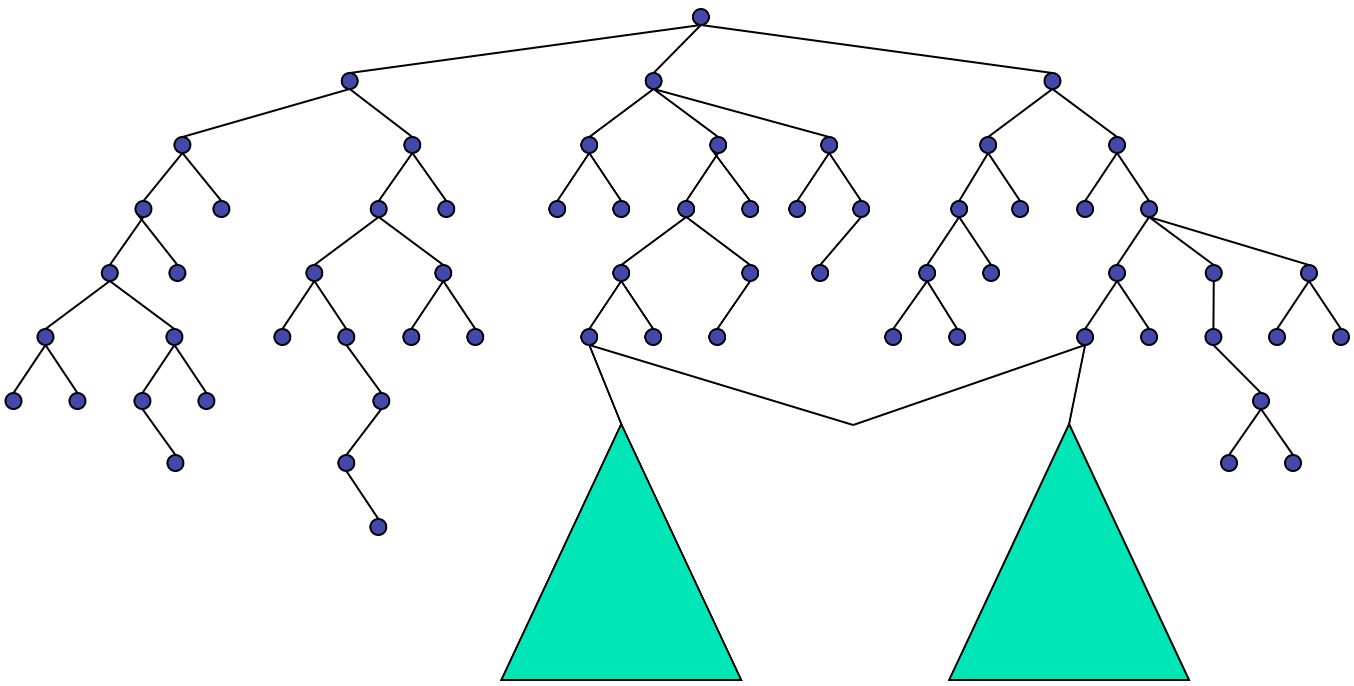
Complexity of AND/OR Tree Search

	AND/OR tree	**OR tree**
Space	O(n)	O(n)
Time	$O(n\ k^h)$ $O(n\ k^{w^* \log n})$ (Freuder & Quinn85), (Collin, Dechter & Katz91), (Bayardo & Miranker95), (Darwiche01)	$O(k^n)$

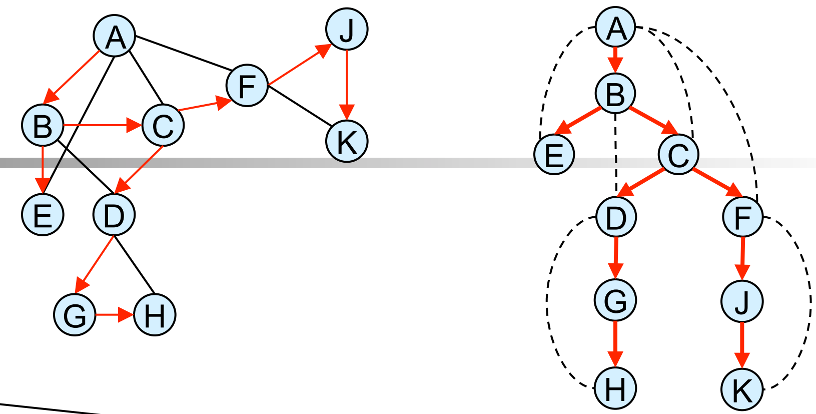
k = domain size
h = depth of pseudo-tree
n = number of variables
w*= treewidth

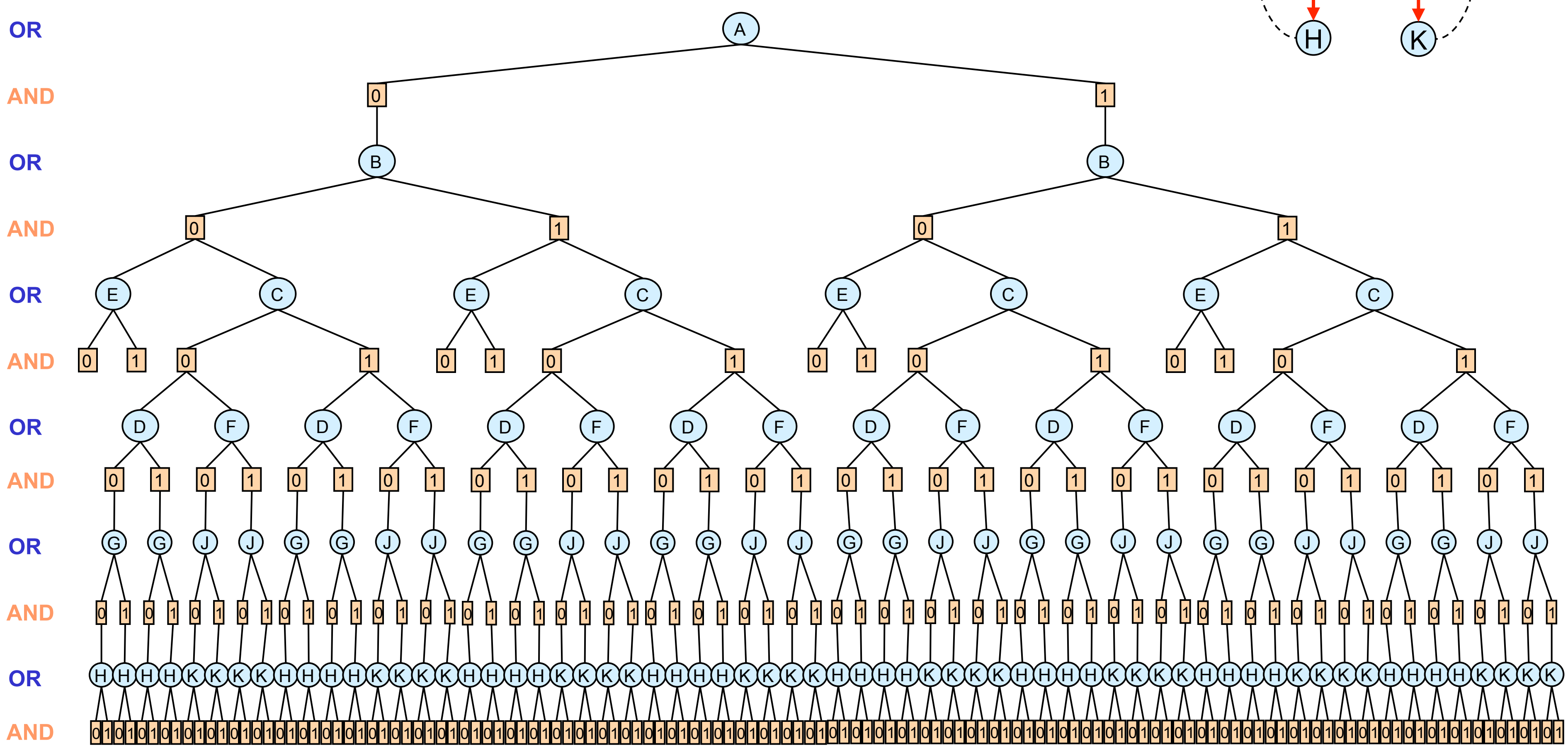
From Search Trees to Search Graphs

- Any two nodes that root identical subtrees (subgraphs) can be merged

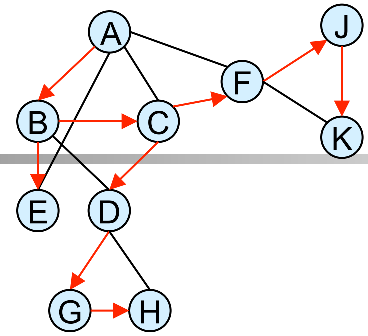


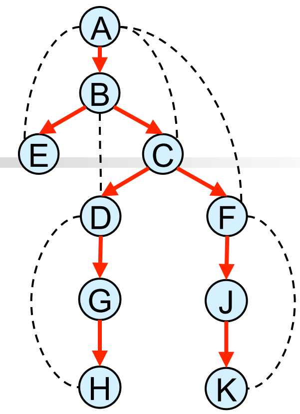
From AND/OR Tree

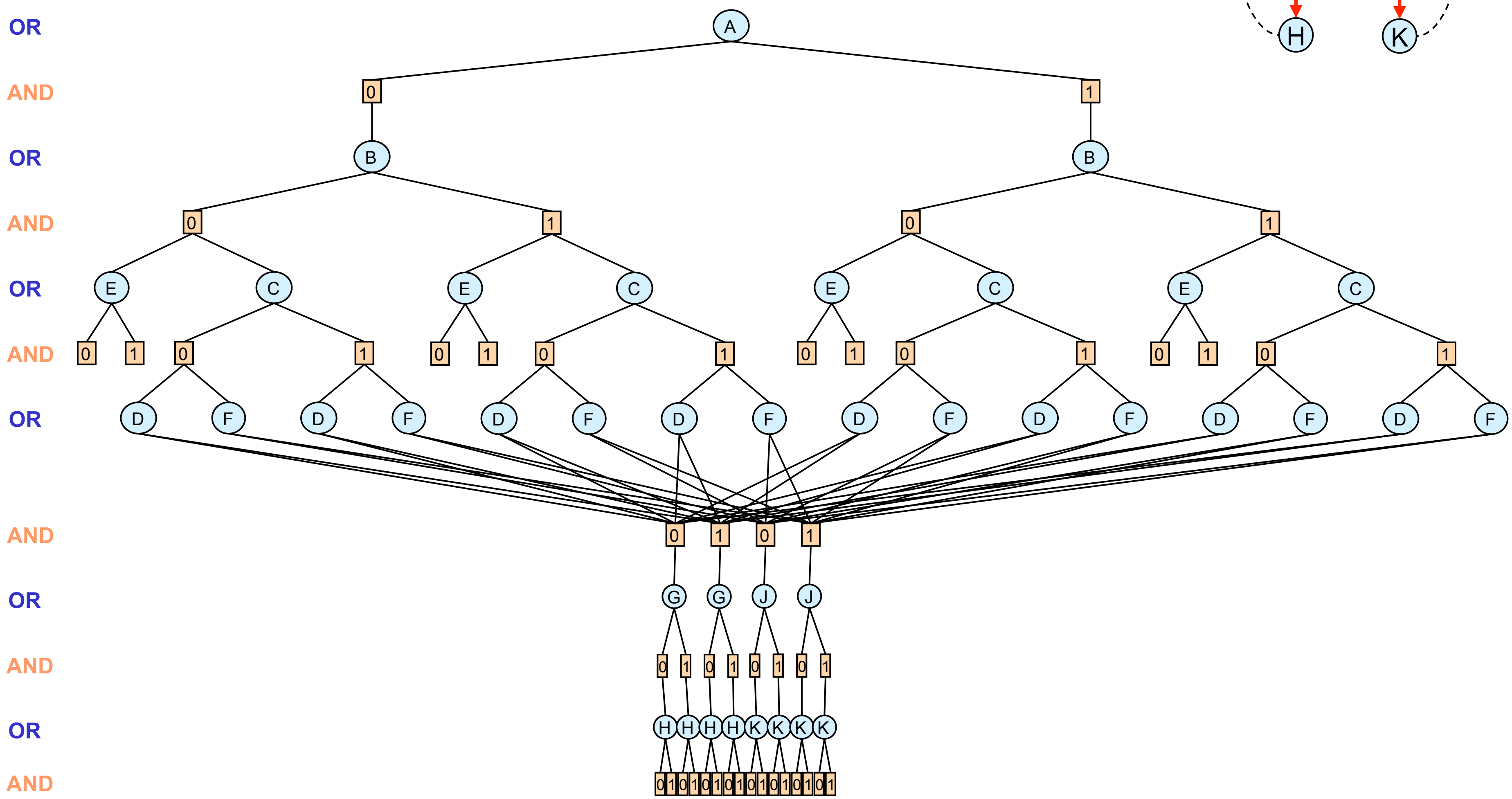




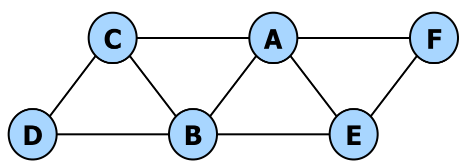
To an AND/OR Graph

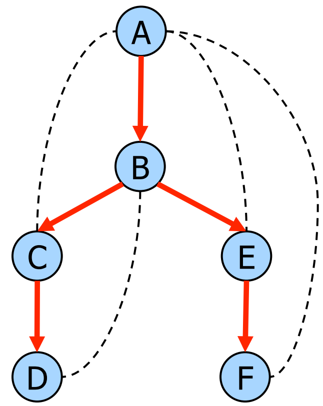






Weighted AND/OR Search Tree for a Cost Network





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

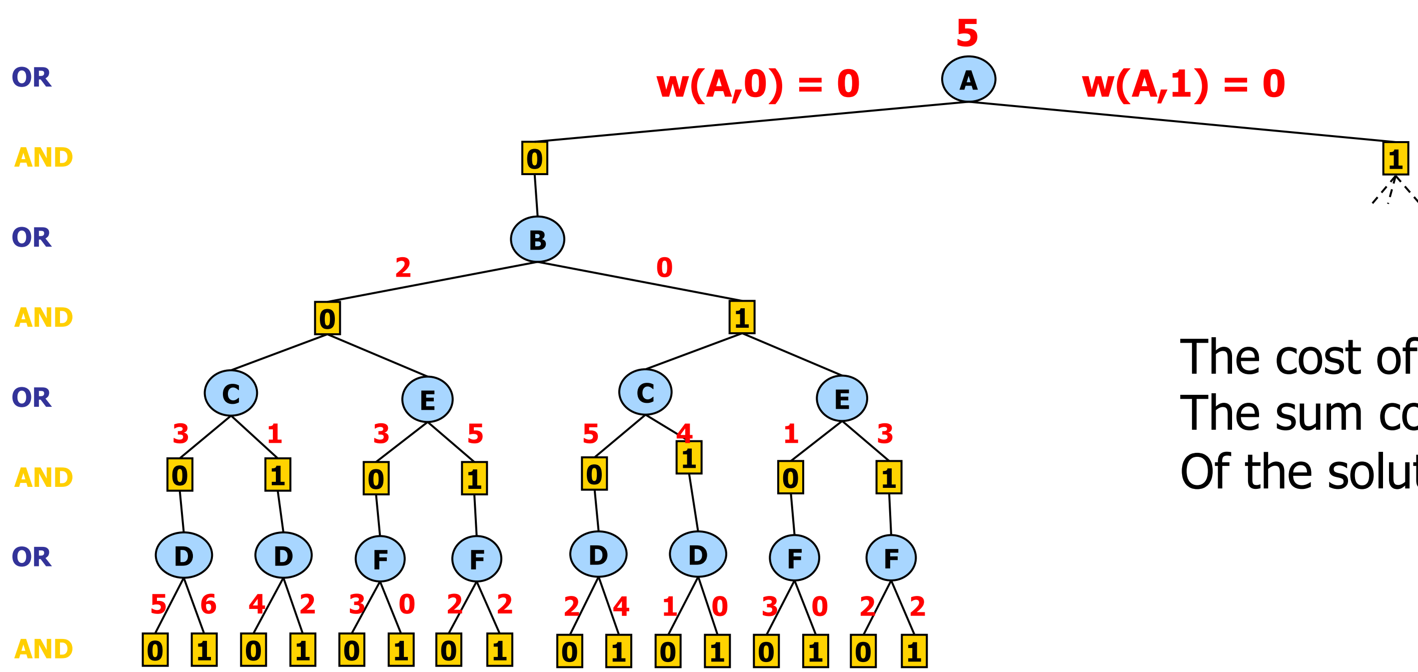
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

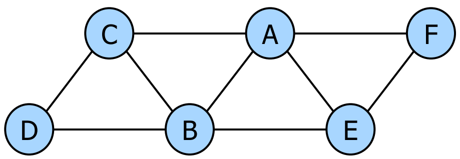
E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

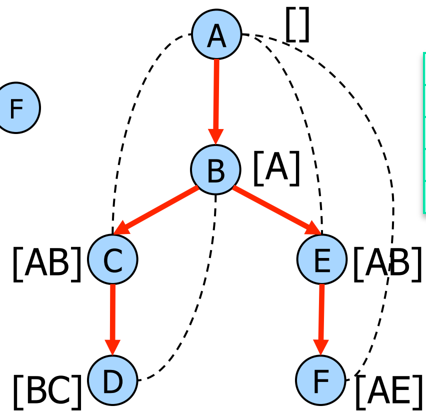
$$f(X) = \sum_{i=1}^{9} f_i(X)$$



The cost of a solution is
The sum cost of weights
Of the solution tree

A Network Weighted AND/OR Search Graph





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

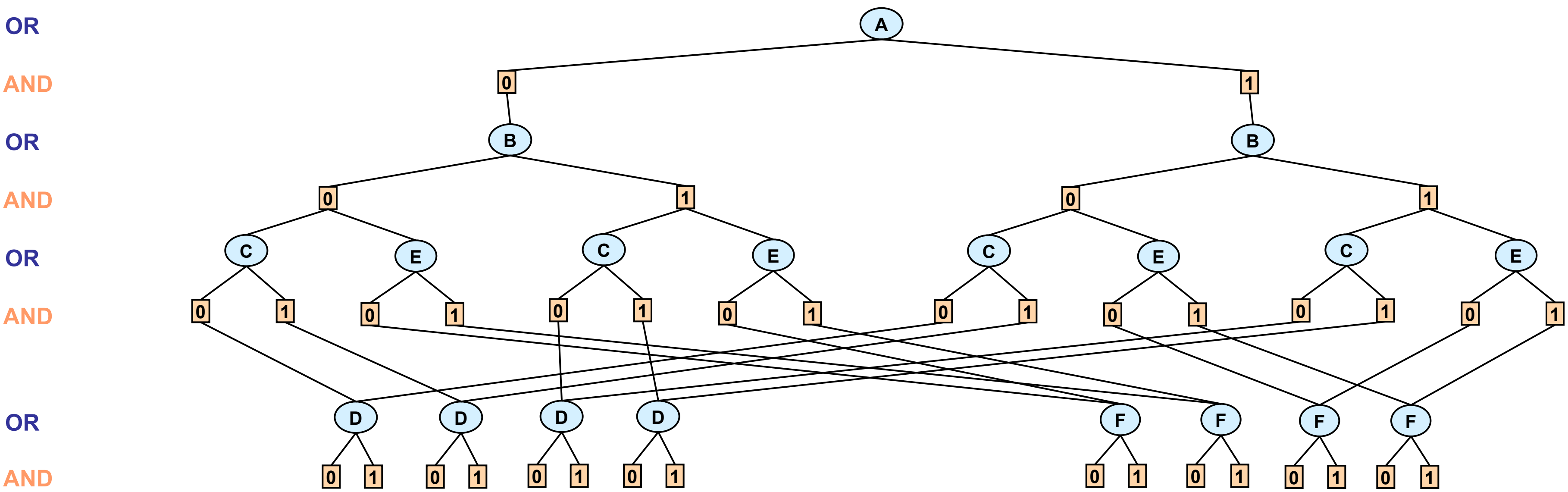
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \sum_{i=1}^{9} f_i(X)$$



context minimal graph

A Bayesian Network AND/OR Search Tree

$P(E \mid A, B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

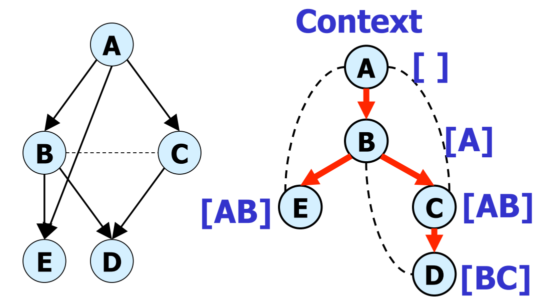
A	B=0	B=1
0	.4	.6
1	.1	.9

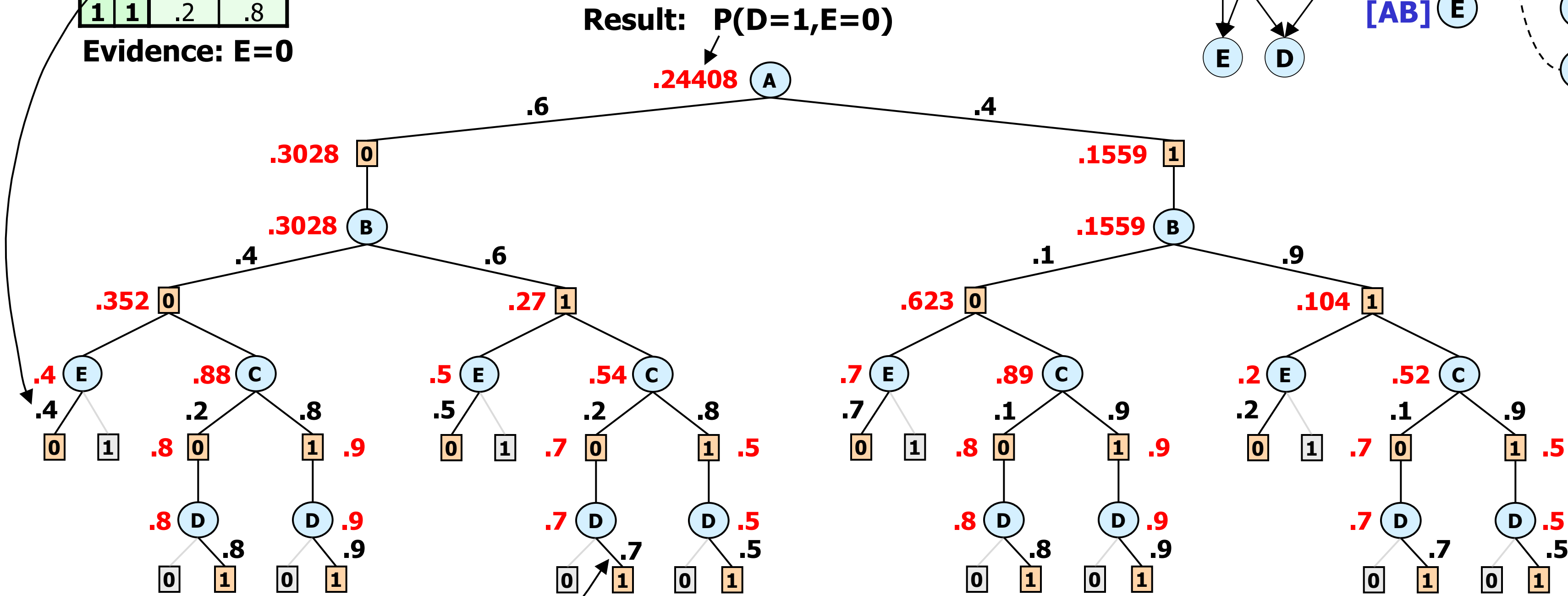
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





$P(D \mid B, C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

OR node: Marginalization operator (summation)

AND node: Combination operator (product)

Value of node = updated belief for sub-problem below

A Bayesian Network AND/OR Search Graph

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

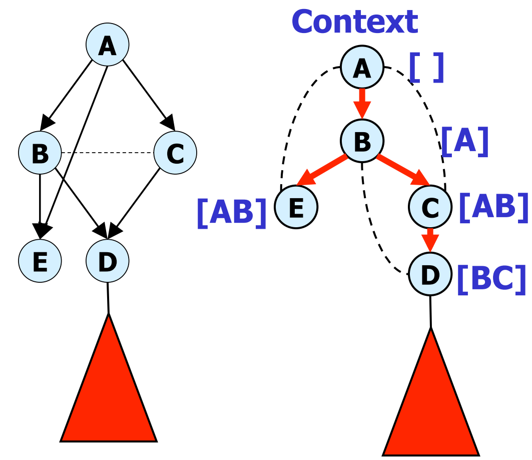
A	B=0	B=1
0	.4	.6
1	.1	.9

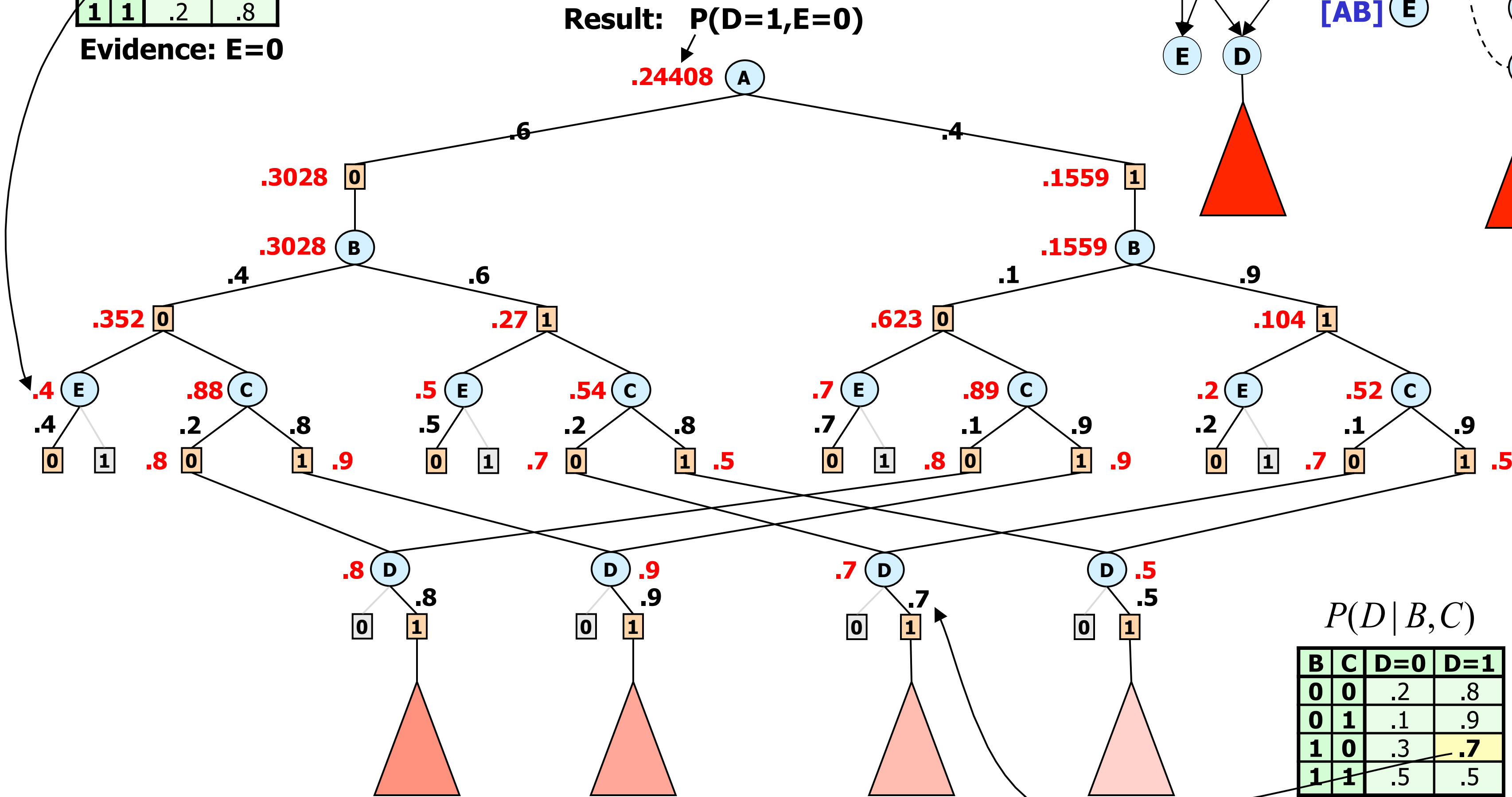
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4





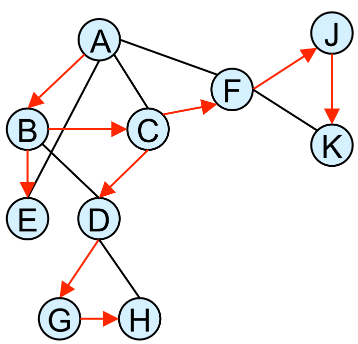
$P(D \mid B,C)$

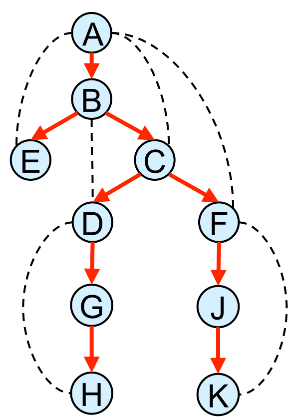
B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

AND/OR Context Minimal Graph

- Caching is possible when context is the same
- context = parent-separator set in induced pseudo-graph
  = current variable +
  ancestors connected to subtree below





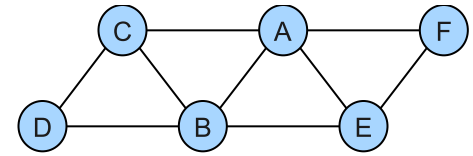
context(B) = {A, B}

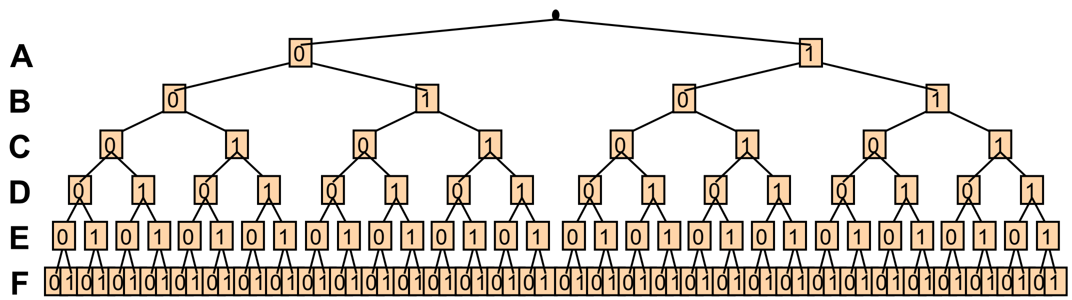
context(C) = {A,B,C}

context(D) = {D}

context(F) = {F}

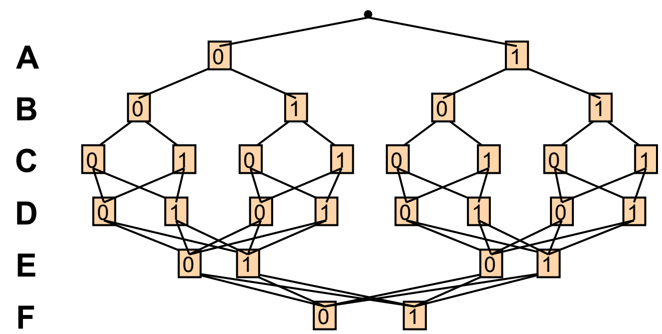
All Four Search Spaces





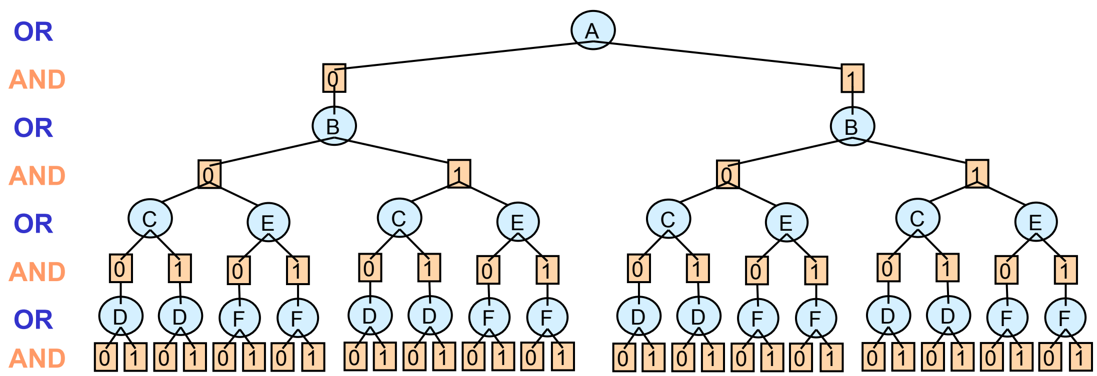
Full OR search tree

126 nodes



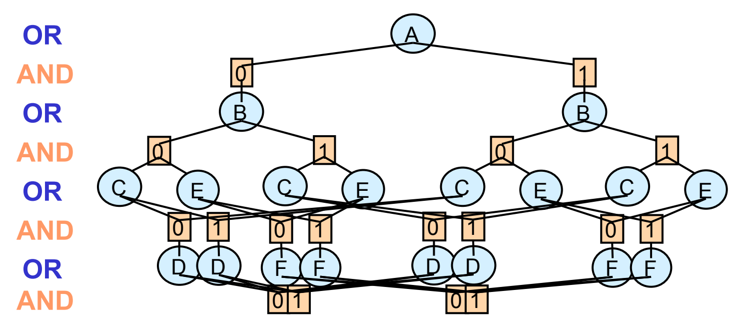
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes

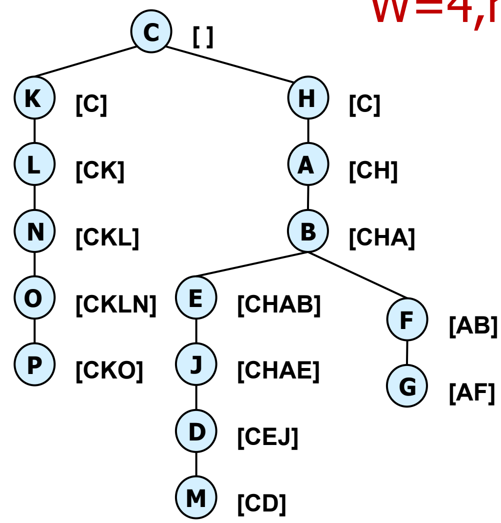


Context minimal AND/OR search graph

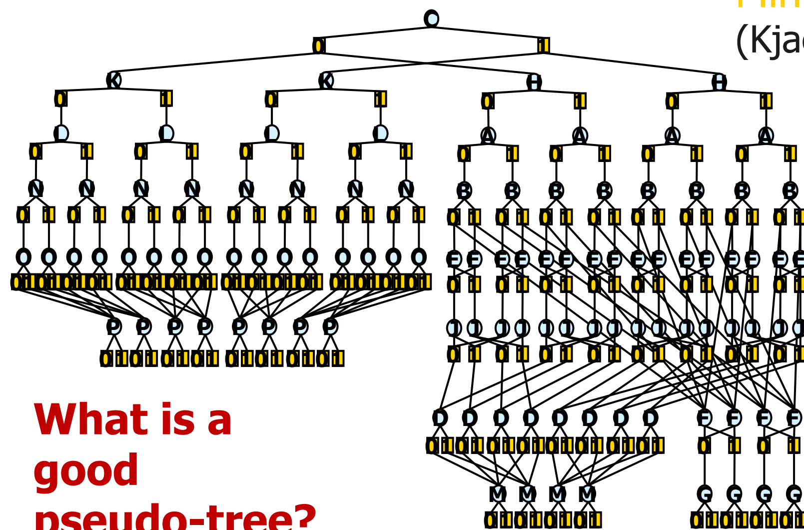
18 AND nodes

Two AND/OR Context-Minimal Graphs

W=4,h=8

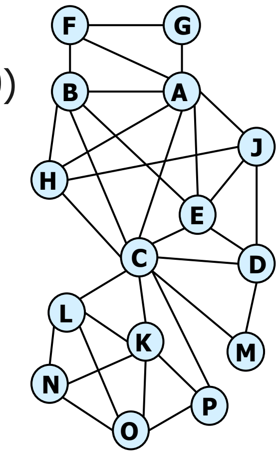


(C K H A B E J L N O D P M F G)



Min-Fill

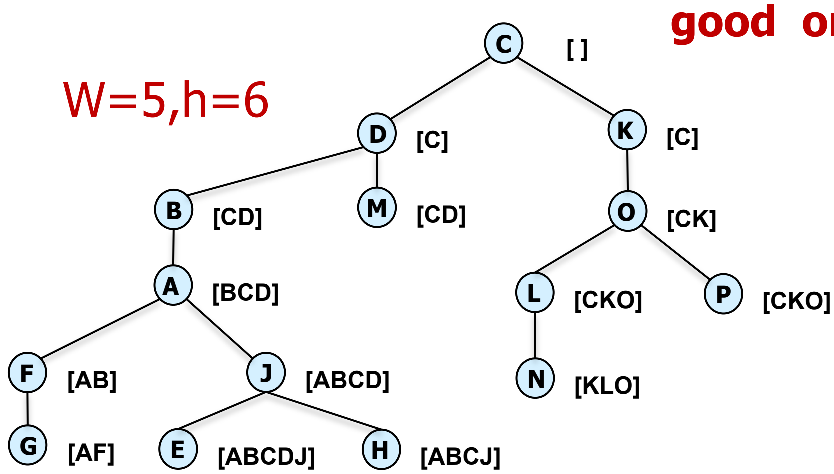
(Kjaerulff90)

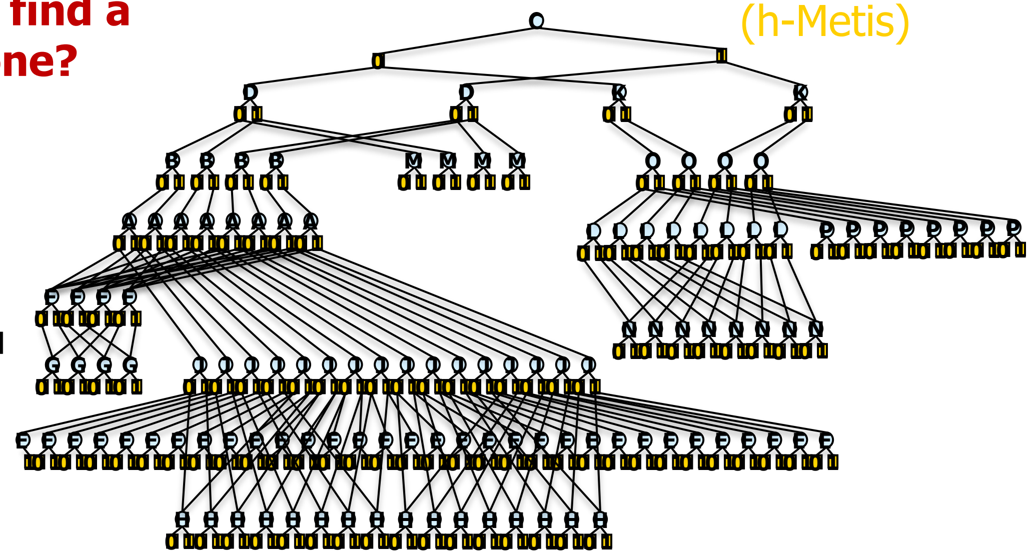


What is a good pseudo-tree? How to find a good one?

Hypergraph Partitioning (h-Metis)

W=5,h=6





HUJI 2012

(C D K B A O M L N P J H E F G)

Complexity of AND/OR Graph Search

	AND/OR graph	OR graph
Space	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$
Time	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$

k = domain size
n = number of variables
w^* = treewidth
pw^* = pathwidth

$w^* \le pw^* \le w^* \log n$

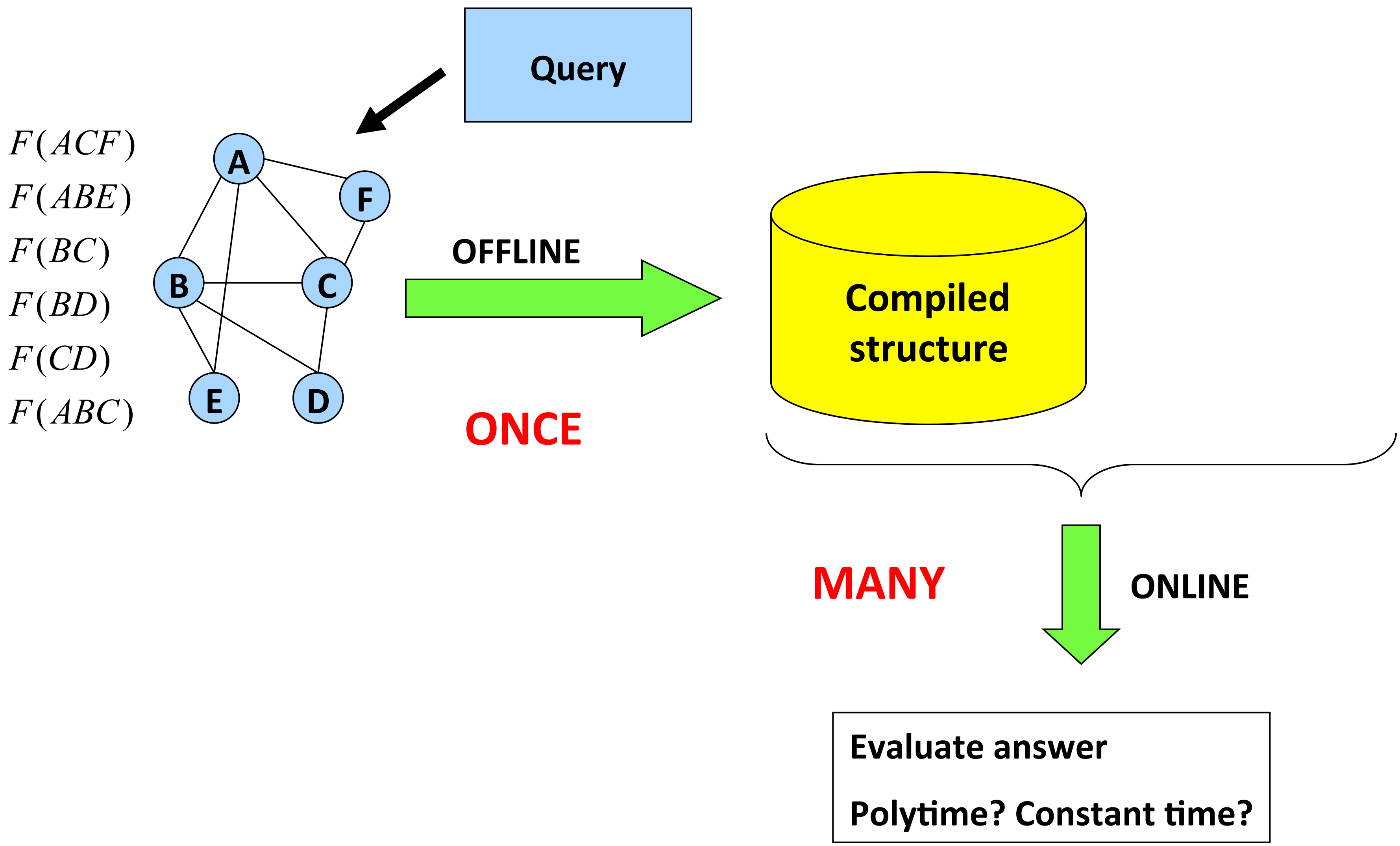
From Context-Minimal to Minimal AND/ORs

- Any two nodes that root identical subtrees/subgraphs (are unifiable) can be merged
- **Minimal AND/OR search graph:** of GM relative to a pseudo-tree T is the closure under merge of its AND/OR search tree, where inconsistent sub-trees are pruned.
- **Canonicity:** The minimal AND/OR search graph is unique (canonical) for all equivalent formulas (Boolean or Constraints) or weighted GM, consistent with its pseudo tree.
- **AOMDD: AND/OR Multi-valued Decision Diagrams are minimal AND/OR search graph representation**
- **Complexity:** Minimal AND/OR GM relative to pseudo-tree T is O(exp(w*)) where w* is the tree-width of GM along T.
- $w^* \leq pw^*$, $pw^* \leq w^* \log n$

Outline

- Motivation
- Background in Graphical models
- AND/OR search trees and Graphs
- Minimal AND/OR graphs
- From AND/OR search graphs to AOMDDs
- Compilation of AOMDDs
- AOMDDs and earlier BDDs

Compilation of Graphical Models

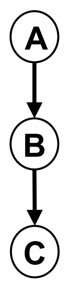


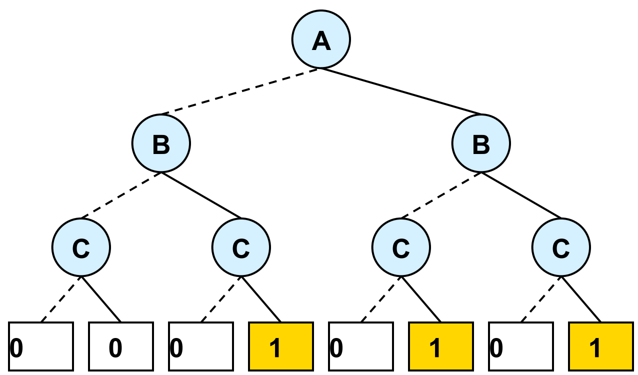
Ordered Binary Decision Diagram

B = {0,1} f : $B^3 \rightarrow B$

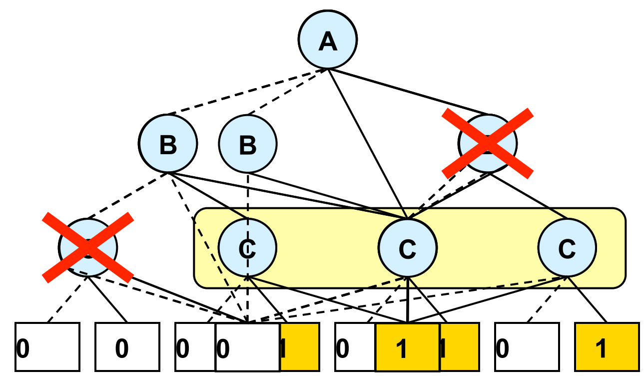
A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Table





Decision tree



1) Merge nodes with identical children

[Bryant86]

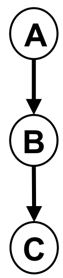
2) Remove redundant nodes

Ordering enables efficient operations

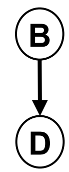
AND/OR CM Graph vs. AOMDD

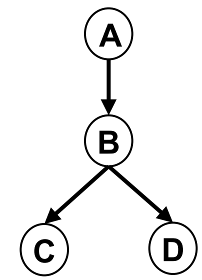
For a constraint network

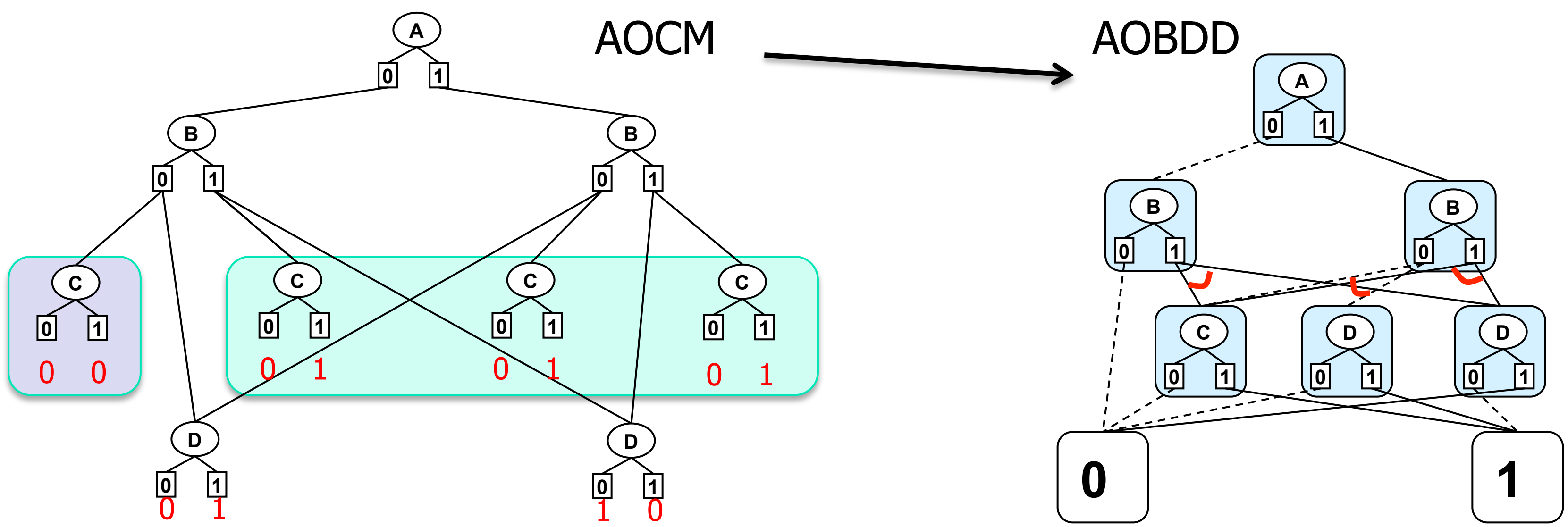
A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



B	D	G(BD)
0	0	0
0	1	1
1	0	1
1	1	0



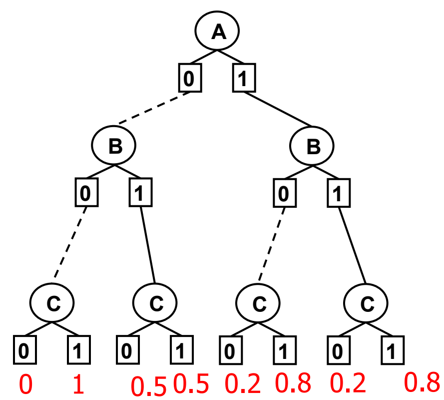


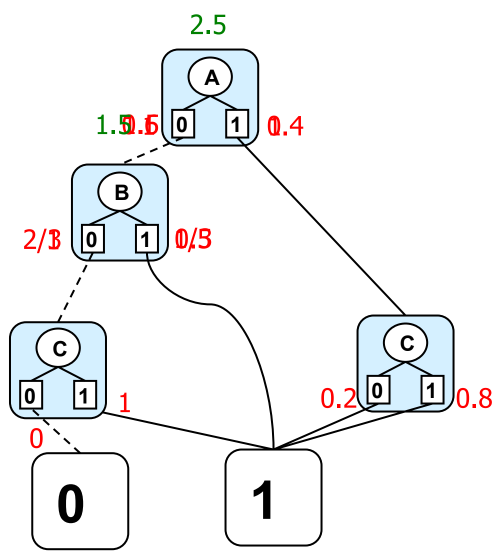


Weighted AND/OR Decision Diagrams

- Example of converting a CPT to a weighted AOMDD, for variable ordering (A,B,C)

A	B	C	P(C|A,B)
0	0	0	**0**
0	0	1	**1**
0	1	0	**0.5**
0	1	1	**0.5**
1	0	0	**0.2**
1	0	1	**0.8**
1	1	0	**0.2**
1	1	1	**0.8**

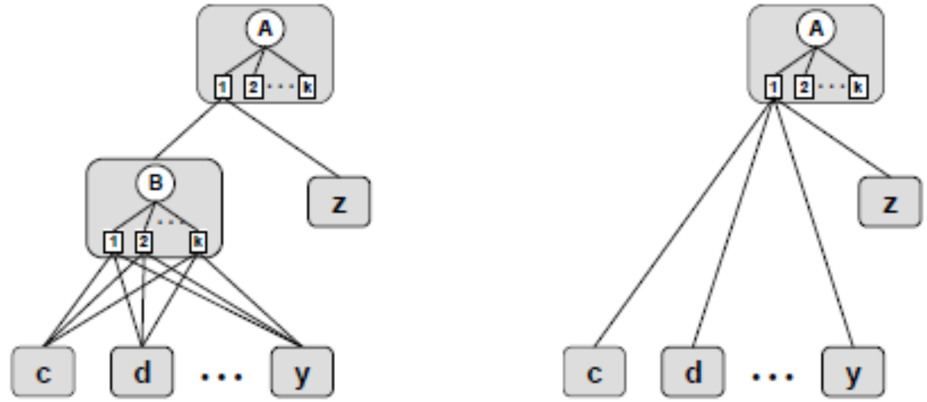




AND/OR Multi-Valued Decision Diagrams

- AOMDDs are:
  - Weighted AND/OR search graphs
  - Canonical representations, given a pseudo tree
  - Defined by two rules:
    - All isomorphic subgraphs are merged
    - There are no redundant (meta) nodes

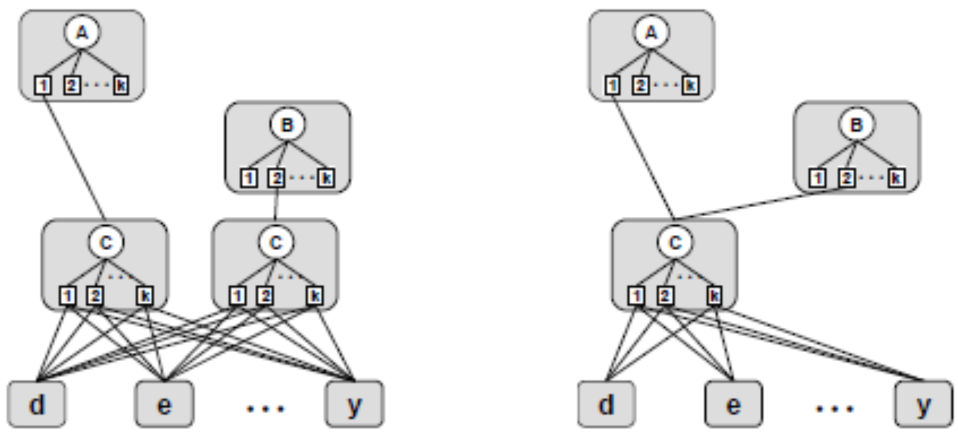
Redundancy and Isomorphism Rules



(a) Fragment of an AOMDD

(b) After eliminating the B meta-node

Figure 13: Redundancy reduction



(a) Fragment of an AOMDD

(b) After merging the isomorphic C meta-nodes

Outline

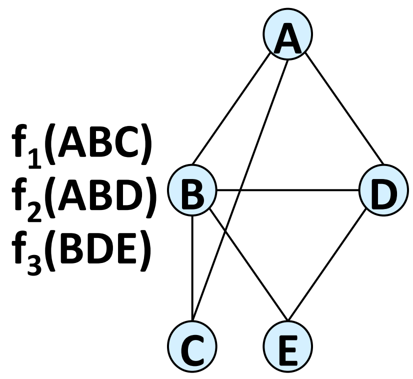
- Motivation
- Background in Graphical models
- AND/OR search trees and Graphs
- Minimal AND/OR graphs
- From AND/OR search graphs to AOMDDs
- Compilation of AOMDDs
  - Top down
  - Bottom up
- AOMDDs and earlier BDDs

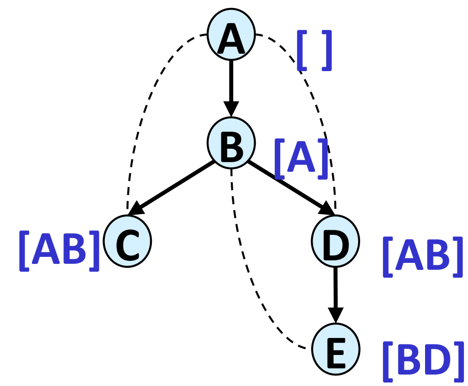
Cost Networks- Weighted AND/OR Tree

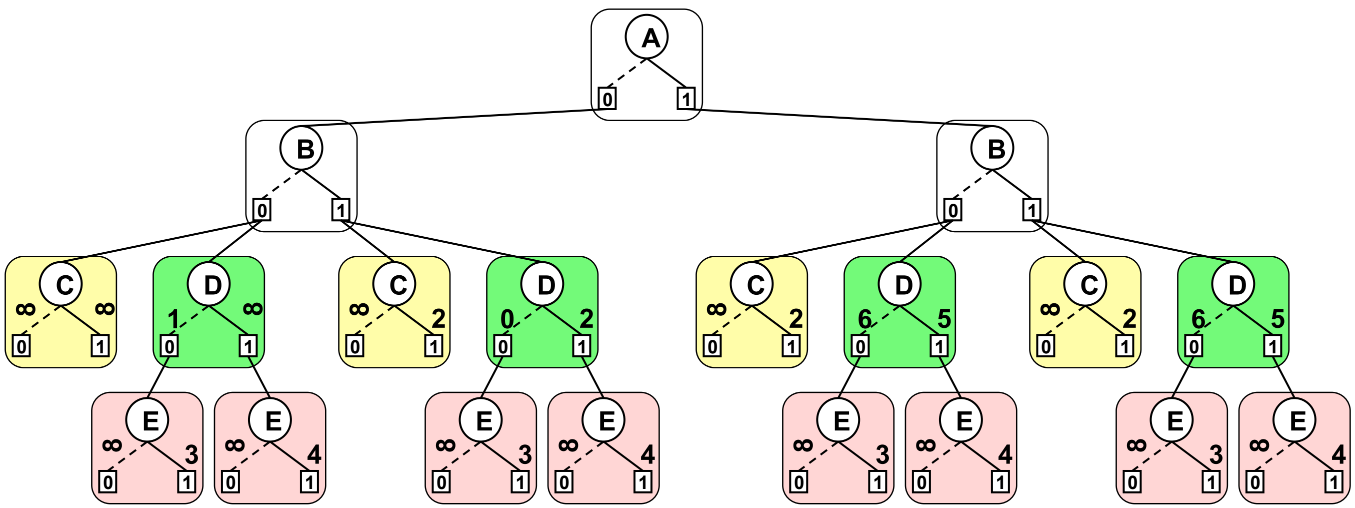
A	B	C	$f_1(ABC)$
0	0	0	∞
0	0	1	∞
0	1	0	∞
0	1	1	2
1	0	0	∞
1	0	1	2
1	1	0	∞
1	1	1	2

A	B	D	$f_2(ABD)$
0	0	0	1
0	0	1	∞
0	1	0	0
0	1	1	2
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

B	D	E	$f_3(BDE)$
0	0	0	∞
0	0	1	3
0	1	0	∞
0	1	1	4
1	0	0	∞
1	0	1	3
1	1	0	∞
1	1	1	4





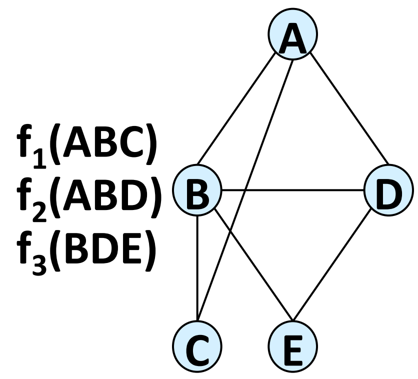


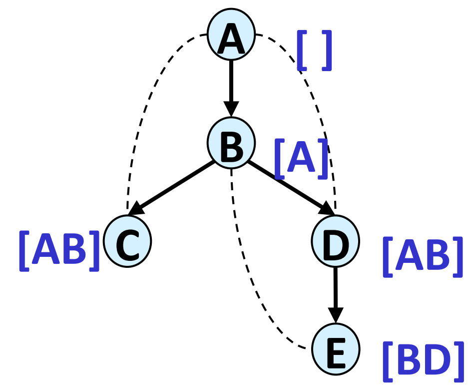
Weighted AND/OR Context Minimal Graph

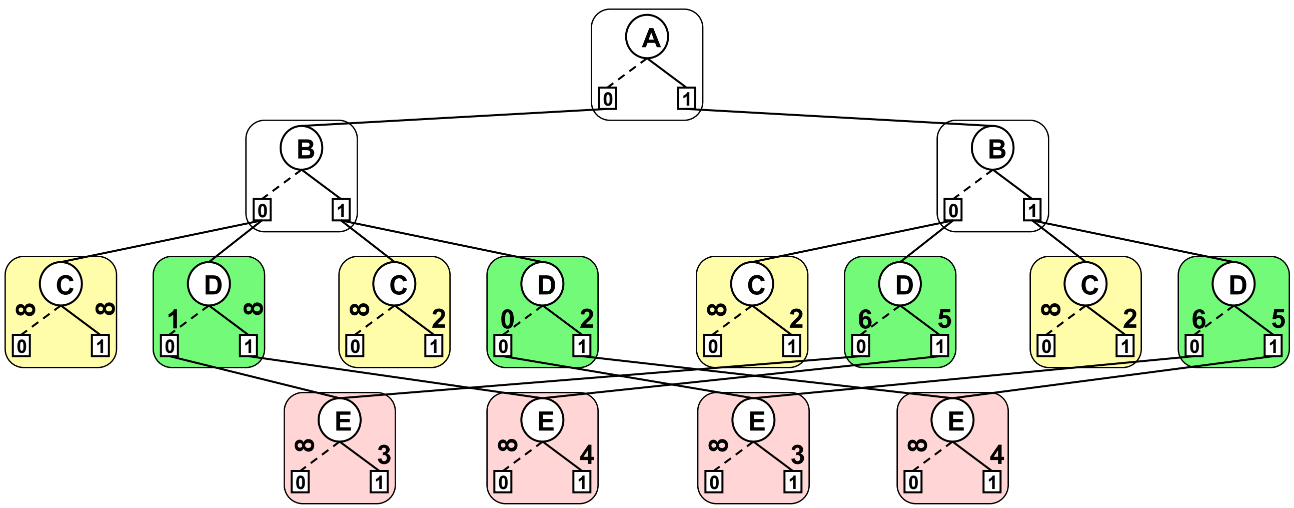
A	B	C	f_1(ABC)
0	0	0	∞
0	0	1	∞
0	1	0	∞
0	1	1	2
1	0	0	∞
1	0	1	2
1	1	0	∞
1	1	1	2

A	B	D	f_2(ABD)
0	0	0	1
0	0	1	∞
0	1	0	0
0	1	1	2
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

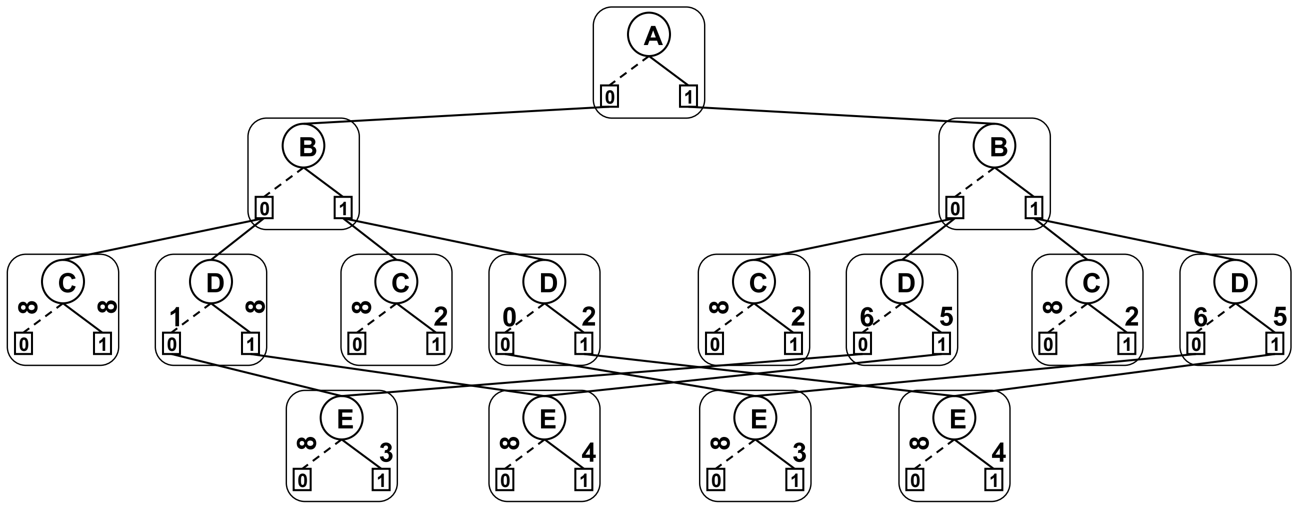
B	D	E	f_3(BDE)
0	0	0	∞
0	0	1	3
0	1	0	∞
0	1	1	4
1	0	0	∞
1	0	1	3
1	1	0	∞
1	1	1	4

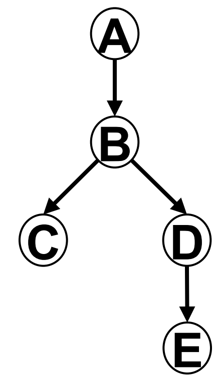




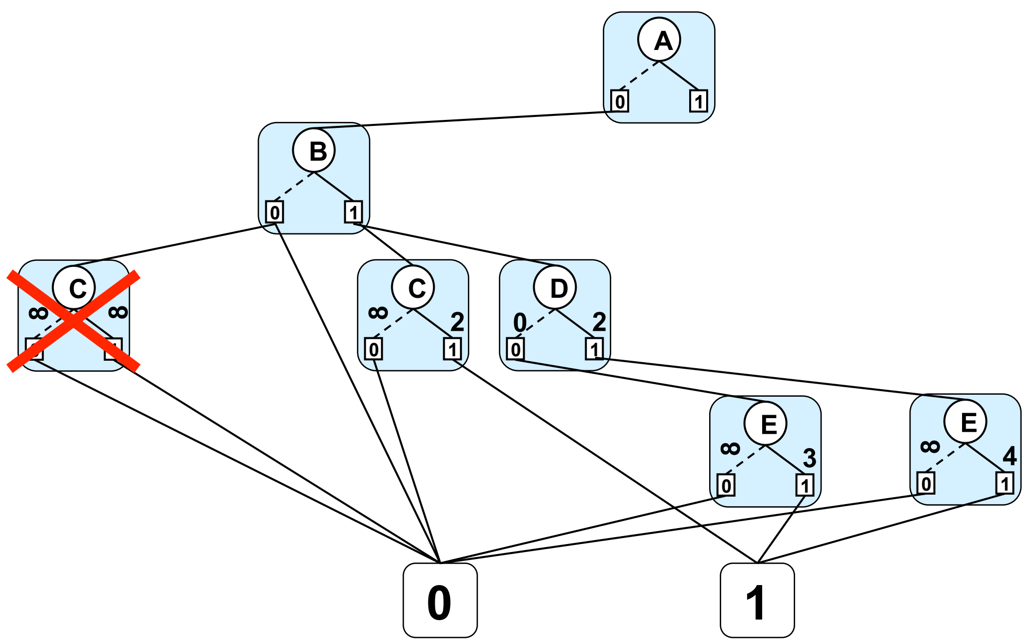


AOMDD – Compilation by Search

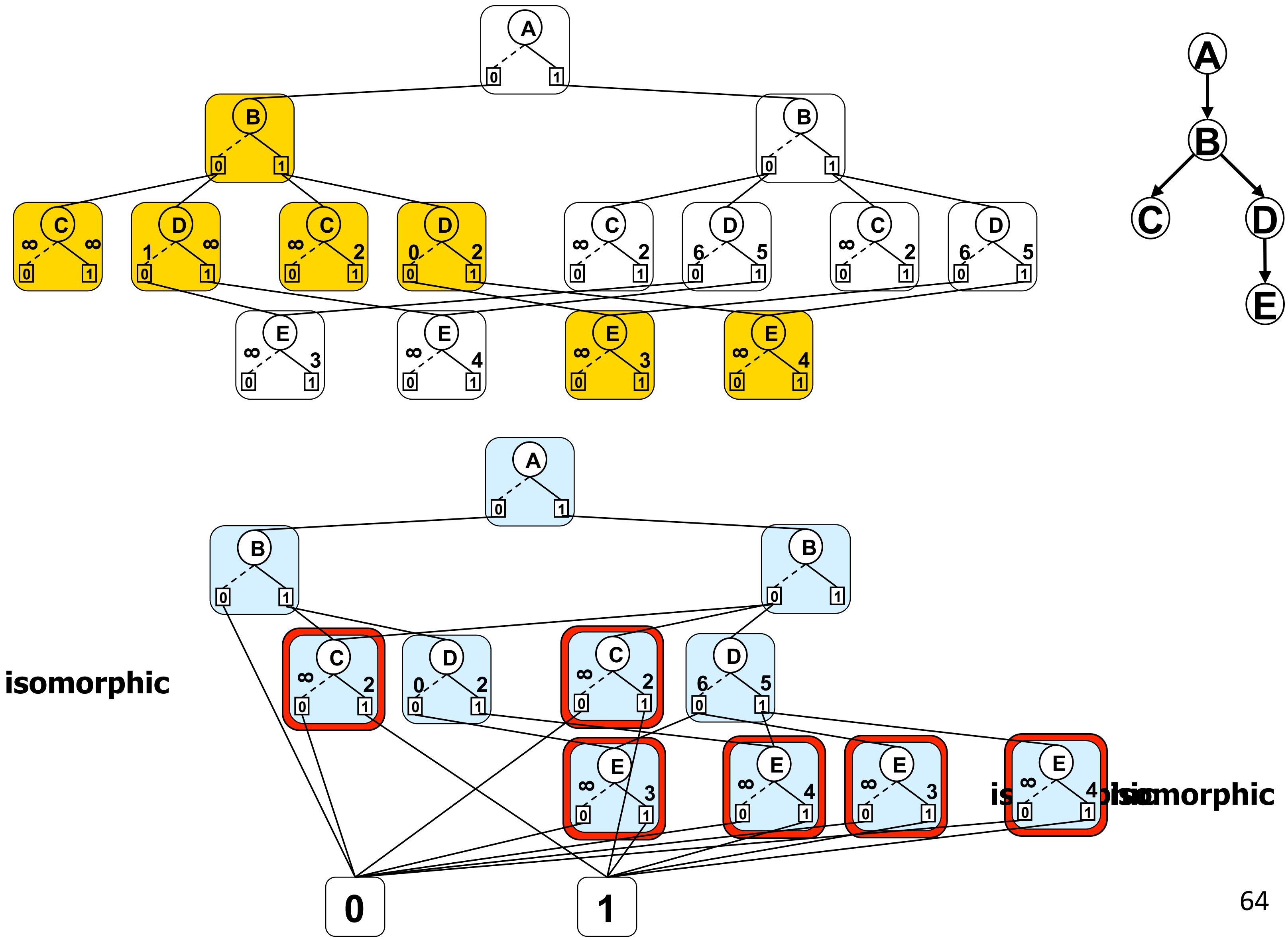




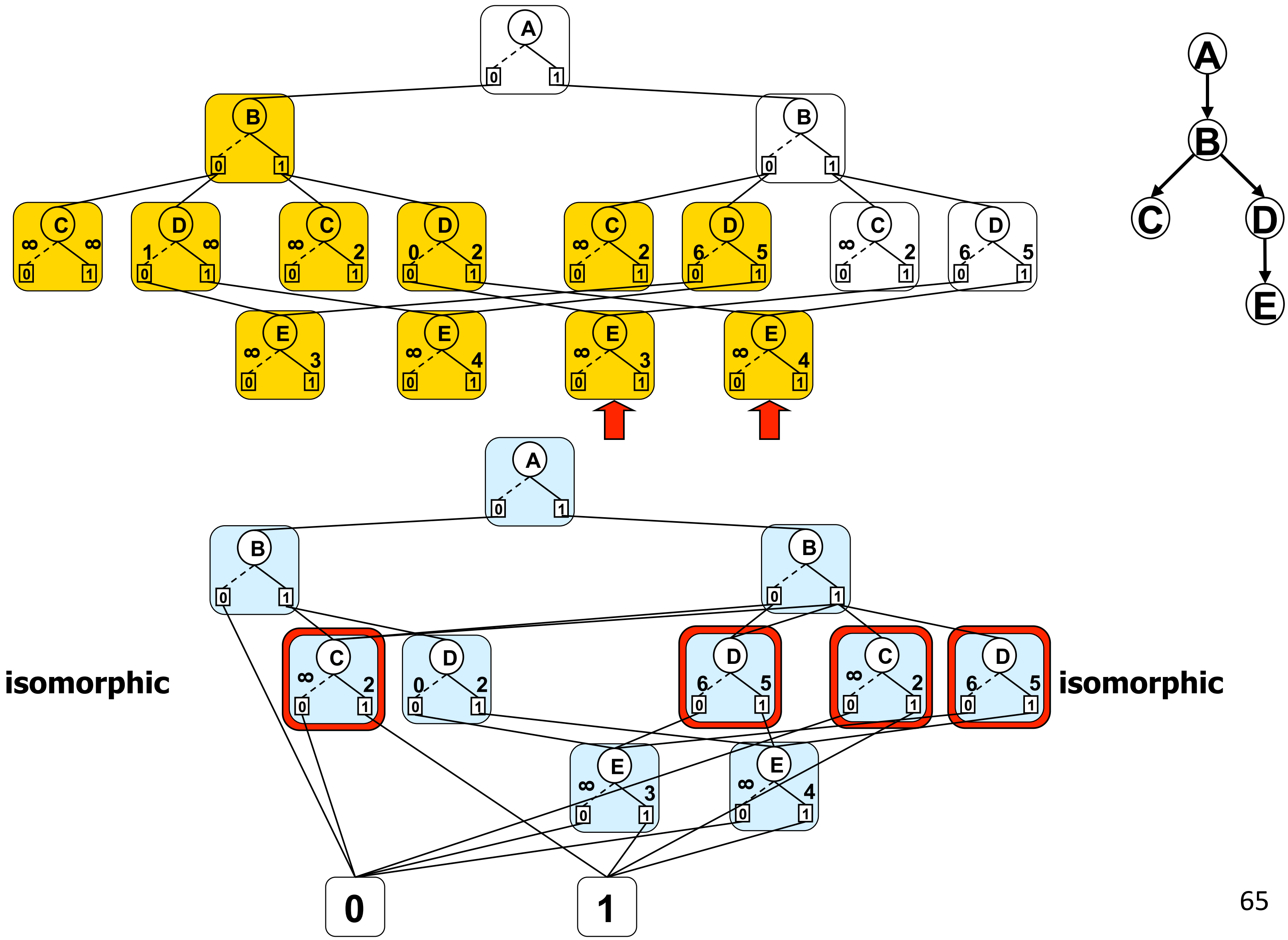
redundant



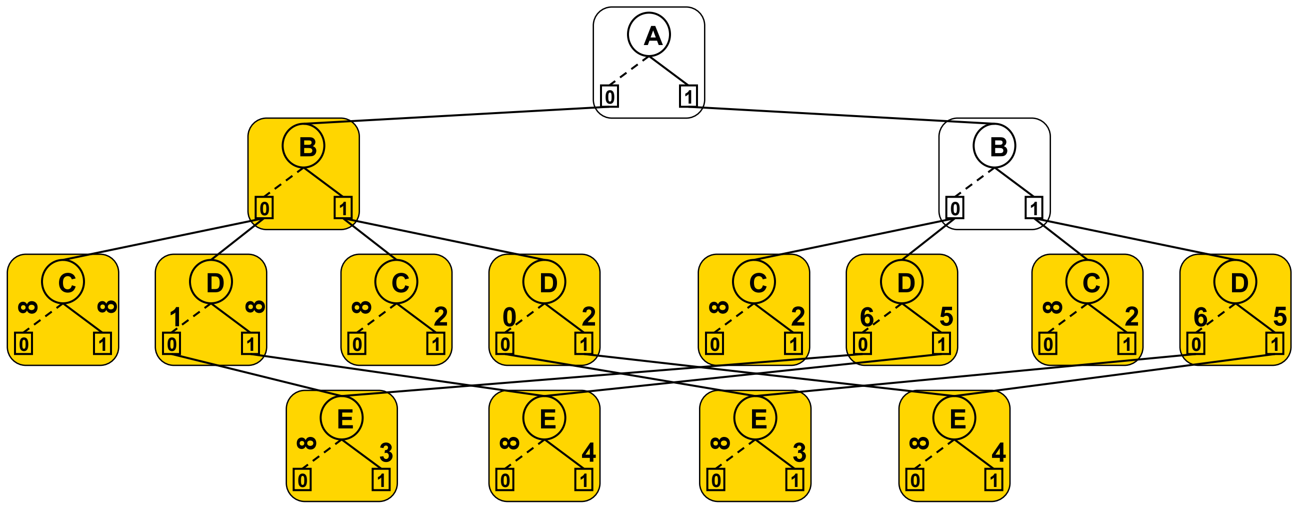
AOMDD – Compilation by Search

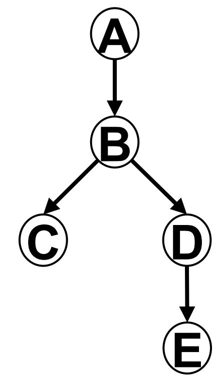


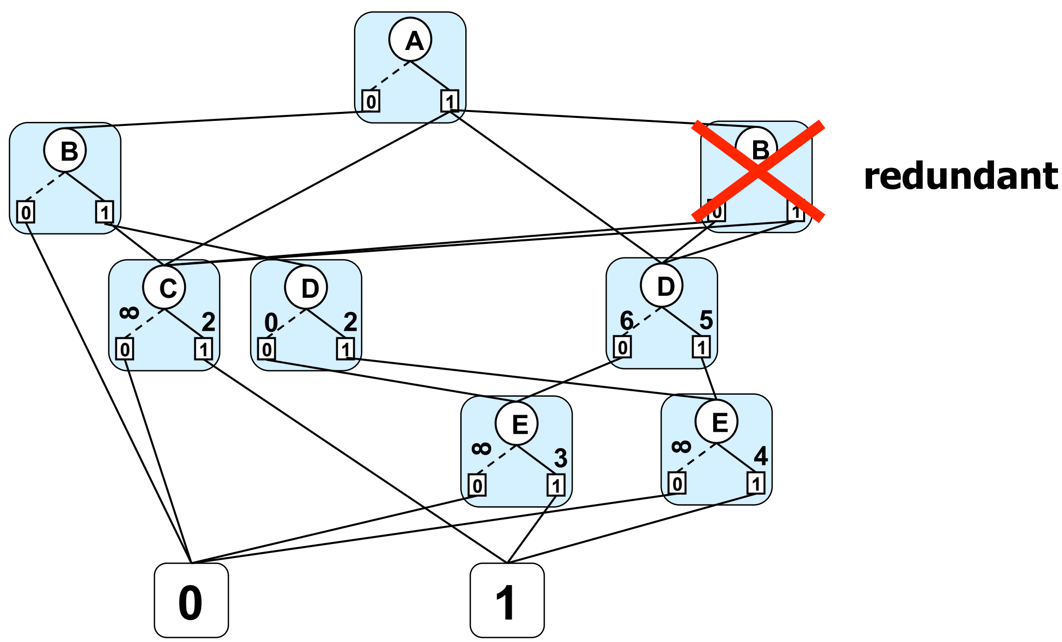
AOMDD – Compilation by Search



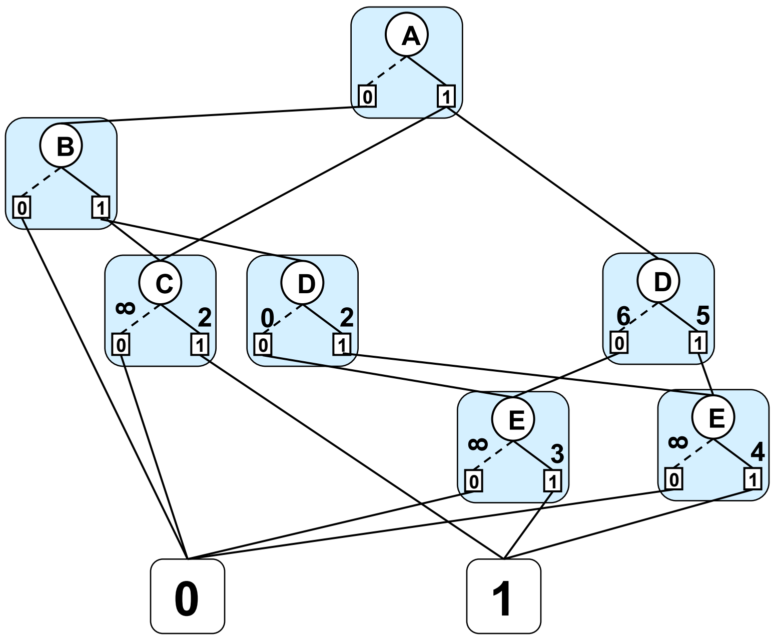
AOMDD – Compilation by Search



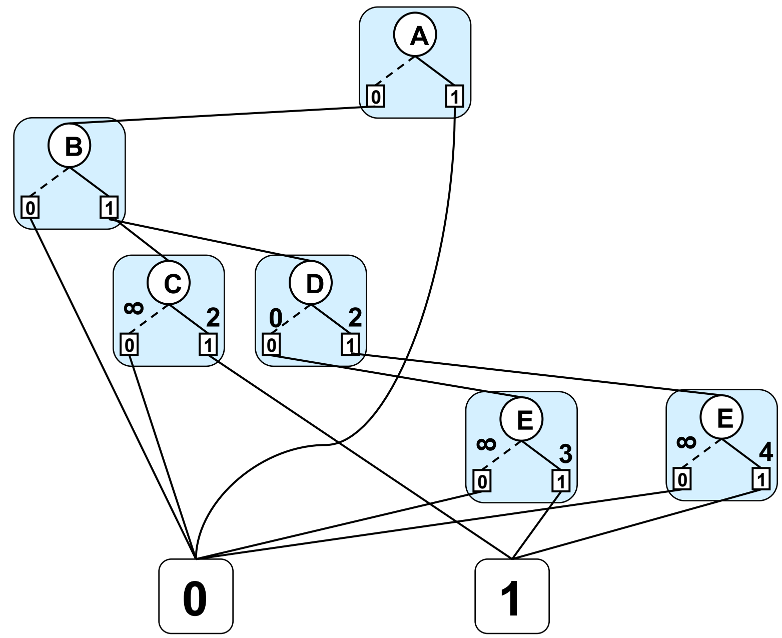




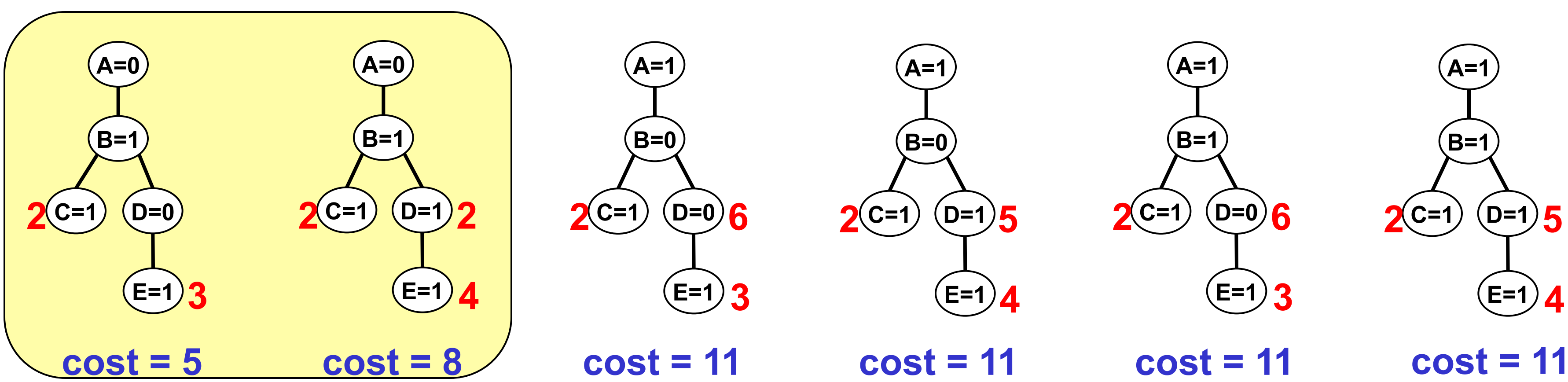
AOMDD for Constraint Optimization



AOMDD for all solutions



AOMDD for two best solutions



Outline

- Background in Graphical models
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- From AND/OR search graphs to AOMDDs
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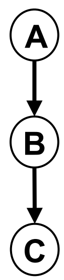
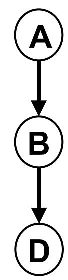
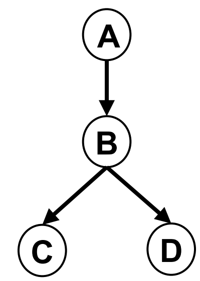
The Apply Operator

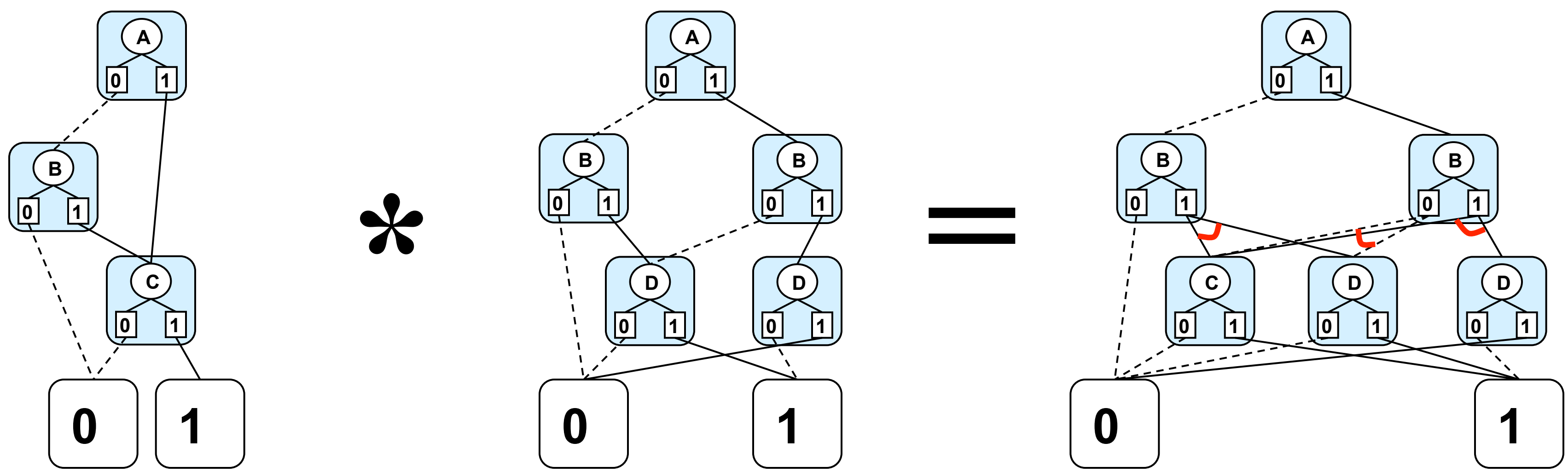
A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



A	B	D	g(ABD)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

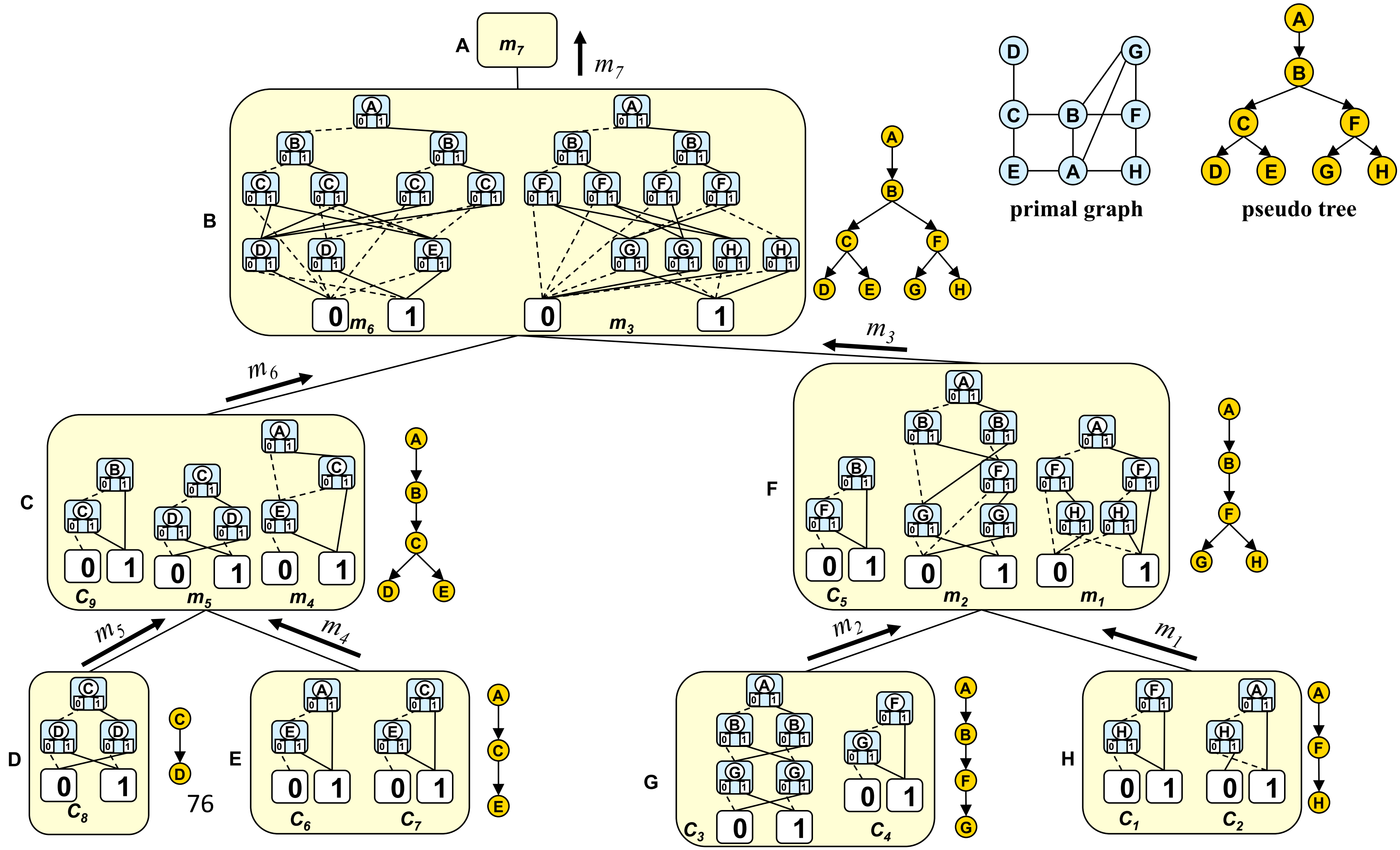




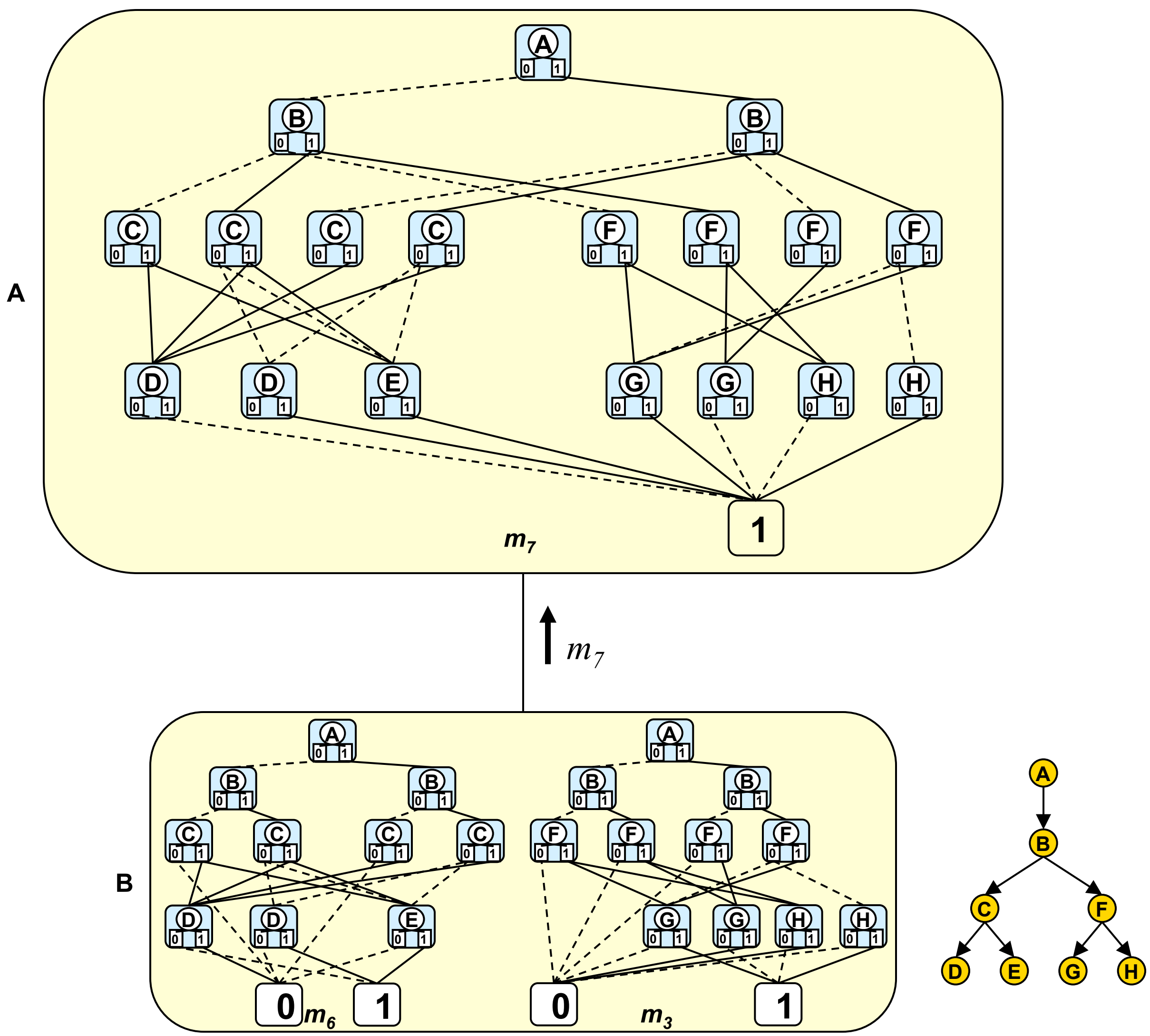


Example:

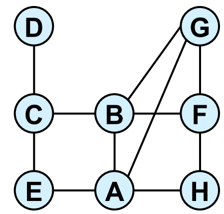
$$(F \lor H) \land (A \lor \neg H) \land (A \neq B \neq G) \land (F \lor G) \land (B \lor F) \land (A \lor E) \land (C \lor E) \land (C \neq D) \land (B \lor C)$$



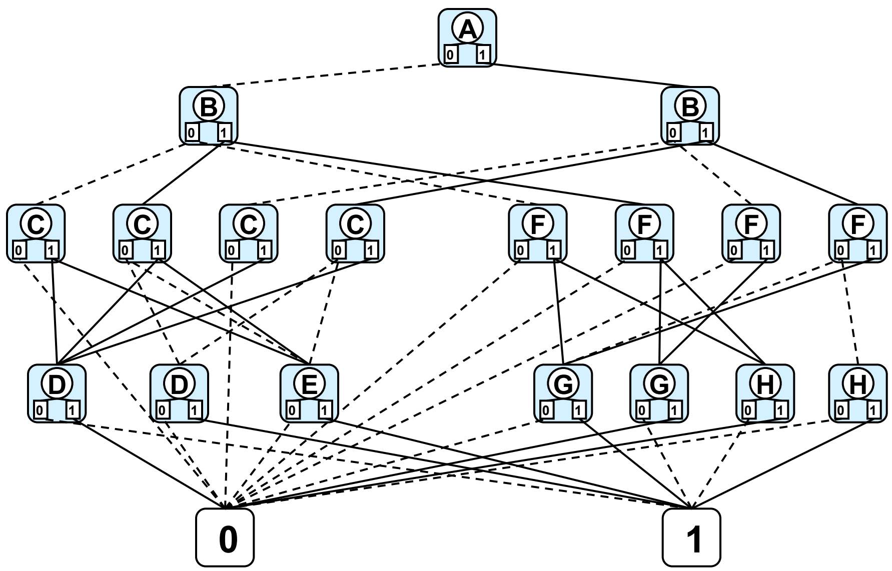
Example (continued)

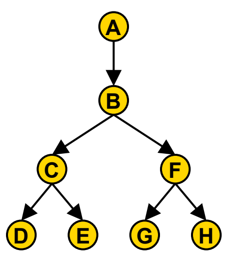


AOBDD vs. OBDD



primal graph

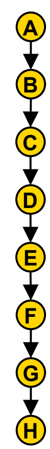


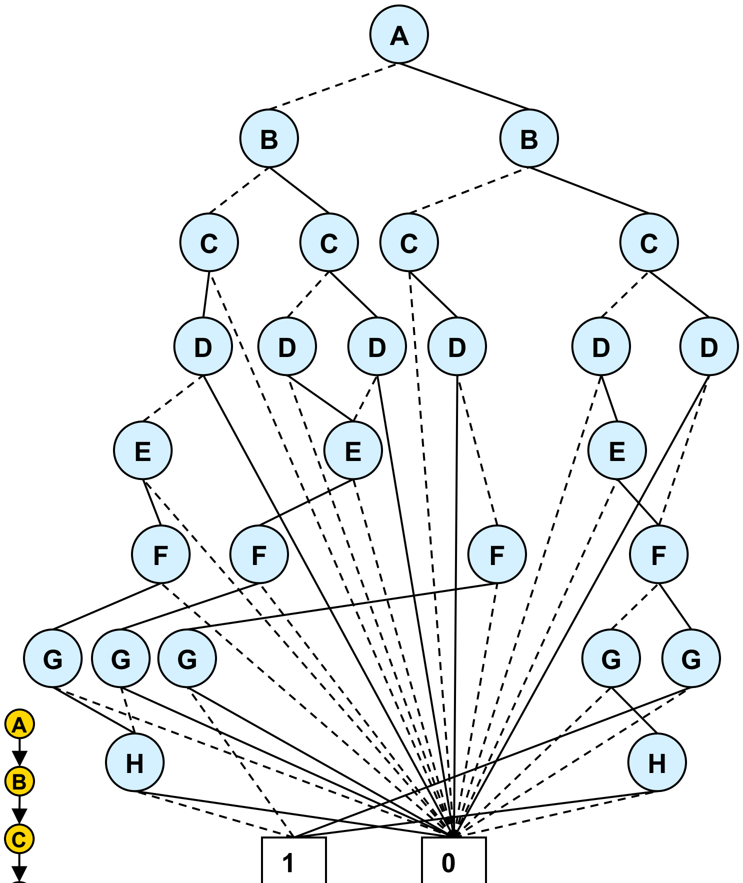


AOBDD

18 nonterminals

47 arcs





OBDD

27 nonterminals

54 arcs

Complexity of Compilation

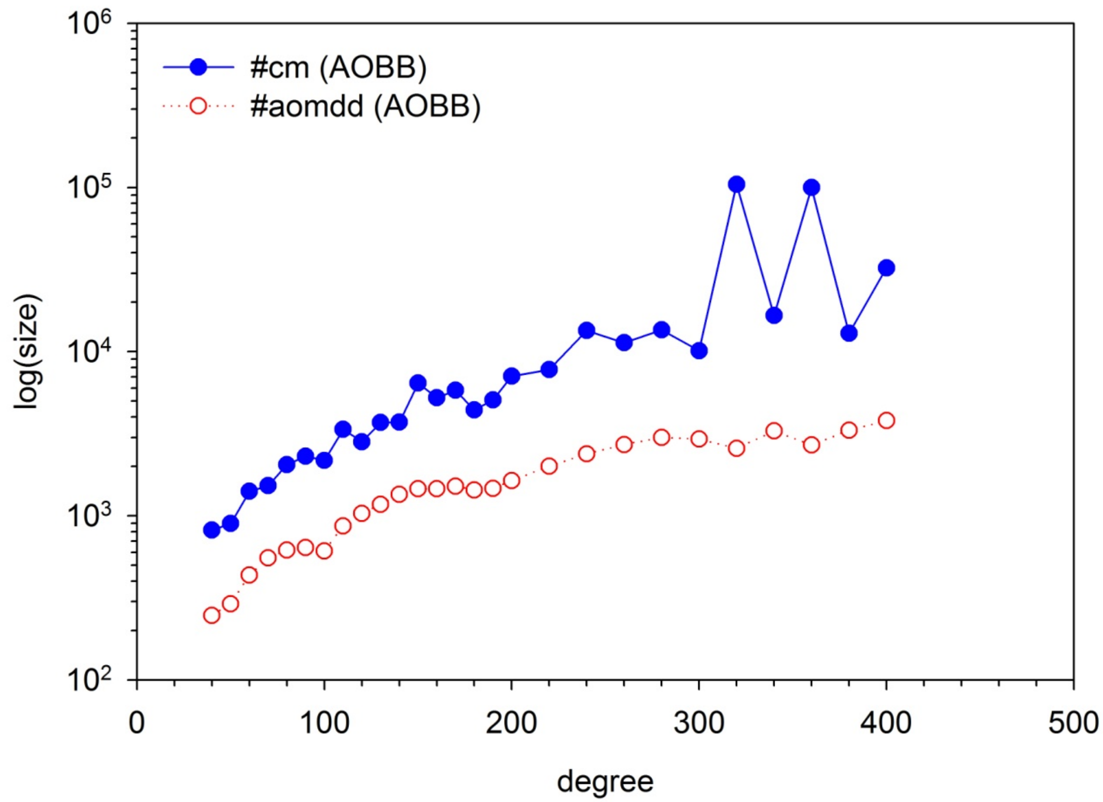
- The size of the AOMDD is $O(n\ k^{w^*})$
- The compilation time is also bounded by $O(n\ k^{w^*})$

k = domain size
n = number of variables
w*= treewidth

Empirical Evaluation

- Bayesian Networks (UAI 2006 evaluation)
- Weighted CSPs
- Randomly generated Bayesian Networks
- Pedigree networorks

MAX-SAT Instances (ILP)



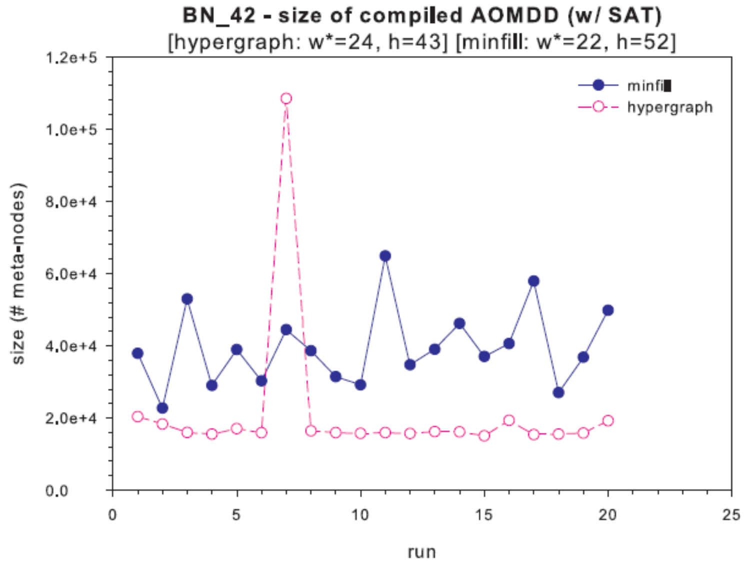
Results for dubois MAX-SAT instances

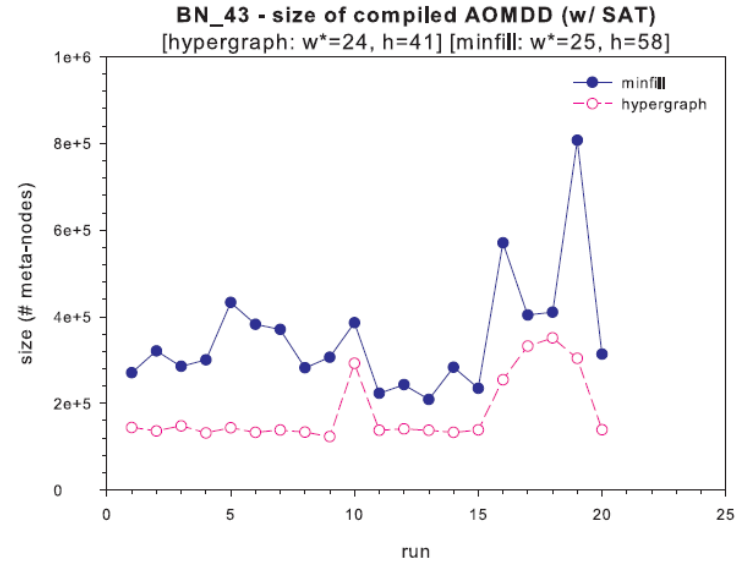
Bayesian Networks Repository

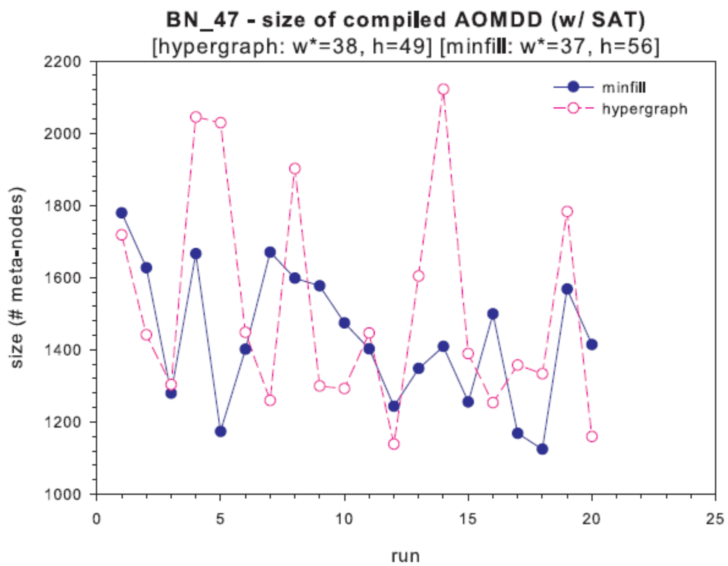
Network	(w*, h)	(n, k)	ACE		MDD w/ BCP			AOMDD w/ BCP			AOMDD w/ SAT		
			#nodes	time	#meta	#cm(OR)	time	#meta	#cm(OR)	time	#meta	#cm(OR)	time
Bayesian Network Repository													
alarm	(4, 13)	(37, 4)	1,511	0.01	208,837	682,195	73.35	**320**	459	0.05	**320**	459	0.22
cpcs54	(14, 23)	(54, 2)	196,933	0.06	-	-	-	**65,158**	66,405	6.97	**65,158**	66,405	6.97
cpcs179	(8, 14)	(179, 4)	67,919	0.05	-	-	-	**9,990**	32,185	46.56	**9,990**	32,185	46.56
cpcs360b	(20, 27)	(360, 2)	**5,258,826**	1.72	-	-	-	-	-	-	-	-	-
diabetes	(4, 77)	(413, 21)	**7,615,989**	1.81	-	-	-	-	-	-	-	-	-
hailfinder	(4, 16)	(56, 11)	8,815	0.01	-	-	-	2,068	2,202	0.34	**1,893**	2,202	1.48
mildew	(4, 13)	(35, 100)	823,913	0.39	-	-	-	73,666	110,284	1367.81	**62,903**	65,599	3776.82
mm	(20, 57)	(1220, 2)	47,171	1.49	-	-	-	38,414	58,144	4.54	**30,274**	52,523	99.55
munin2	(9, 32)	(1003, 21)	2,128,147	1.91	-	-	-	-	-	-	-	-	-
munin3	(9, 32)	(1041, 21)	1,226,635	1.27	-	-	-	-	-	-	-	-	-
munin4	(9, 32)	(1044, 21)	2,423,009	4.44	-	-	-	-	-	-	-	-	-
pathfinder	(6, 11)	(109, 63)	18,250	0.05	610,854	1,303,682	352.18	6,984	16,267	30.71	**2,265**	15,963	50.36
pigs	(11, 26)	(441, 3)	636,684	0.19	-	-	-	261,920	294,101	174.29	**198,284**	294,101	1277.72
water	(10, 15)	(32, 4)	59,642	0.52	707,283	1,138,096	95.14	18,744	20,926	2.02	**18,503**	19,225	7.45

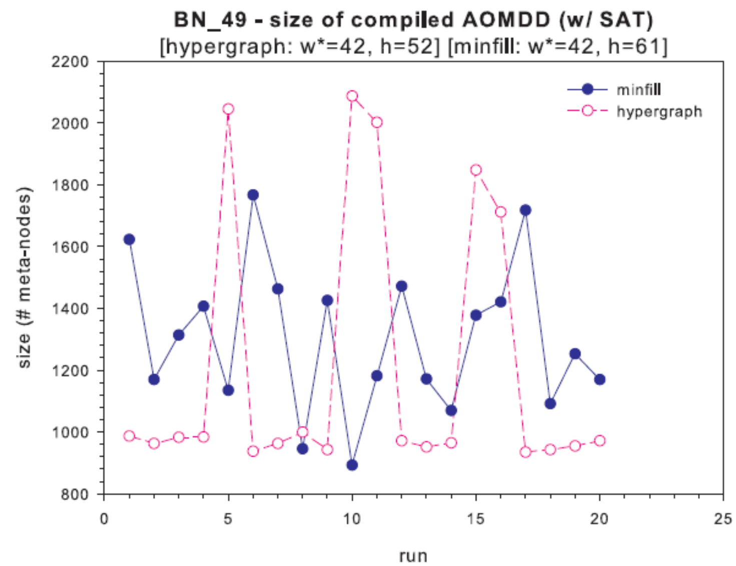
Size (number of nodes), time (seconds)

Effect of Variable Ordering









AOMDD Compilation Results

name	n	w	h	k	# functions	time (s)	CM OR	AOMDD Meta	CM AND	AOMDD AND	Effective semantic width	Max UniqueTable Memory (MB)	Max Operation Cache Memory (MB)	Compiled AOMDD memory (MB)
BN_42	850	20	50	2	879	93	5623680	25901	11237360	51802	10.35	203.5	189.65	5.41
BN_43	850	21	50	2	881	484	22731586	148255	45463172	296510	13.76	1181.3	1024	30.88
BN_44	850	21	53	2	880	394	11681649	80878	23363298	161756	13.58	962.73	822.8	16.81
BN_45	850	21	56	2	875	140	15778481	122816	31556962	245632	13.58	292.29	305.16	25.1
BN_46	850	19	47	2	499	268	4277086	4352	8554172	8704	8	618.04	492.24	0.93

(Lam and Dechter CP 2012)

Recent Experiments (Lam and Dechter cp 2012)

name	n	w	h	k	# functions	time (s) [BE-AOMDD+R] [AOMDD-BCP]	CM OR	Metanodes [BE-AOMDD+R] [AOMDD-BCP]	Memory Usage (MB)	Compiled AOMDD mem (MB)
BN_42	850	20	50	2	879	10 36	5623680	**25841** 95963	405.21	8.12
BN_43	850	21	50	2	881	73 647	22731586	**148184** 629027	2132.53	46.37
BN_45	850	21	56	2	875	17 142	15778481	**122763** 260917	646.25	34.44

Table 1. Compilation results on UAI 2006 benchmarks (ISCAS circuits). Note that many instances are not shown here, which BE-AOMDD+R fails to compile due to memory limitations.

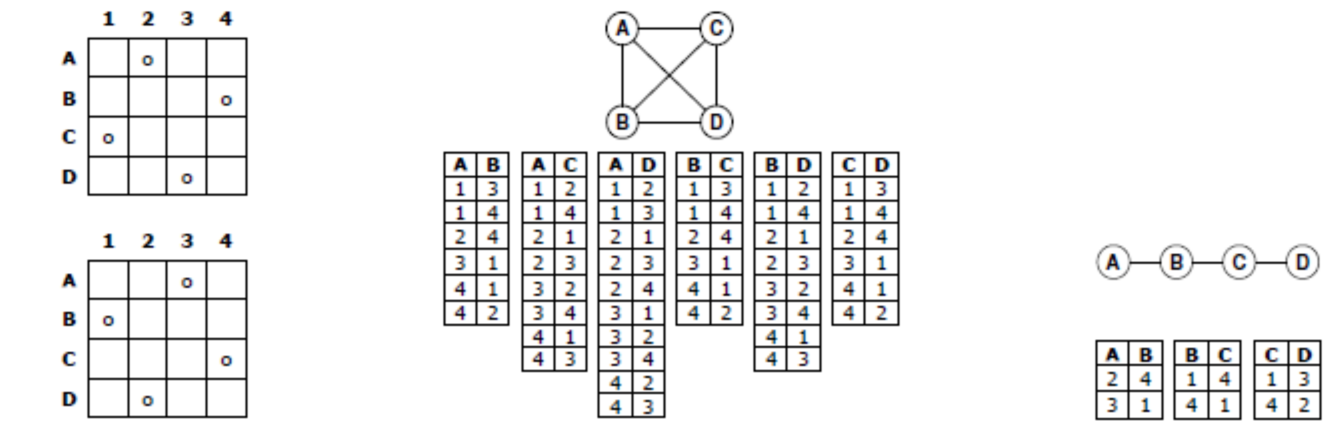
name	n	w	h	k	# functions	time (s)	CM OR	Metanodes [BE-AOMDD+R]	Max Memory Usage (MB)	Compiled AOMDD memory (MB)
pdb1fna	75	6	18	81	218	136	1983522	56377	467.61	44.44
pdb1j8e	39	6	12	81	119	294	2714323	258198	950.33	238.32
pdb1pef	17	6	11	81	55	430	4123288	342367	4499.79	772.83
pdb1rb9	42	7	14	81	128	1127	13370233	1163424	3789.48	1751.98
pdb2igd	50	6	19	81	146	1295	33711674	451081	3396.36	1132.93

Table 2. Compilation results on protein networks using BE-AOMDD+R.

Outline

- Background in Graphical models
- AND/OR search trees and Graphs
- Minimal AND/OR graphs
- From AND/OR search graphs to AOMDDs
- Compilation of AOMDDs
- Semantic Width
- Learning AOMDDs

Semantic Width



(a) The two solutions (b) First model (c) Second model

Figure 23: The 4-queen problem

Semantic Treewidth

- Given a graphical model, there may exist a simpler equivalent graphical model
- (of a pseudo tree) The smallest treewidth over equivalent graphical models that can have that pseudo tree
- (of a graphical model) The smallest treewidth over all equivalent graphical models with any legal pseudo tree

Semantic Width

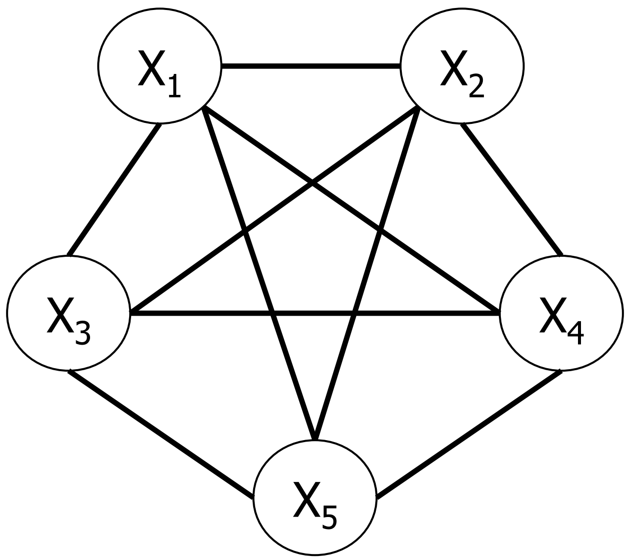
DEFINITION **27 (semantic treewidth)** *The* semantic treewidth of a graphical model $\mathcal{M}$ relative to a pseudo tree $\mathcal{T}$ *denoted by* $sw_{\mathcal{T}}(\mathcal{M})$, *is the smallest treewidth taken over all models* $\mathcal{R}$ *that are equivalent to* $\mathcal{M}$, *and accept the pseudo tree* $\mathcal{T}$. *Formally, it is defined by* $sw_{\mathcal{T}}(\mathcal{M}) = min_{\mathcal{R},u(\mathcal{R})=u(\mathcal{M})} w_{\mathcal{T}}(\mathcal{R})$, *where* $u(\mathcal{M})$ *is the universal function of* $\mathcal{M}$, *and* $w_{\mathcal{T}}(\mathcal{R})$ *is the induced width of* $\mathcal{R}$ *along* $\mathcal{T}$. *The* semantic treewidth *of a graphical model,* $\mathcal{M}$, *is the minimal semantic treewidth over all the pseudo trees that can express its universal function.*

Computing the semantic treewidth can be shown to be NP-hard.

Proposition 7 *The size of the AOMDD of a graphical model* $\mathcal{M}$ *is bounded by* $O(n\ k^{sw_{\mathcal{T}}(\mathcal{M})})$, *where* n *is the number of variables,* k *is the maximum domain size and* $sw_{\mathcal{T}}(\mathcal{M})$ *is the semantic treewidth of* $\mathcal{M}$ *along the pseudo tree* $\mathcal{T}$.

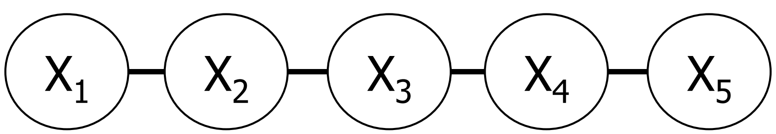
Semantic Treewidth

An extreme example…



Treewidth = 4

$$\psi(x_i, x_j) = \begin{cases} 1 & \text{if } x_i = x_j \\ 0 & \text{otherwise} \end{cases}$$



Treewidth = 1

Compiled AOMDDs for each will be the same!

AOMDD Compilation Results

name	n	w	h	k	# functions	time (s)	CM OR	AOMDD Meta	CM AND	AOMDD AND	Effective semantic width	Max UniqueTable Memory (MB)	Max Operation Cache Memory (MB)	Compiled AOMDD memory (MB)
BN_42	850	20	50	2	879	93	5623680	25901	11237360	51802	10.35	203.5	189.65	5.41
BN_43	850	21	50	2	881	484	22731586	148255	45463172	296510	13.76	1181.3	1024	30.88
BN_44	850	21	53	2	880	394	11681649	80878	23363298	161756	13.58	962.73	822.8	16.81
BN_45	850	21	56	2	875	140	15778481	122816	31556962	245632	13.58	292.29	305.16	25.1
BN_46	850	19	47	2	499	268	4277086	4352	8554172	8704	8	618.04	492.24	0.93

Outline

- Background in Graphical models
- AND/OR search trees and Graphs
- Minimal AND/OR graphs
- From AND/OR search graphs to AOMDDs
- Compilation of AOMDDs
- Semantic Width
- Learning AOMDDs

Learning Weighted AOMDDs

- Gogate, Webb, Domingo, 2010 “Learning efficient Markov Networks
- Idea: Assume a relative small weighted AND/OR graph and learn the weights from data
- Where does the graph comes from: start from the context-minimal graph (if you know the structure of the model) and remove nodes randomly (like Hinton’s dropout idea)
- Future work in my group. Want to join?

Publications

- **Rina Dechter and Robert Mateescu.** "AND/OR Search Spaces for Graphical Models". *Artificial Intelligence 171 (2-3), pp. 73-106, 2007.*
- **Robert Mateescu, Rina Dechter and Radu Marinescu.** "AND/OR Multi-Valued Decision Diagrams (AOMDDs) for Graphical Models *(JAIR), 2008.*
- **Robert Mateescu, Radu Marinescu and Rina Dechter.** "AND/OR Multi-Valued Decision Diagrams (AOMDDs) for Constraint Optimization". *In CP 2007*
- **Robert Mateescu and Rina Dechter.** "AND/OR Multi-Valued Decision Diagrams (AOMDDs) for Weighted Graphical Models". *In UAI'07.*
- **William Lam and Rina Dechter.** "Empirical Evaluation of AND/OR Multivalued Decision Diagrams for Inference" *in Doctoral Programme of CP 2012*.

Thank You !!