BREN:ICS INFORMATION AND COMPUTER SCIENCES UNIVERSITY of CALIFORNIA IRVINE

Modern Exact and Approximate Combinatorial Optimization Algorithms: Max-Product and Max-Sum-product

In the pursuit of universal solver

Rina Dechter

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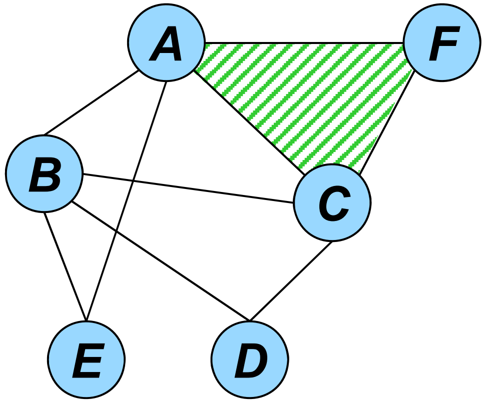

Main collaborators:
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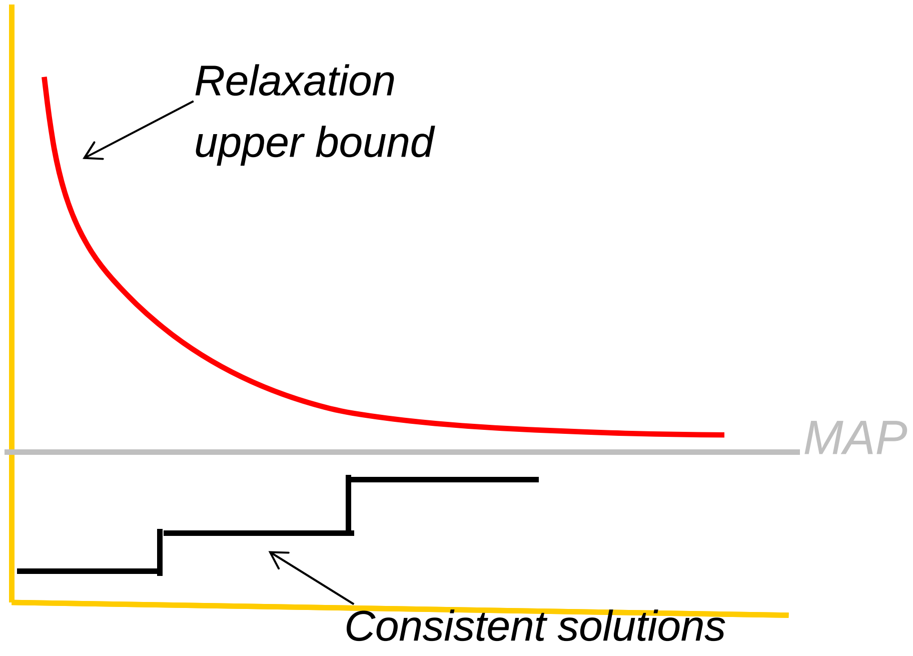
Lars Otten
Alex Ihler
Kalev Kask
Robert Mateescu

ISAIM 2016

How to Design a Good Solver

- Heuristic Search
- The core of a good search algorithm
  - A compact search space
  - A good heuristic evaluation function
  - A good traversal strategy
- Anytime search yields a good approximation.

Bounding from Above and Below



For a maximization probelm

Relaxation provides upper bound
Any configuration: lower bound

Outline

- Graphical models, Queries, Algorithms
- Inference Algorithms: bucket-elimination
- Bounded Inference: a) mini-bucket, b) cost-shifting
- AND/OR search spaces and AND/OR BnB
- Evaluation, Software
- Summary and future work

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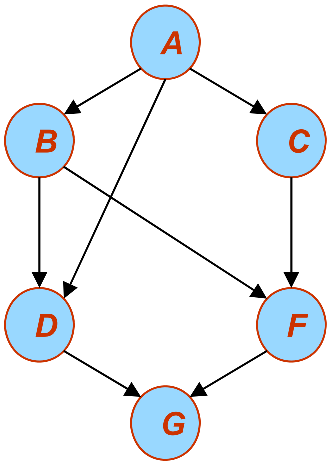
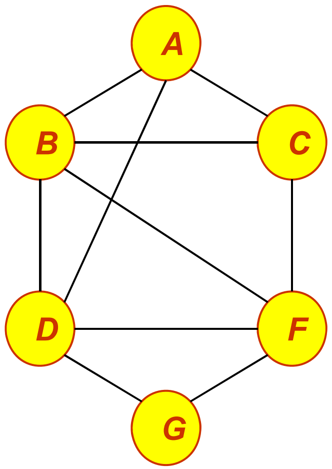
- Conclusions

Graphical Models

P(A)
P(B|A)
P(C|A)
P(D|A,B)
P(F|B,C)
P(G|D,F)

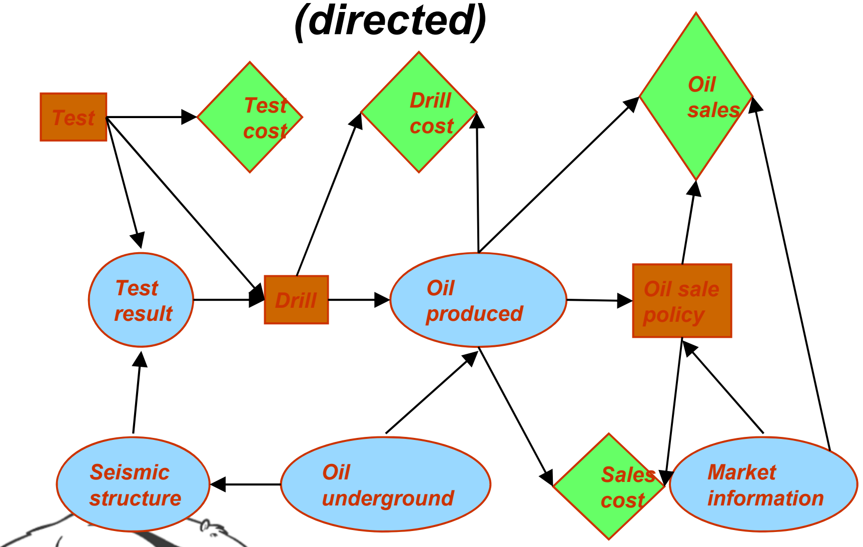


a) Belief network (directed)

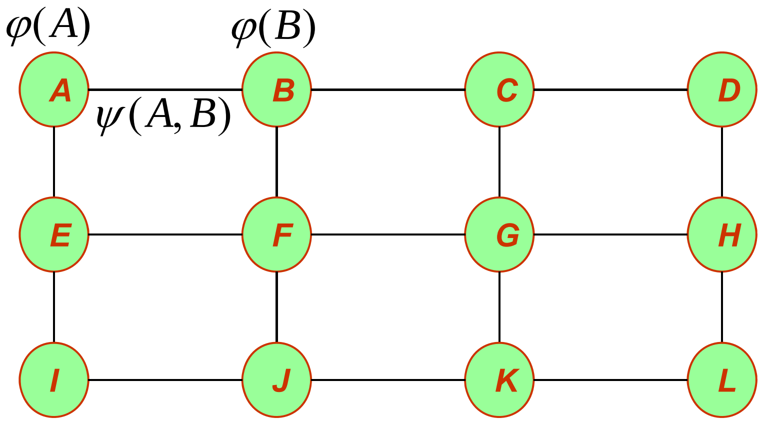


R(A)
R(A,B)
R(A,C)
R(A,B,D)
R(B,C,F)
R(D,F,G)

b) Constraint network (undirected)



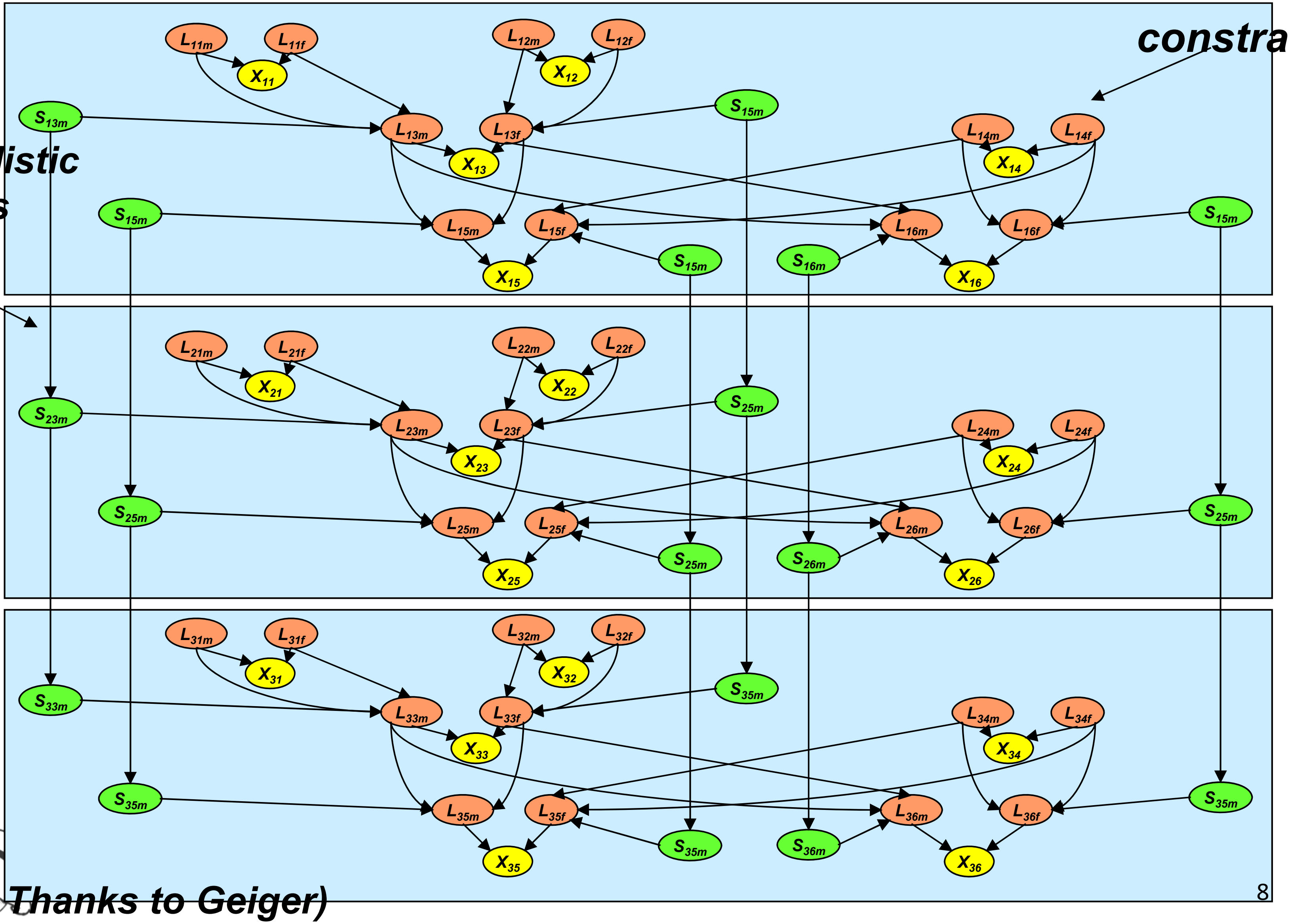
c) Influence diagram



d) Markov network

Many Many applications

Linkage Analysis: Pedigree: 6 People, 3 Markers



constraints

Probabilistic relations

(Slide, Thanks to Geiger)

Graphical Models

- A graphical model **(X,D,F)**:
  - □ **X** = $\{X_1,\ldots X_n\}$ variables
  - □ **D** = $\{D_1, \ldots D_n\}$ domains
  - □ **F** = $\{f_1,\ldots,f_r\}$ functions
    (constraints, CPTS, CNFs …)

- Operators:
  - □ combination : Sum, product, join
  - □ Elimination: projection, sum, max/min

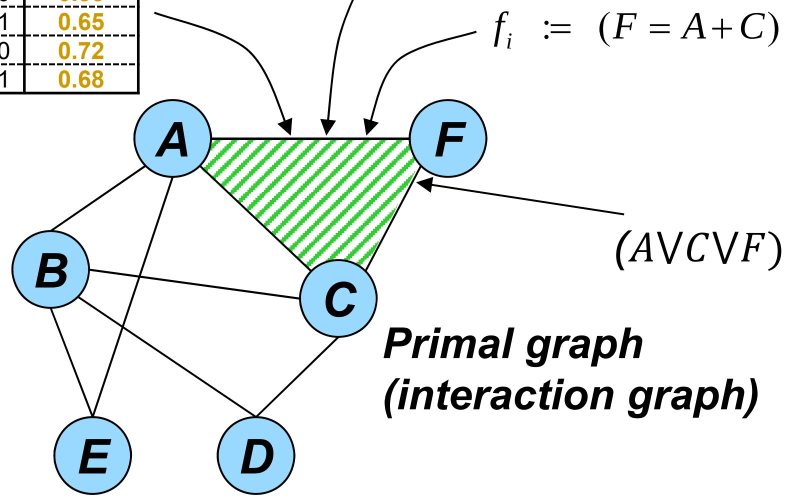
- Tasks:
  - □ **Belief updating**: $\Sigma_{X\backslash Y}\prod_j P_i$
  - □ **MPE\MAP**: $\max_X \prod_j P_j$
  - □ **Marginal MAP**: $\max_Y \sum_{X\backslash Y}\prod_j P_j$
  - □ **CSP**: $\prod_X \times_j C_j$
  - □ **Max-CSP**: $\min_X \Sigma_j F_j$

Conditional Probability Table (CPT)

A	C	F	P(F|A,C)
0	0	0	0.14
0	0	1	0.96
0	1	0	0.40
0	1	1	0.60
1	0	0	0.35
1	0	1	0.65
1	1	0	0.72
1	1	1	0.68

Relation

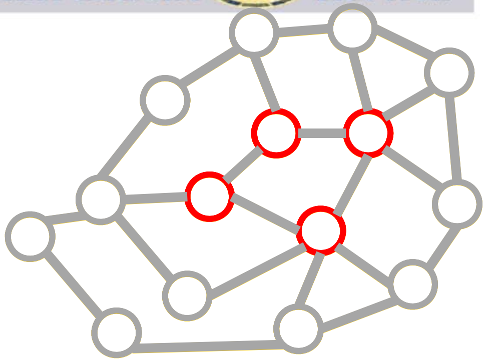
A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue



Primal graph (interaction graph)

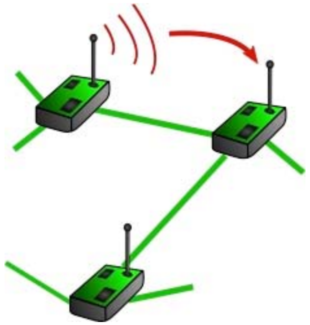
- ***All these tasks are NP-hard***
  - ***exploit problem structure***
  - ***identify special cases***
  - ***approximate***

Why Marginal MAP?

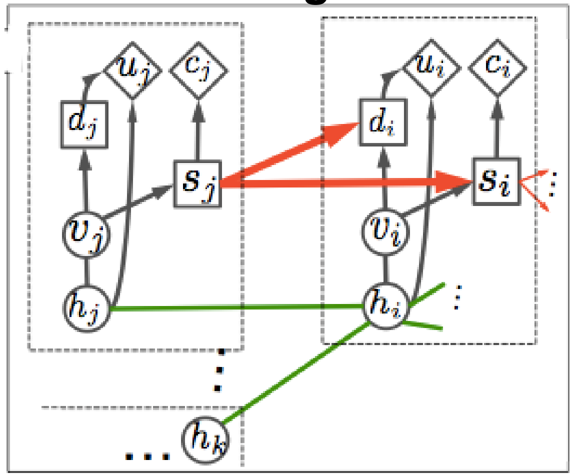


- Often, Marginal MAP is the "right" task:
  - We have a model describing a large system
  - We care about predicting the state of some part
- Example: decision making
  - Sum over random variables (random effects, etc.)
  - Max over decision variables (specify action policies)

Sensor network



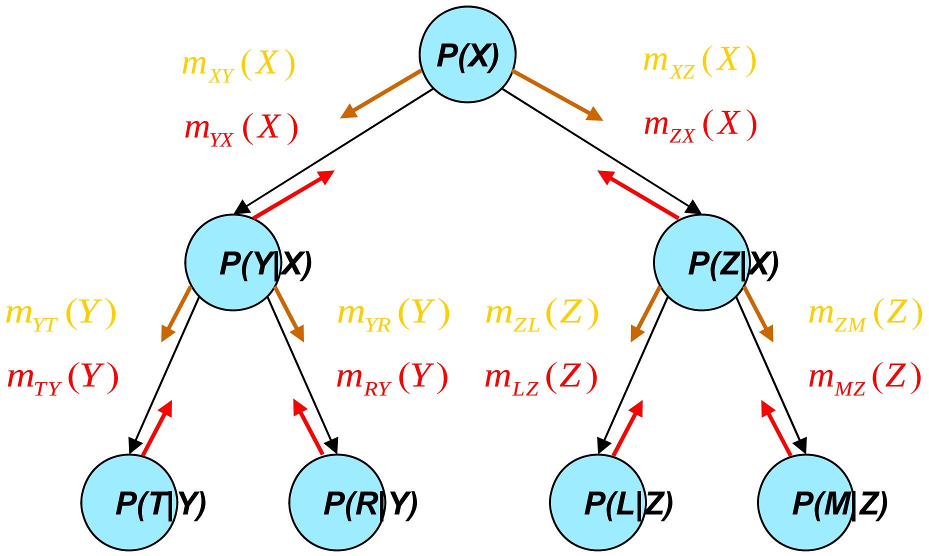
Influence diagram:



Tree-solving is easy

Belief updating (sum-prod)

CSP – consistency (projection-join)



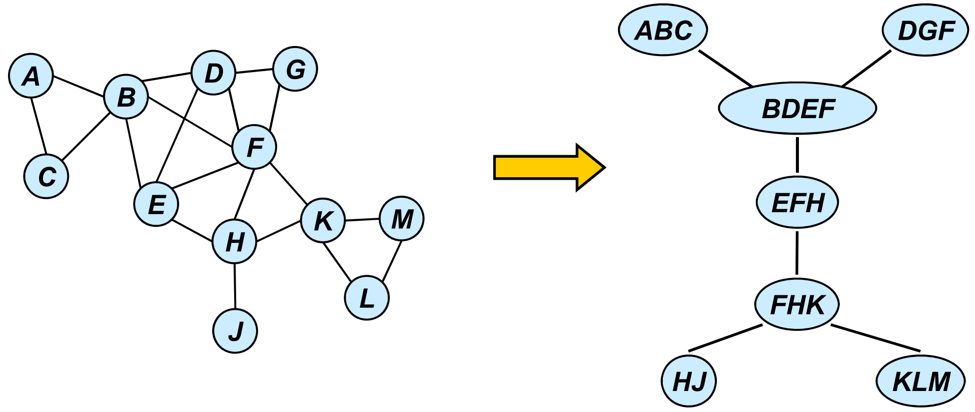
Marginal Map is not Easy even for trees

MPE (max-prod)

#CSP (sum-prod)

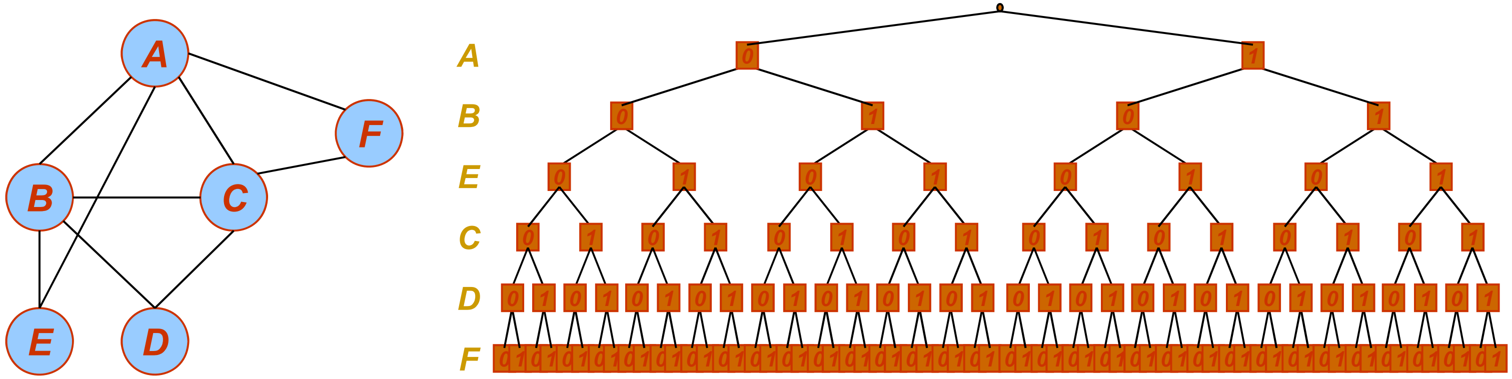
Trees are processed in linear time and memory

Inference vs Conditioning-Search



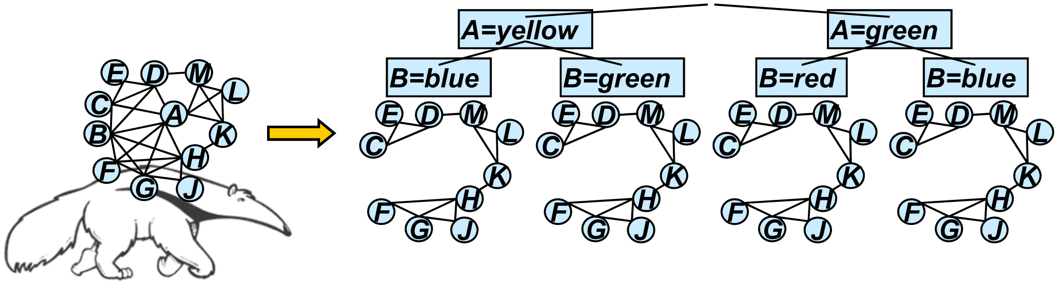
Inference

exp(w) time/space*



Search

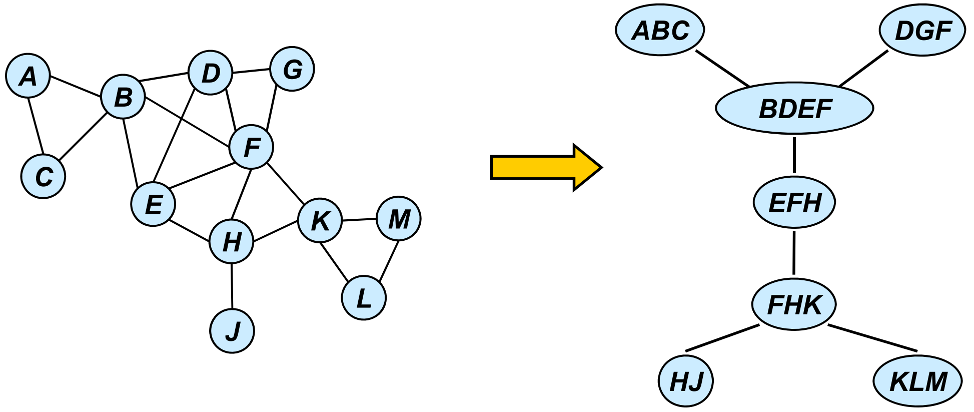
Exp(n) time
O(n) space



Search+inference:
Space: exp(w)
Time: exp(w+c(w))

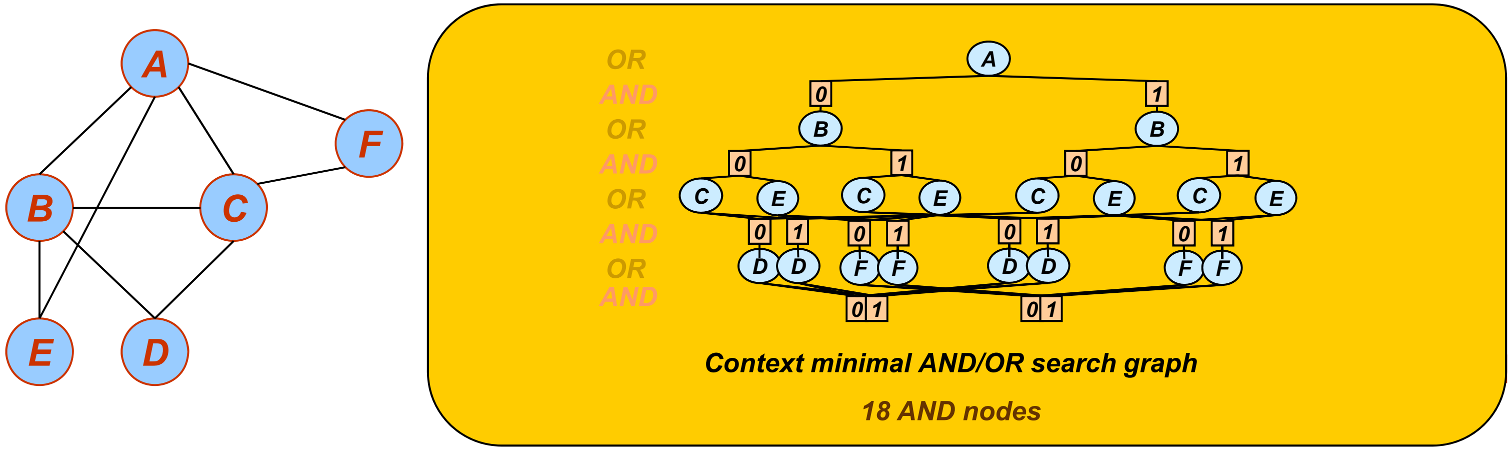
w: user controlled

Inference vs conditioning-search



Inference

exp(w*) time/space

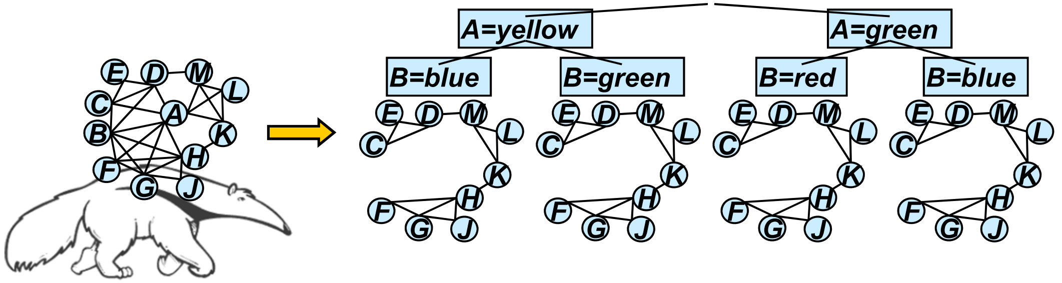


Context minimal AND/OR search graph

18 AND nodes

Search

Exp(w*) time
O(w*) space



Search+inference:
Space: exp(q)
Time: exp(q+c(q))

q: user controlled

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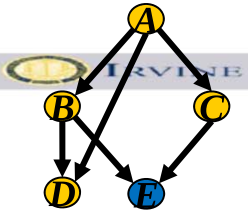
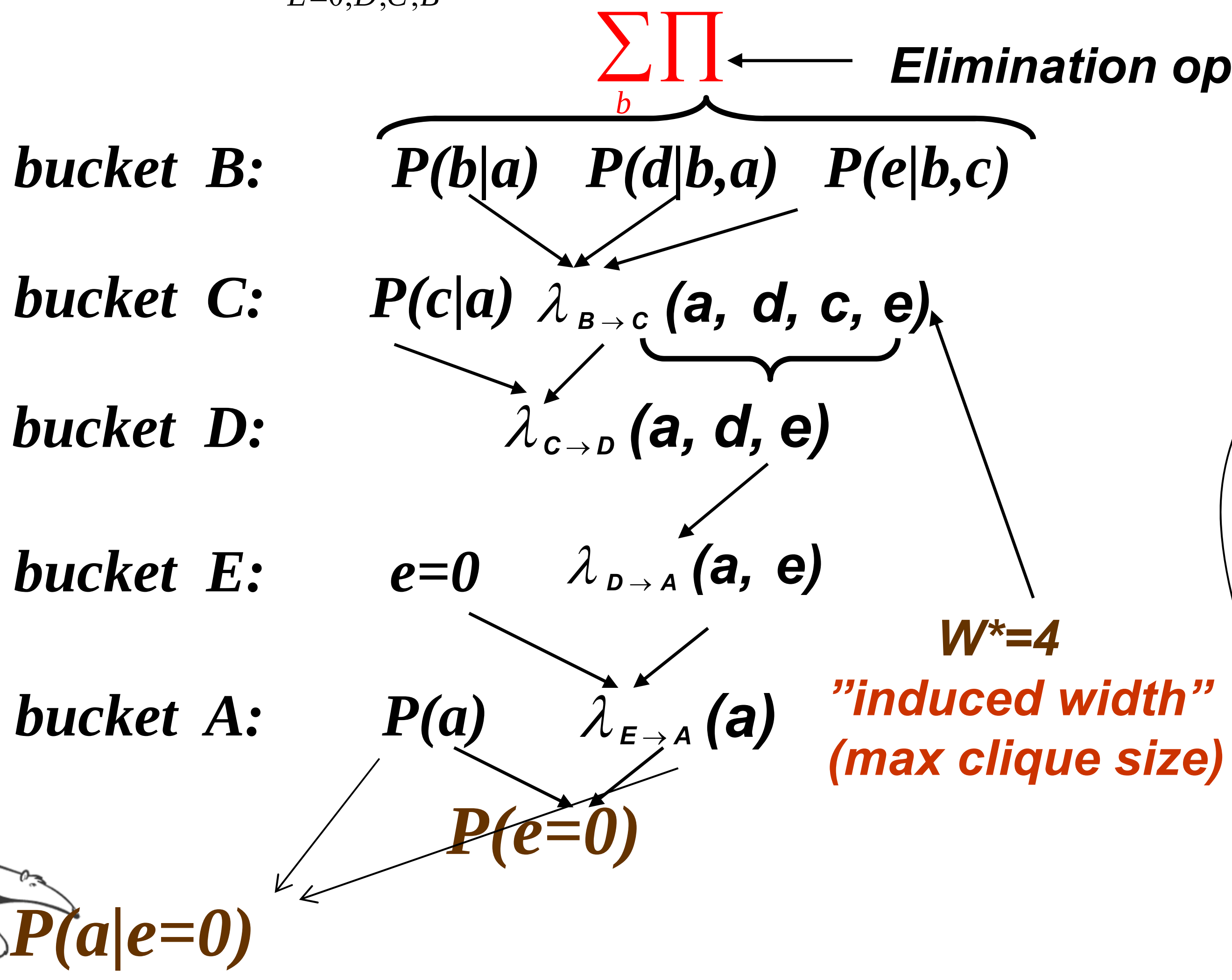
Likelihood queries: Inference (sum-product)



Finding Marginals by Bucket elimination

Algorithm *BE-bel* (Dechter 1996)

$$P(A|E=0)=\alpha\sum_{E=0,D,C,B} P(A)\cdot P(B|A)\cdot P(C|A)\cdot P(D|A,B)\cdot P(E|B,C)$$



$\sum_b \prod$ Elimination operator

bucket B: $\overbrace{P(b|a)\quad P(d|b,a)\quad P(e|b,c)}$

bucket C: $P(c|a)\quad \lambda_{B\rightarrow C}(a,\ \underbrace{d,\ c,\ e})$

bucket D: $\lambda_{C\rightarrow D}(a,\ d,\ e)$

bucket E: $e=0\quad \lambda_{D\rightarrow A}(a,\ e)$

bucket A: $P(a)\quad \lambda_{E\rightarrow A}(a)$

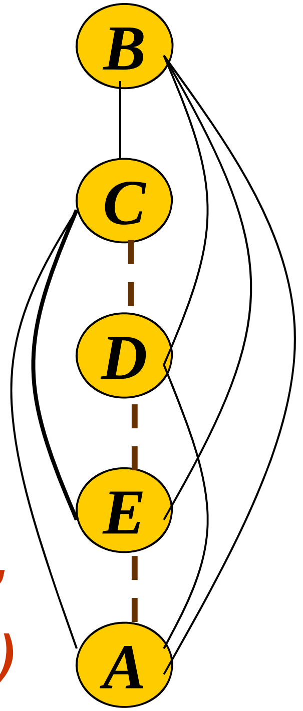
$P(e=0)$

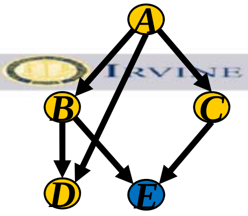
$P(a|e=0)$

W=4*

"induced width"

(max clique size)





Finding Marginals by Bucket elimination

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$$P(A|E=0)=\alpha\sum_{E=0,D,C,B}P(A)\cdot P(B|A)\cdot P(C|A)\cdot P(D|A,B)\cdot P(E|B,C)$$

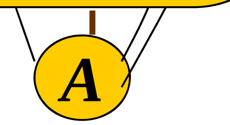
$\sum_b \prod$ ← ***Elimination operator***

Time and space exponential in the induced-width / treewidth

$$O(nk^{w*+1})$$

bucket A: $P(a)$ $\lambda_{E\to A}(a)$

induced width (max clique size)

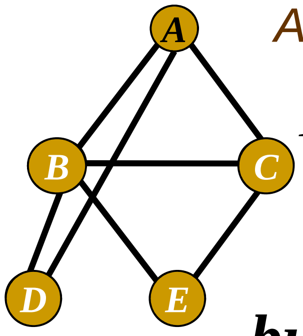


$P(e=0)$

$P(a|e=0)$

Optimization queries: Inference

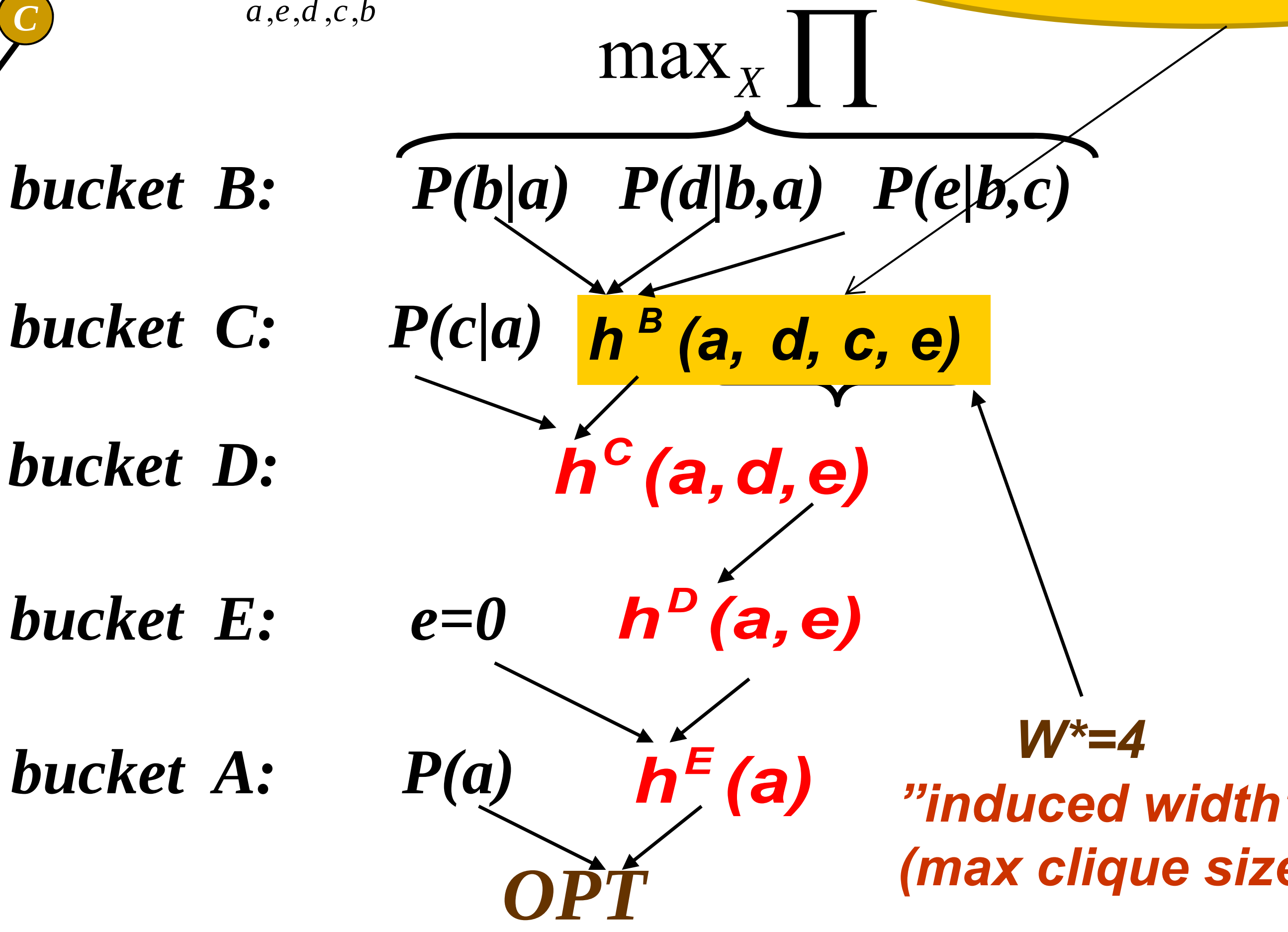
Finding MAP by Bucket Elimination



Algorithm BE-mpe (Dechter 199

$$= \max_b P(b|a)\cdot P(d|b,a)\cdot P(e|b,c)$$

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)$$



$$\overbrace{\max_X \prod}$$

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $h^B(a,d,c,e)$

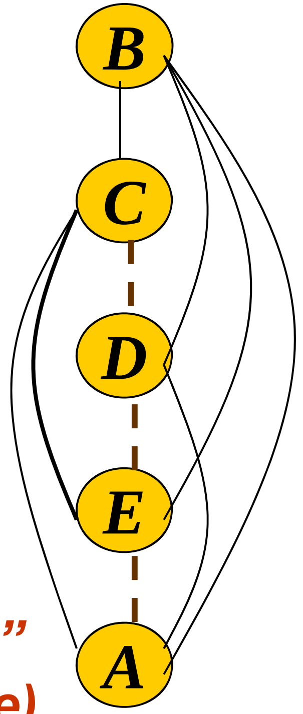
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $h^D(a,e)$

bucket A: $P(a)$ $h^E(a)$

OPT

W=4*
"induced width"
(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_{b} P(b \mid a') \times \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_{c} P(c \mid a') \times \times h^{B}(a', d', c, e')$

3. $d' = \arg\max_{d} h^{C}(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_{a} P(a) \cdot h^{E}(a)$

B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

C: $P(c|a)$ $h^{B}(a, d, c, e)$

D: $h^{C}(a, d, e)$

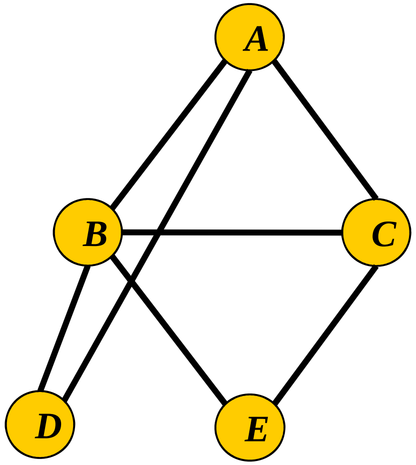
E: $e=0$ $h^{D}(a, e)$

A: $P(a)$ $h^{E}(a)$

Return (a', b', c', d', e')

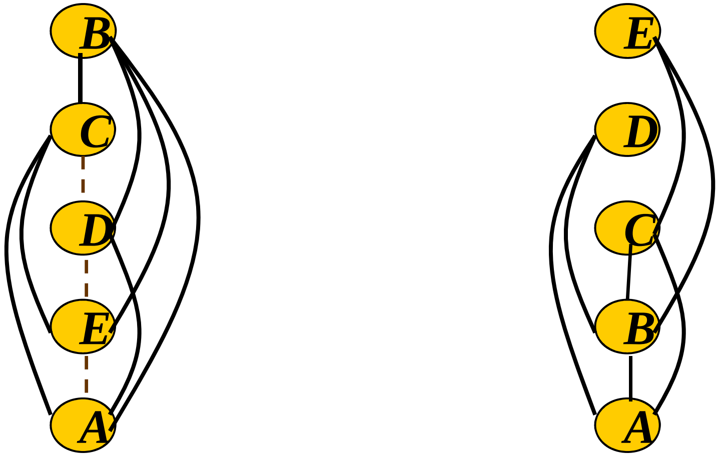
The Induced-Width/Treewidth

$w^*(d)$ – the induced width of graph along ordering d



"Moral" graph

The effect of the ordering:

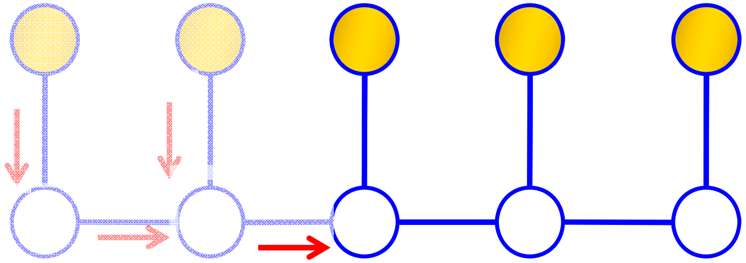


$w^*(d_1) = 4$ $w^*(d_2) = 2$

Marginal Map: Inference

Variable Elimination (max-sum-product)

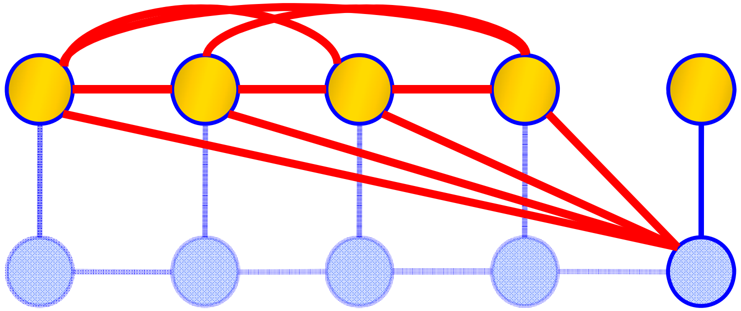
- Pure MAP or summation tasks
  - □ Dynamic programming
  - □ Ex: efficient on trees



- Marginal MAP
  - □ Operations do not commute:

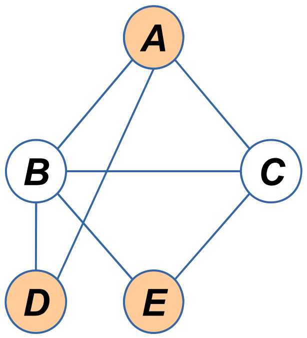
$$\sum \max \neq \max \sum$$

  - □ Sum must be done first!



Bucket Elimination for MMAP

Bucket Elimination

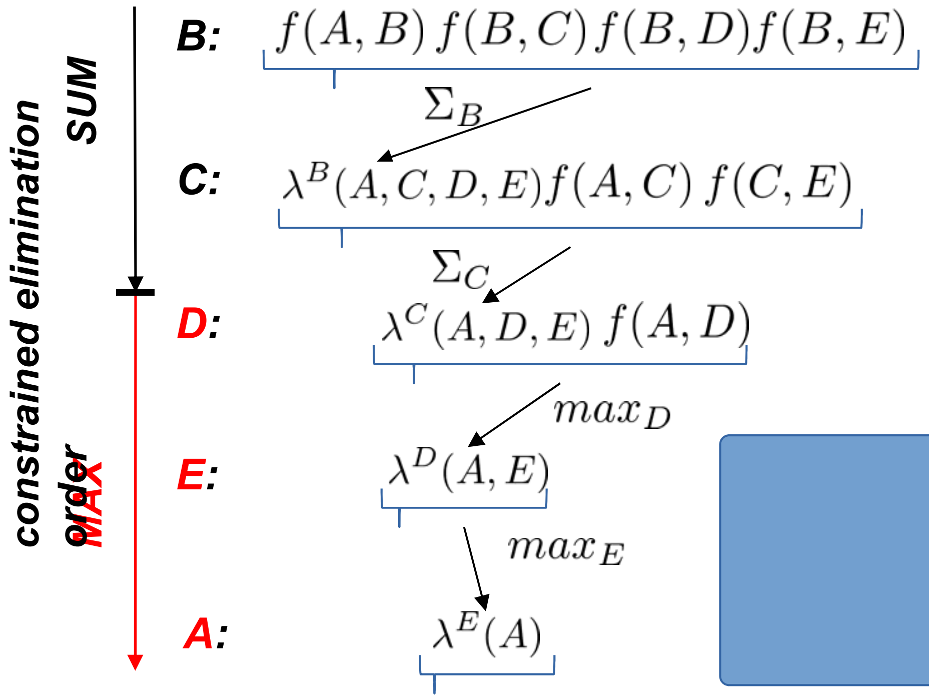


$$\mathbf{X}_M = \{A, D, E\}$$

$$\mathbf{X}_S = \{B, C\}$$

$$\max_{\mathbf{X}_M} \sum_{\mathbf{X}_S} P(\mathbf{X})$$

[Dechter, 1999]



B: $\underbrace{f(A,B) f(B,C) f(B,D) f(B,E)}$ $\xrightarrow{\Sigma_B}$

C: $\underbrace{\lambda^B(A,C,D,E) f(A,C) f(C,E)}$ $\xrightarrow{\Sigma_C}$

D: $\underbrace{\lambda^C(A,D,E) f(A,D)}$ $\xrightarrow{max_D}$

E: $\underbrace{\lambda^D(A,E)}$ $\xrightarrow{max_E}$

A: $\underbrace{\lambda^E(A)}$

MAP* is the marginal MAP value

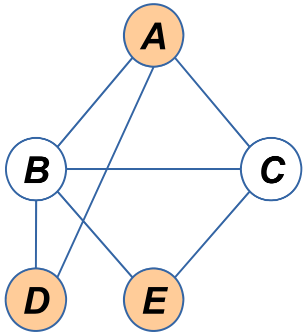
Bucket Elimination for MMAP

Bucket Elimination

$$\max_X \sum_Y \phi \leq \sum_Y \max_X \phi$$

exact

upper bound



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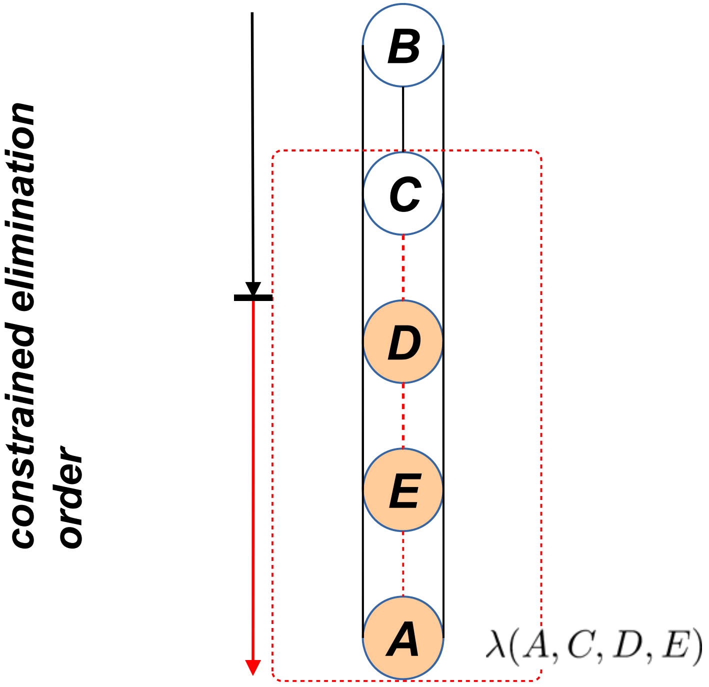
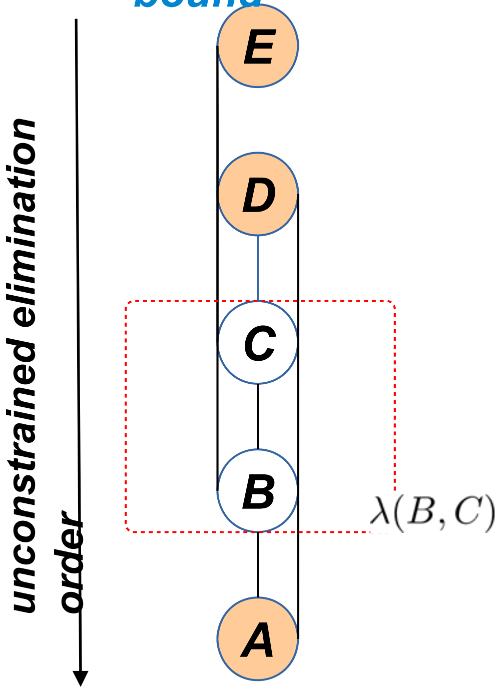


$w^* = 4$



$w^* = 2$

In practice, constrained induced is much larger!

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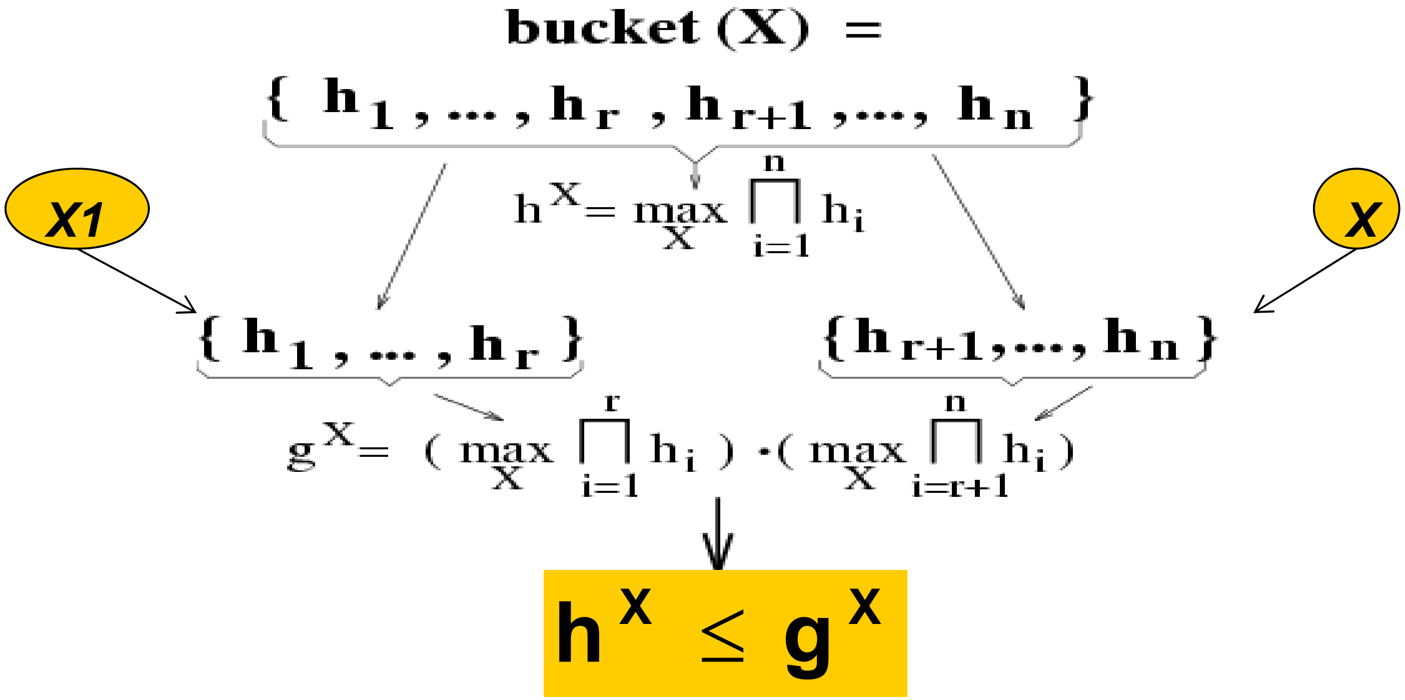
Mini-bucket Approximation: Relaxation

(Dechter and Rish, 1997, 2003)

Split a bucket into mini-buckets =>bound complexity



$$\text{bucket}(X) = \underbrace{\{ h_1, \ldots, h_r, h_{r+1}, \ldots, h_n \}}$$

$$h^X = \max_X \prod_{i=1}^{n} h_i$$

X1

X

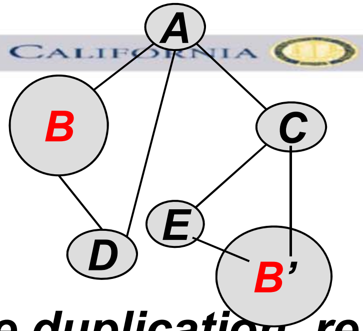
$$\underbrace{\{ h_1, \ldots, h_r \}} \qquad \underbrace{\{ h_{r+1}, \ldots, h_n \}}$$

$$g^X = \left(\max_X \prod_{i=1}^{r} h_i \right) \cdot \left(\max_X \prod_{i=r+1}^{n} h_i \right)$$

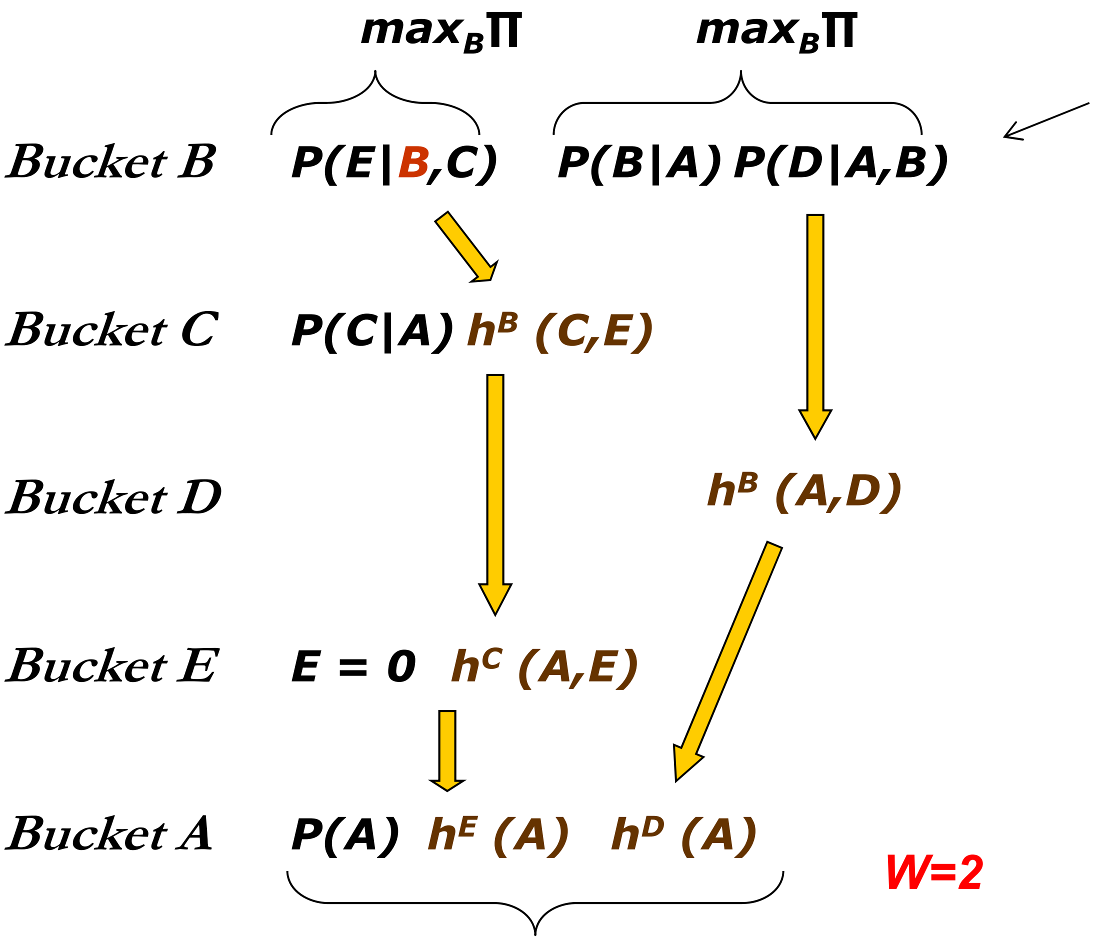
$$h^X \le g^X$$

Exponential complexity decrease : $O(e^n) \rightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Elimination



Node duplication, renaming



$\max_B \Pi$ $\max_B \Pi$

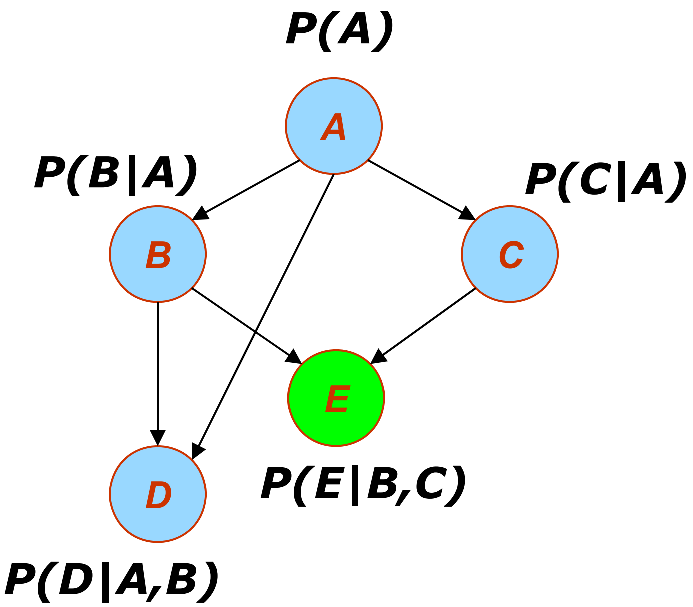
Bucket B **P(E|B,C) P(B|A) P(D|A,B)**

Bucket C **P(C|A) h^B (C,E)**

Bucket D **h^B (A,D)**

Bucket E **E = 0 h^C (A,E)**

Bucket A **P(A) h^E (A) h^D (A)**



W=2

MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

Mini-Bucket Decoding

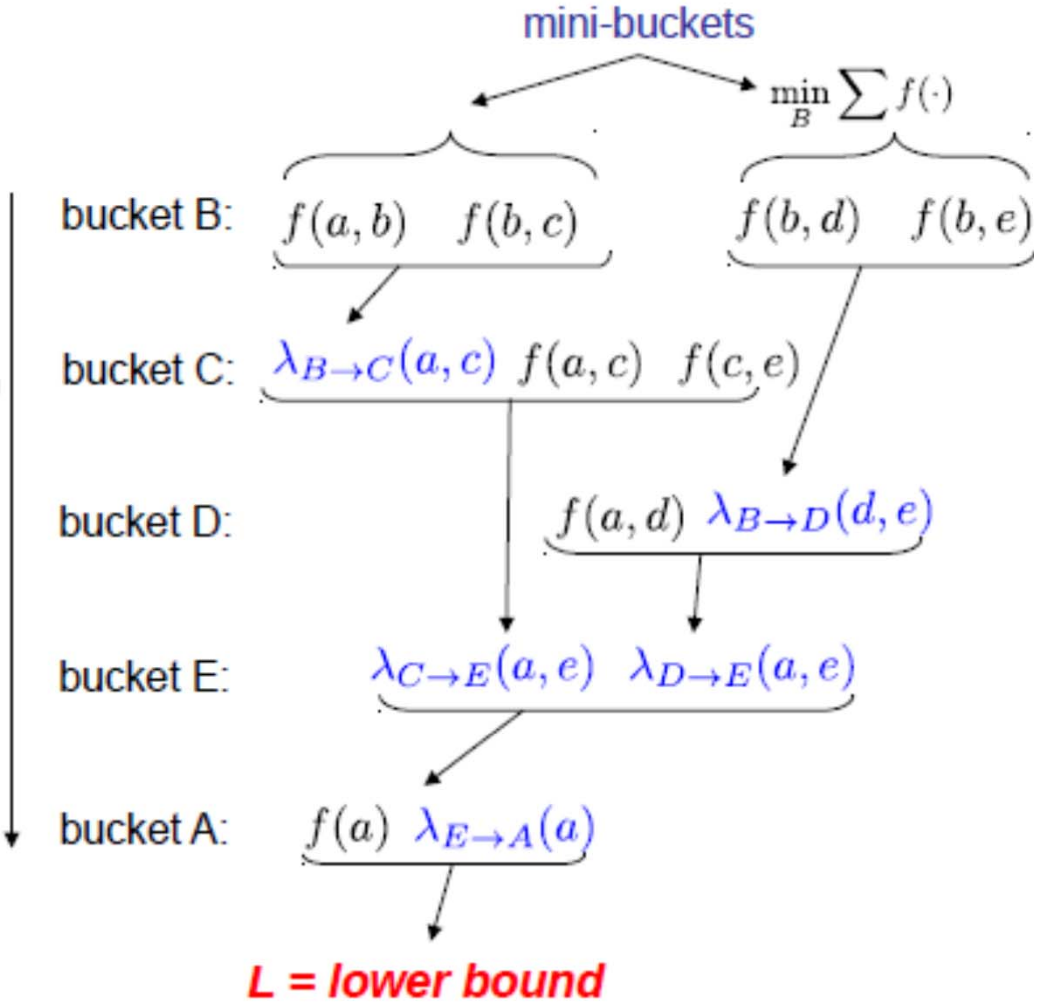
$$\hat{\mathbf{b}} = \arg\min_{\mathbf{b}} f(\hat{a},b) + f(b,\hat{c}) + f(b,\hat{d}) + f(b,\hat{e})$$

$$\hat{\mathbf{c}} = \arg\min_{\mathbf{c}} \lambda_{B\to C}(\hat{a},c) + f(c,\hat{a}) + f(c,\hat{e})$$

$$\hat{\mathbf{d}} = \arg\min_{\mathbf{d}} f(\hat{a},d) + \lambda_{B\to D}(d,\hat{e})$$

$$\hat{\mathbf{e}} = \arg\min_{\mathbf{e}} \lambda_{C\to E}(\hat{a},e) + \lambda_{D\to E}(\hat{a},e)$$

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} f(a) + \lambda_{E\to A}(a)$$



mini-buckets

$\min_B \sum f(\cdot)$

bucket B: $f(a,b)$ $f(b,c)$ $f(b,d)$ $f(b,e)$

bucket C: $\lambda_{B\to C}(a,c)$ $f(a,c)$ $f(c,e)$

bucket D: $f(a,d)$ $\lambda_{B\to D}(d,e)$

bucket E: $\lambda_{C\to E}(a,e)$ $\lambda_{D\to E}(a,e)$

bucket A: $f(a)$ $\lambda_{E\to A}(a)$

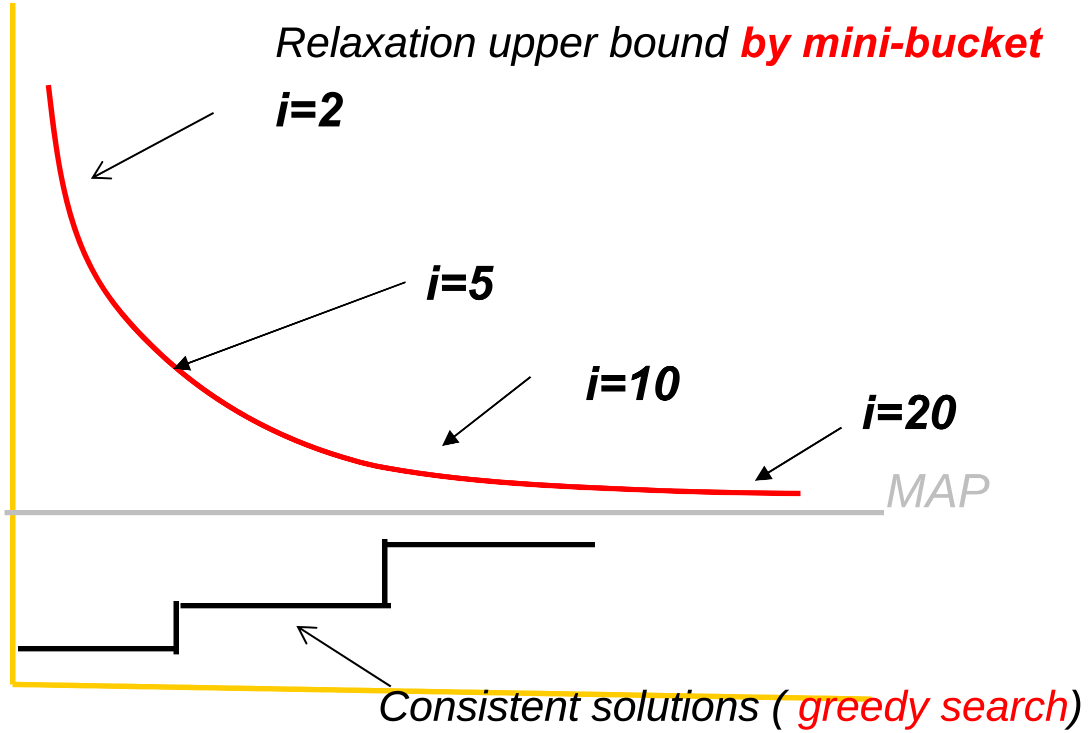
L = lower bound

Greedy configuration = upper bound

Properties of Mini-Bucket Eliminaton

- **Complexity:** *O(r exp(i))* time and *O(exp(i)) space*.
- **Accuracy:** determined by Upper/Lower bound.
- As ***i*** increases, both accuracy and complexity increase.
- Possible use of mini-bucket approximations:
  - As anytime algorithms
  - As heuristics in search

Bounding from above and below


Relaxation upper bound by mini-bucket
i=2
i=5
i=10
i=20
MAP
Consistent solutions (greedy search)

Relaxation provides upper bound
Any configuration: lower bound

Outline

- Graphical models, Queries, Algorithms
- Inference Algorithms: bucket-elimination
- Bounded Inference: a) mini-bucket, b) cost-shifting
- AND/OR search spaces and AND/OR BnB
- Evaluation, Software
- Conclusions

Cost-Shifting

(Reparameterization)

$+\lambda(B)$

A	B	f(A,B)
b	b	6 + 3
b	g	0 - 1
g	b	0 + 3
g	g	6 - 1

+

$-\lambda(B)$

B	C	f(B,C)
b	b	6 - 3
b	g	0 - 3
g	b	0 + 1
g	g	6 + 1

A	B	C	f(A,B,C)
b	b	b	12
b	b	g	6
b	g	b	0
b	g	g	6
g	b	b	6
g	b	g	0
g	g	b	6
g	g	g	12

= 0 + 6

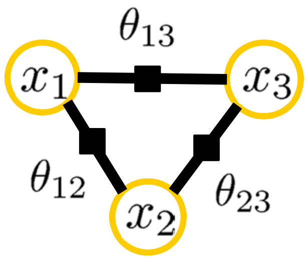
Modify the individual functions

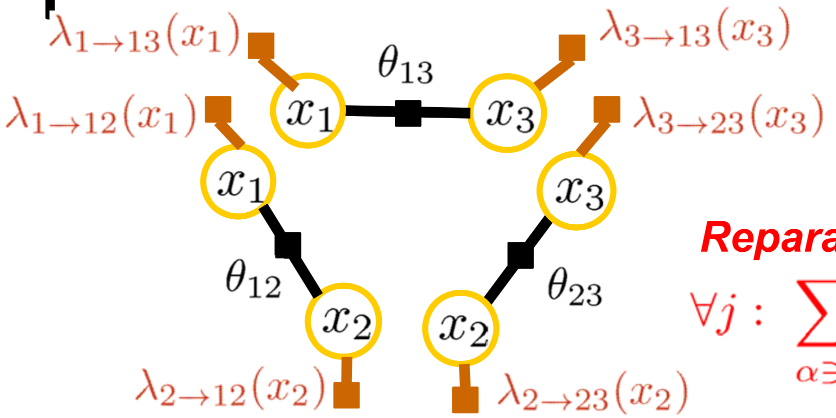
But

keep the sum of functions unchanged

B	$\lambda(B)$
b	3
g	-1

Ex: Dual Decomposition





Reparameterization:

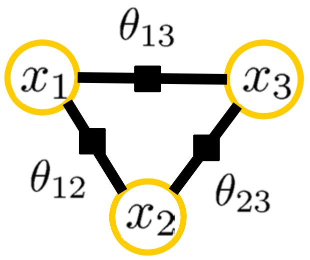
$$\forall j : \sum_{\alpha \ni j} \lambda_{j\to\alpha}(x_j) = 0$$

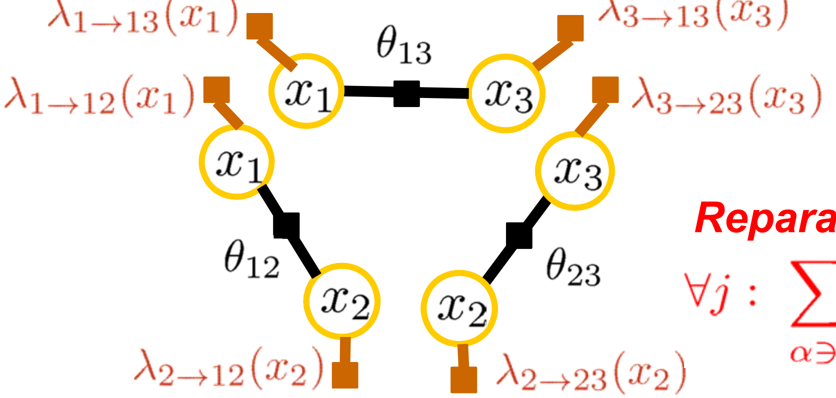
$$\log f(\mathbf{x}^*) = \max_{\mathbf{x}} \sum_{\alpha} \theta_\alpha(\mathbf{x}_\alpha) \quad \le \min_{\{\lambda_{i\to\alpha}\}} \sum_{\alpha} \max_{\mathbf{x}_\alpha} \left[\theta_\alpha(\mathbf{x}_\alpha) + \sum_{i\in\alpha} \lambda_{i\to\alpha}(x_i) \right]$$

- Bound solution using decomposed optimization
- Solve independently: optimistic bound
- Exact if all copies agree
- Tighten the bound by reparameterization
  - Enforces lost equality constraints using Lagrange multipliers

Add factors that "adjust" each local term, but cancel out in total

Ex: Dual Decomposition





Reparameterization:

$$\forall j : \sum_{\alpha \ni j} \lambda_{j\to\alpha}(x_j) = 0$$

$$\log f(\mathbf{x}^*) = \max_{\mathbf{x}} \sum_{\alpha} \theta_\alpha(\mathbf{x}_\alpha) \quad \leq \min_{\{\lambda_{i\to\alpha}\}} \sum_{\alpha} \max_{\mathbf{x}_\alpha} \left[\theta_\alpha(\mathbf{x}_\alpha) + \sum_{i\in\alpha} \lambda_{i\to\alpha}(x_i) \right]$$

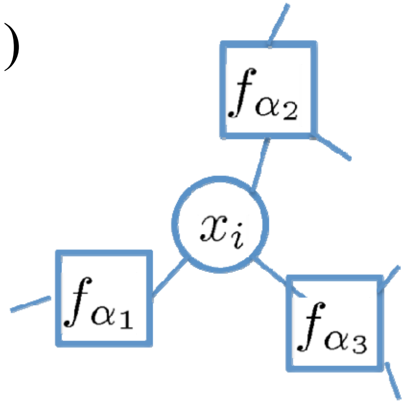
Many names for the same class of bounds:

- Dual decomposition [Komodakis et al. 2007]
- TRW, MPLP [Wainwright et al. 2005, Globerson & Jaakkola 2007]
- Soft arc consistency [Cooper & Schiex 2004]
- Max-sum diffusion [Warner 2007]

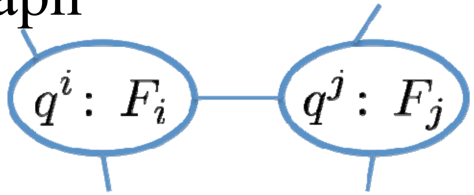
Various Update Schemes

- Can use any decomposition updates
  - (message passing, subgradient, augmented, etc.)

- FGLP: Update the original factors



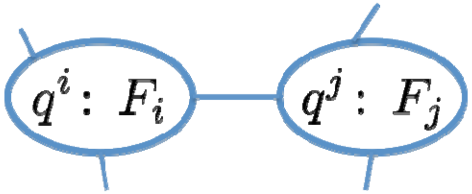
- JGLP: Update clique function of the join graph



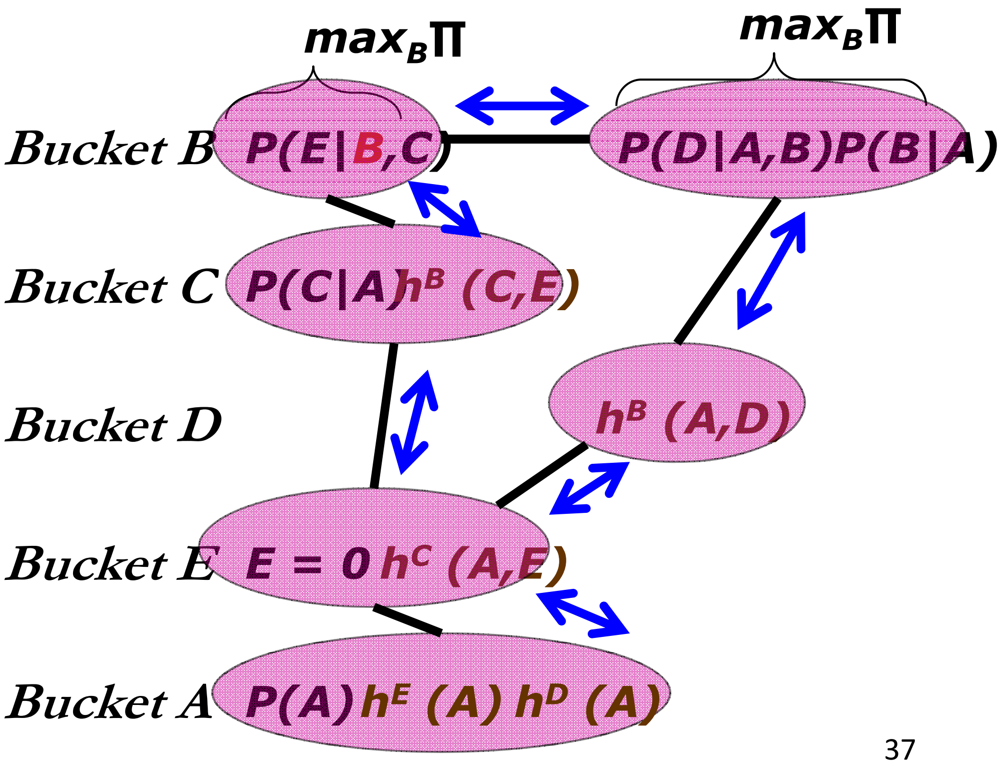
- MBE-MM Update within each bucket only

JGLP: Fixed-Point Updates

- **JGLP:** Update clique function of the join graph
  - □ Use MBE to generate the join graph
  - □ Define function F_i for each clique (mini-bucket) q^i
  - □ Update each edge over separator set

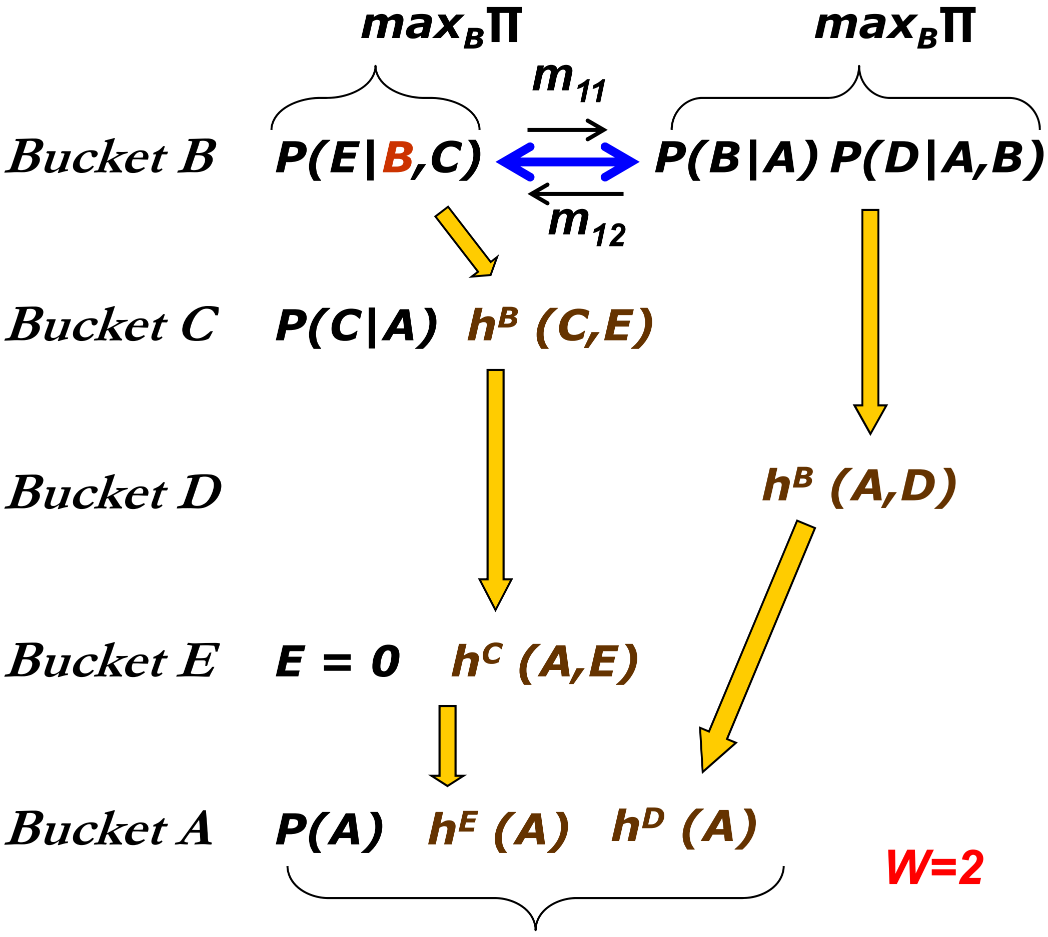


$$F_i \leftarrow F_i + \frac{1}{2}(\gamma_j(x_s) - \gamma_i(x_s))$$



MBE-MM: MBE with Moment Matching

m_{11},m_{12}- moment-matching messages



$\max_B \prod$ $\max_B \prod$

m_{11} m_{12}

Bucket B $\underbrace{P(E|B,C)}$ $\longleftrightarrow$ $\underbrace{P(B|A)\ P(D|A,B)}$

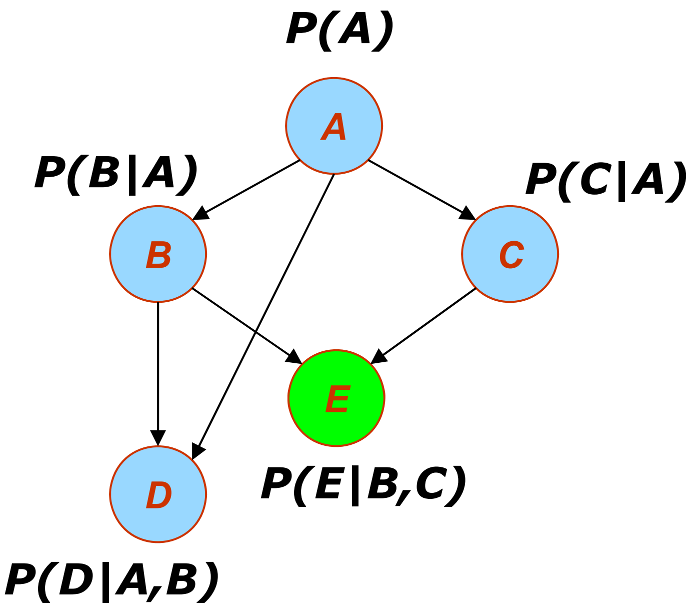
Bucket C $P(C|A)$ $h^B(C,E)$

Bucket D $h^B(A,D)$

Bucket E $E = 0$ $h^C(A,E)$

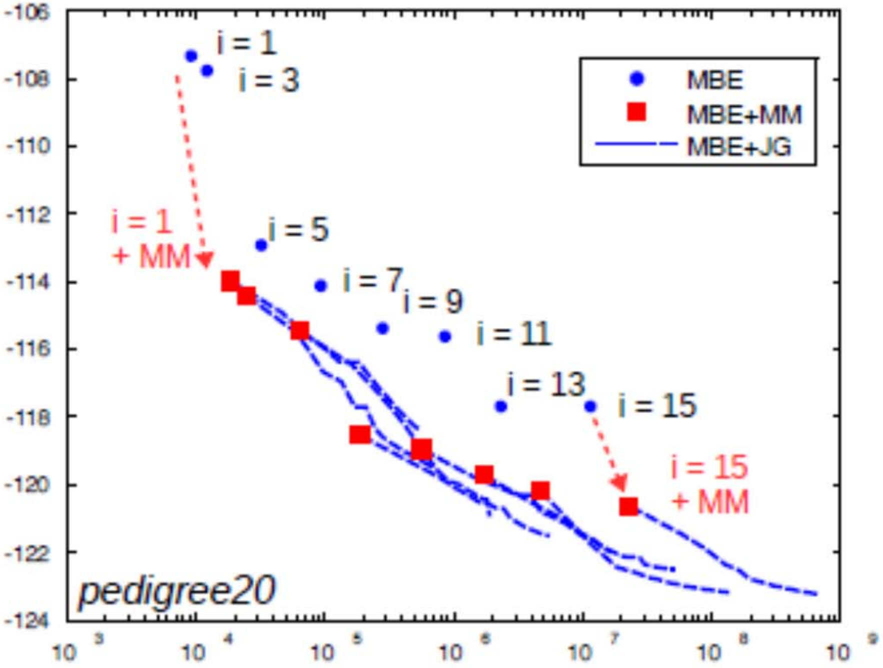
Bucket A $\underbrace{P(A)\ h^E(A)\ h^D(A)}$

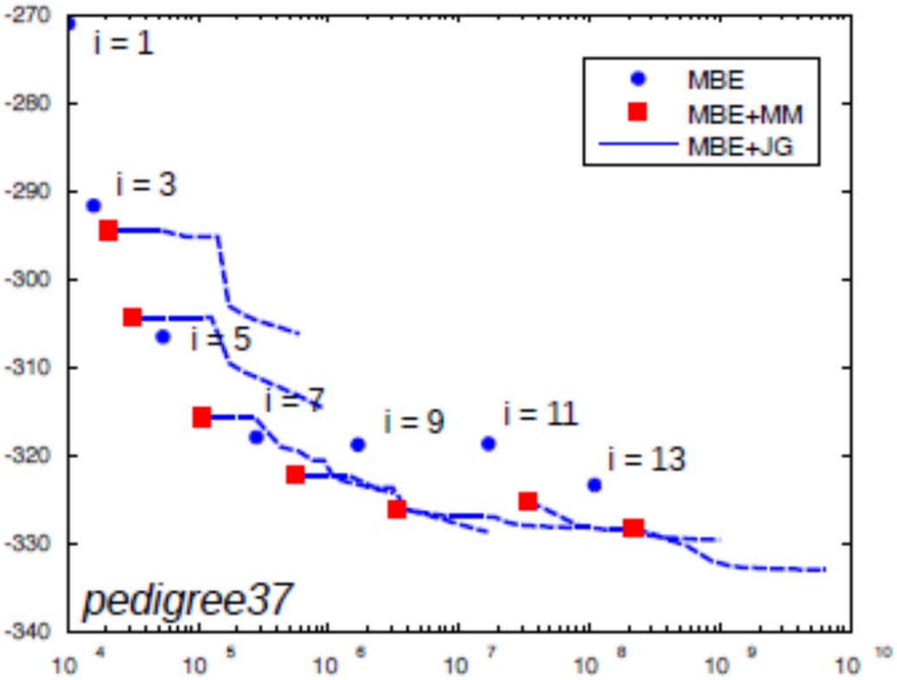
W=2



MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

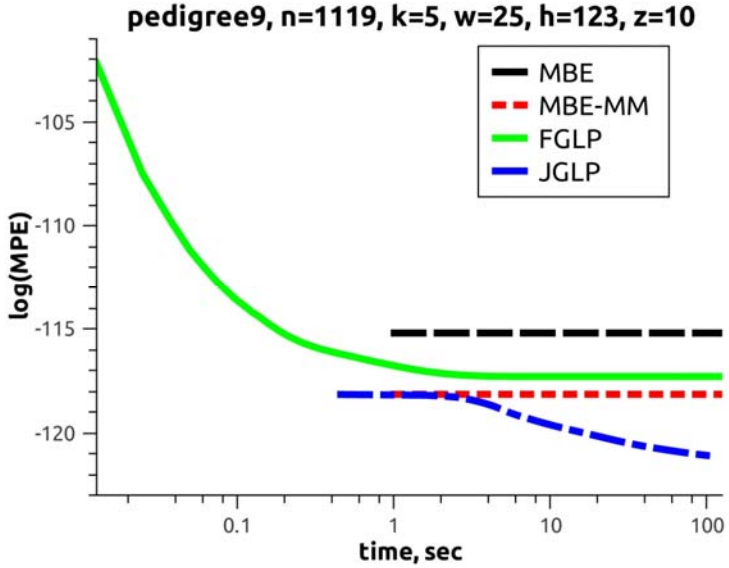
Anytime Approximation

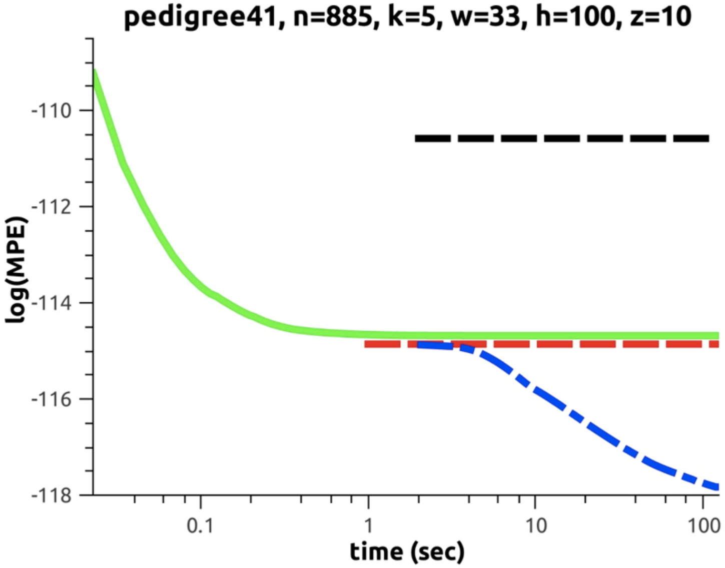


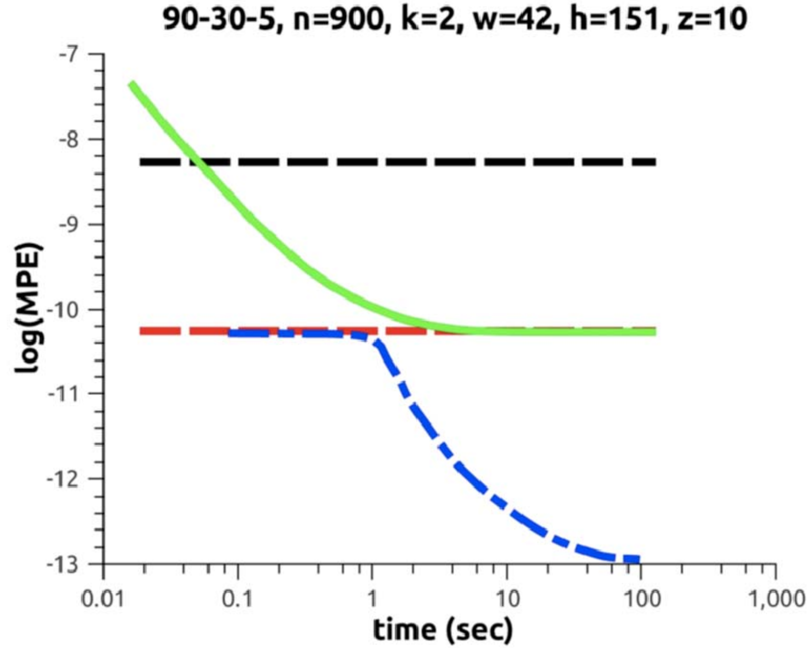


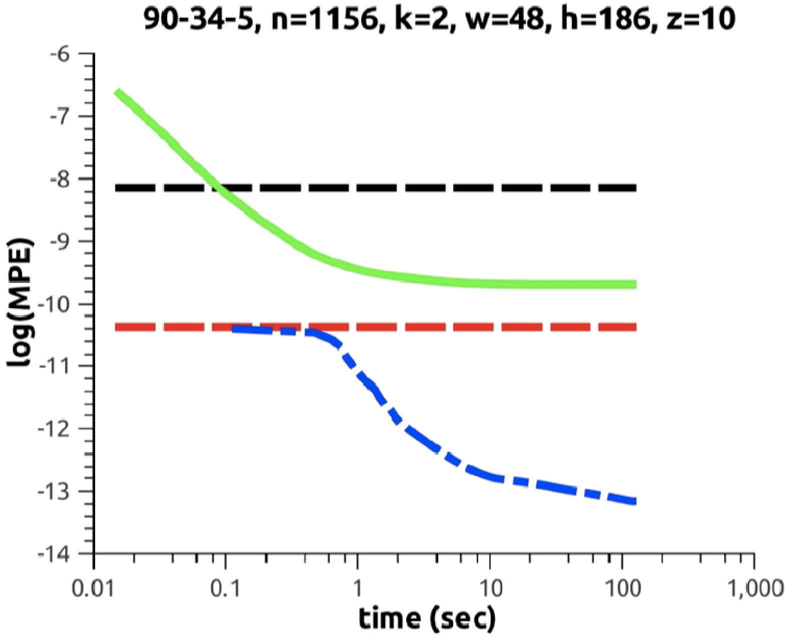
- Can tighten the bound in various ways
  - Cost-shifting (improve consistency between cliques)
  - Increase i-bound (higher order consistency)
- Simple moment-matching step improves bound significantly

Iterative Tightening as Bounding Schemes


pedigree9, n=1119, k=5, w=25, h=123, z=10


pedigree41, n=885, k=5, w=33, h=100, z=10


90-30-5, n=900, k=2, w=42, h=151, z=10

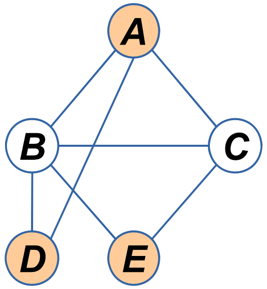

90-34-5, n=1156, k=2, w=48, h=186, z=10

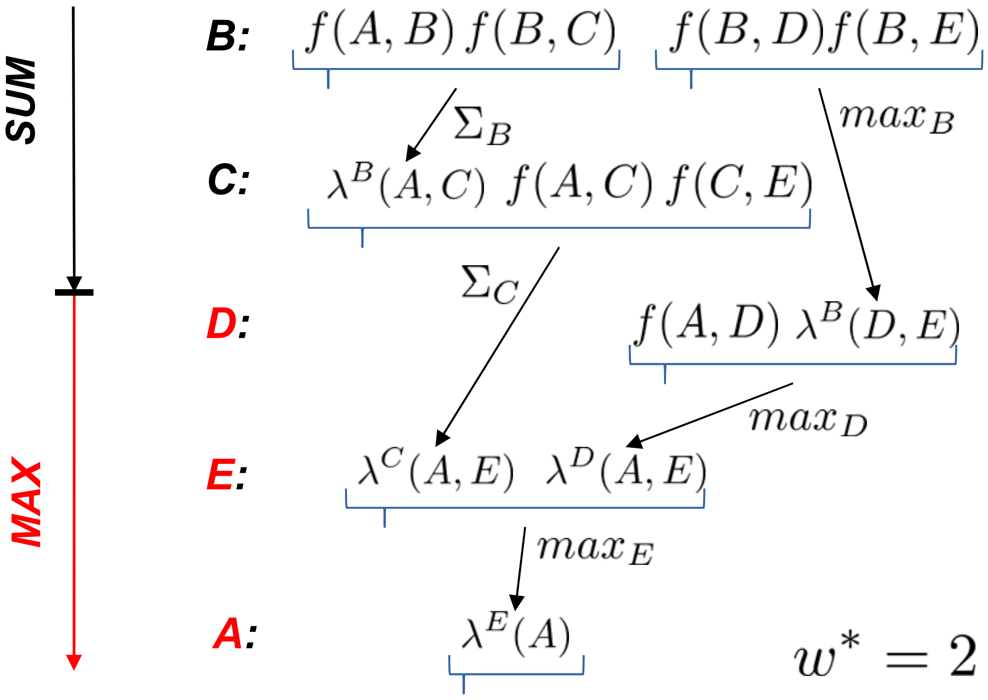
Bounded Inference for sum-product and max-sum-oproduct Inference:

Mini-bucket and weighted mini-bucket

Marginal (sum) and Marginal MAP (max-sum-product)

Partition a bucket into mini-buckets with i variables





B: $\underbrace{f(A,B)\,f(B,C)}$ $\underbrace{f(B,D)f(B,E)}$

Σ_B max_B

C: $\underbrace{\lambda^B(A,C)\ f(A,C)\,f(C,E)}$

Σ_C

D: $\underbrace{f(A,D)\ \lambda^B(D,E)}$

max_D

E: $\underbrace{\lambda^C(A,E)\ \ \lambda^D(A,E)}$

max_E

A: $\underbrace{\lambda^E(A)}$

$$w^* = 2$$

$$\mathbf{X}_M = \{A, D, E\}$$
$$\mathbf{X}_S = \{B, C\}$$

[Dechter and Rish, 2001]

MAP* is an upper bound on the marginal MAP value

Weighted Mini-Bucket

(for summation bounds)

Exact bucket elimination:

$$\lambda_B(a,c,d,e) = \sum_b \left[\, f(a,b)\cdot f(b,c)\cdot f(b,d)\cdot f(b,e) \,\right]$$

$$\leq \Big[\sum_b^{w_1} f(a,b)f(b,c)\Big] \cdot \Big[\sum_b^{w_2} f(b,d)f(b,e)\Big]$$

$$= \lambda_{B\to C}(a,c) \qquad \cdot\, \lambda_{B\to D}(d,e)$$

(mini-buckets)

where $\sum_x^{w} f(x) = \left[\sum_x f(x)^{1/w}\right]^w$

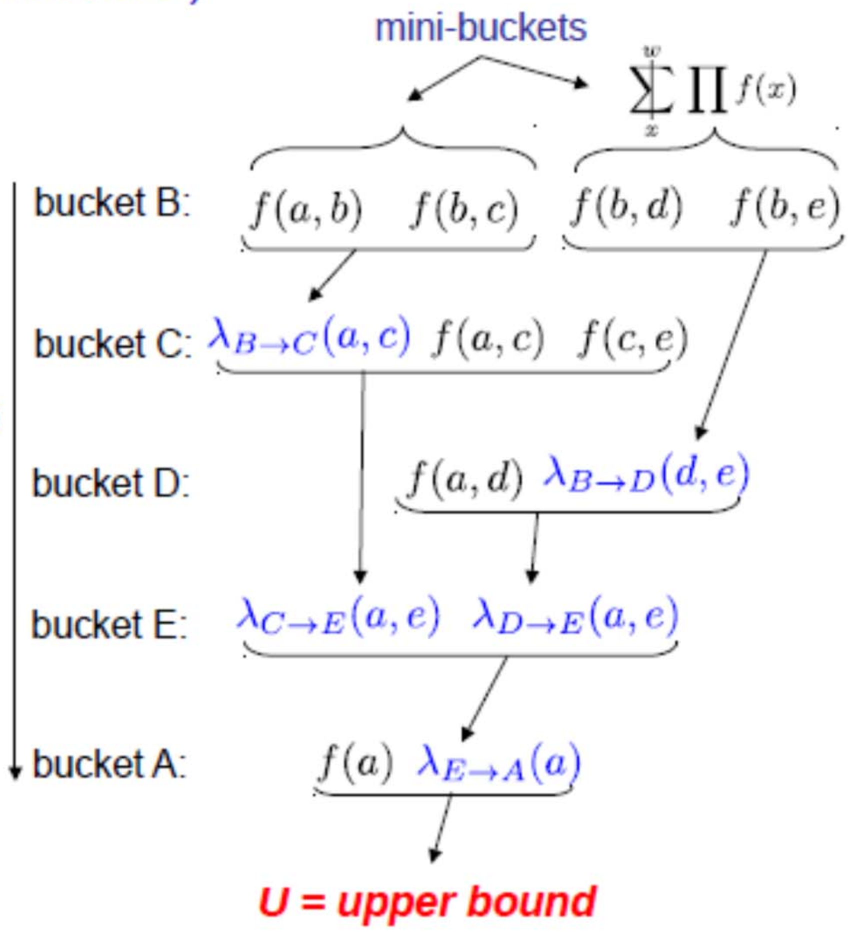
is the weighted or "power" sum operator

By Holder's inequality,

$$\sum_x^{w} f_1(x)f_2(x) \leq \left[\sum_x^{w_1} f_1(x)\right]\left[\sum_x^{w_2} f_2(x)\right]$$

where $w_1 + w_2 = w$ and $w_1 > 0,\ w_2 > 0$

(lower bound if $w_1 > 0,\ w_2 < 0$)



mini-buckets

$\sum_x^{w} \prod f(x)$

bucket B: $f(a,b)$ $f(b,c)$ $f(b,d)$ $f(b,e)$

bucket C: $\lambda_{B\to C}(a,c)$ $f(a,c)$ $f(c,e)$

bucket D: $f(a,d)$ $\lambda_{B\to D}(d,e)$

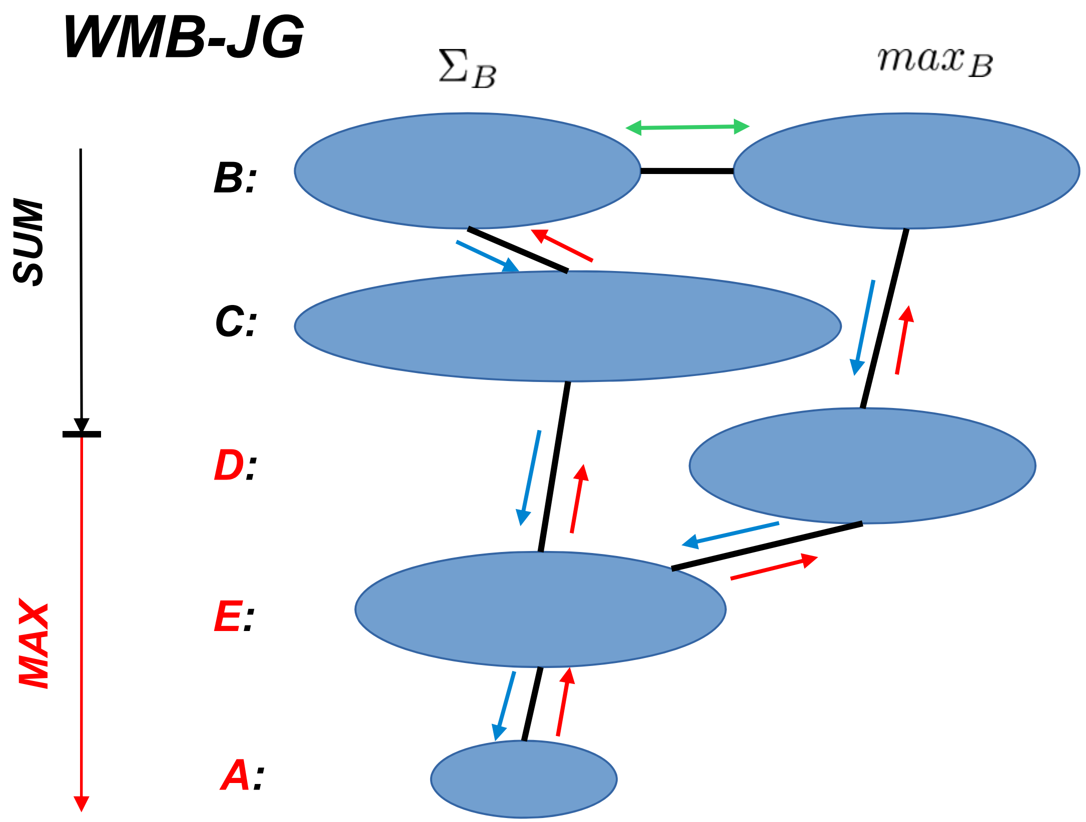
bucket E: $\lambda_{C\to E}(a,e)$ $\lambda_{D\to E}(a,e)$

bucket A: $f(a)$ $\lambda_{E\to A}(a)$

U = upper bound

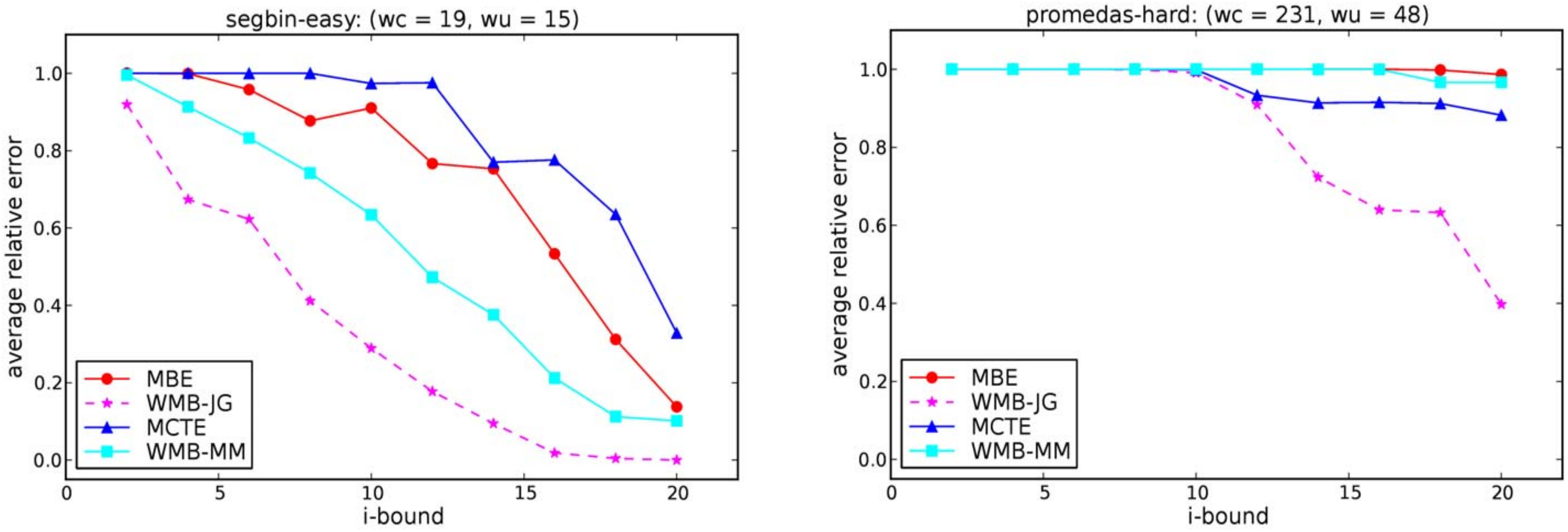
[Liu & Ihler 2011]

WMB on Join-Graphs



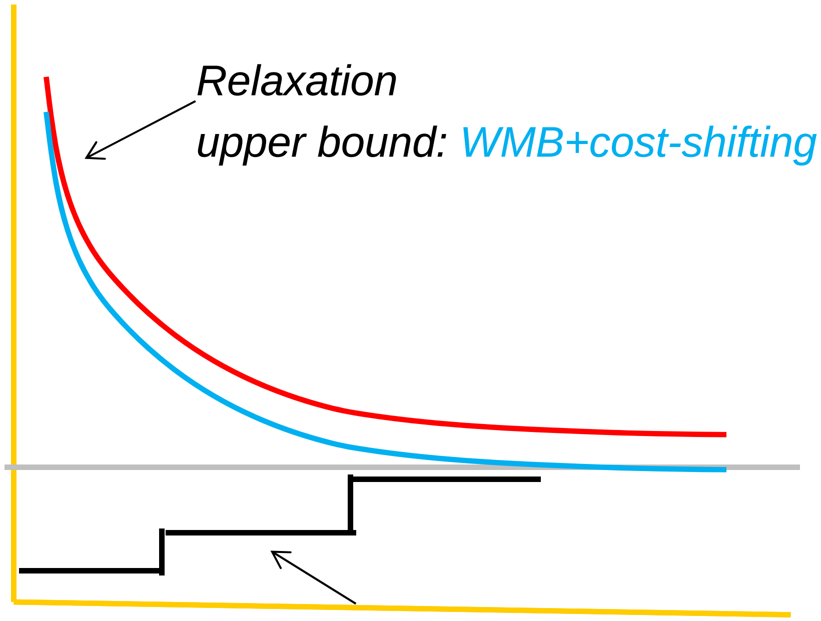
- ***WMB defines a join-graph***
- ***Propagate downward / upward messages until convergence***
- ***Downward messages***
  - ***e.g. from B to D***
- ***Upward messages***
  - ***e.g. from D to B***
  - ***used during cost-shifting***
- ***Cost-shifting within a bucket***
- ***In practice, yields a much tighter upper bound than WMB***

MMAP: Quality of the Upper Bounds



Average relative error wrt tightest upper bound. 10 iterations for WMB-JG(i).

Bounding from Above and Below



Relaxation
upper bound: WMB+cost-shifting

Marginal MAP

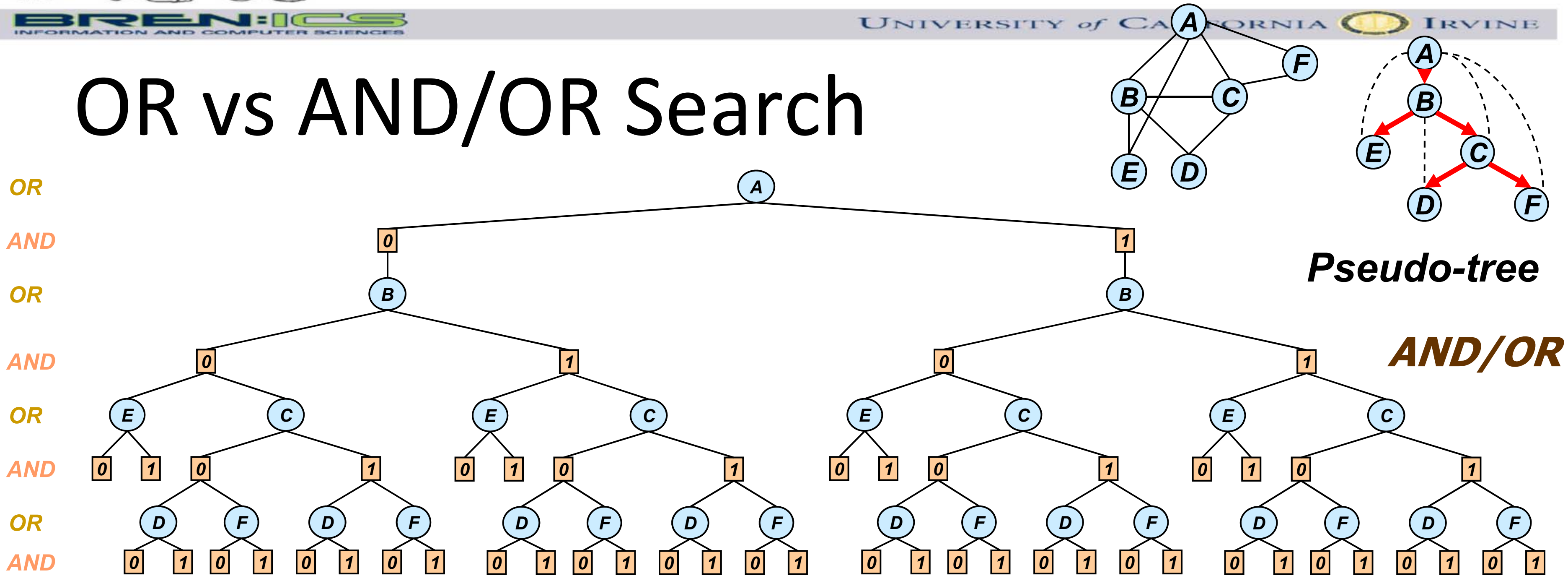
Consistent solutions (greedy search)

Relaxation provides upper bound
Any configuration: lower bound but NP-hard

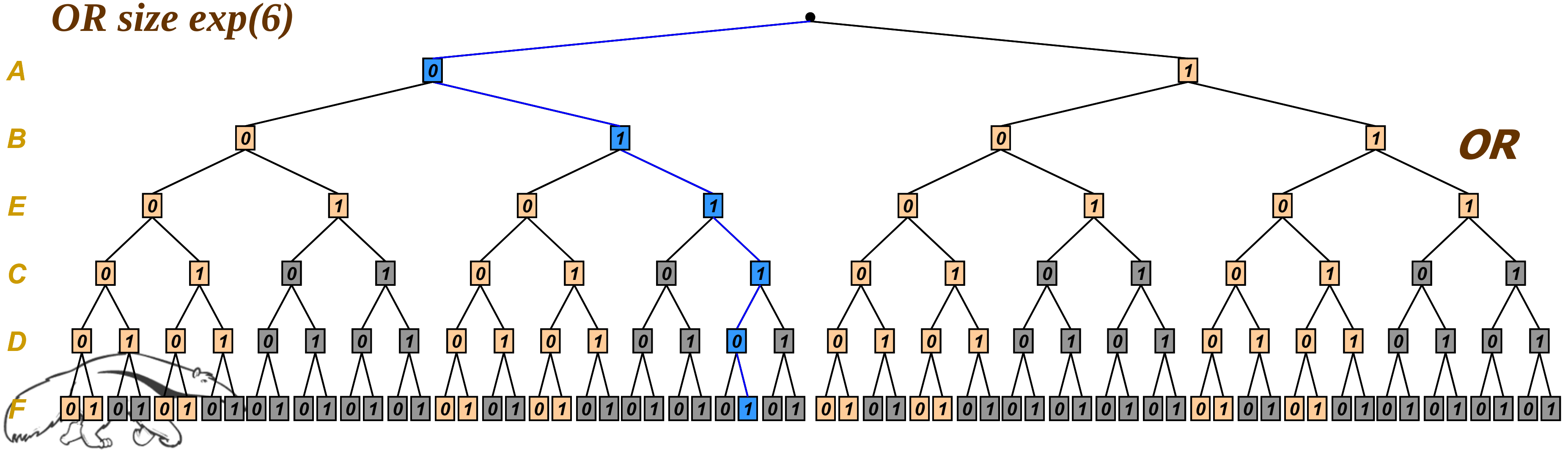
Outline

- Graphical models, Queries , Algorithms
- Inference Algorithms
- Bounded Inference: mini-bucket, cost-shifting
- AND/OR search spaces and AND/OR BnB
- Evaluation, Software
- Conclusions

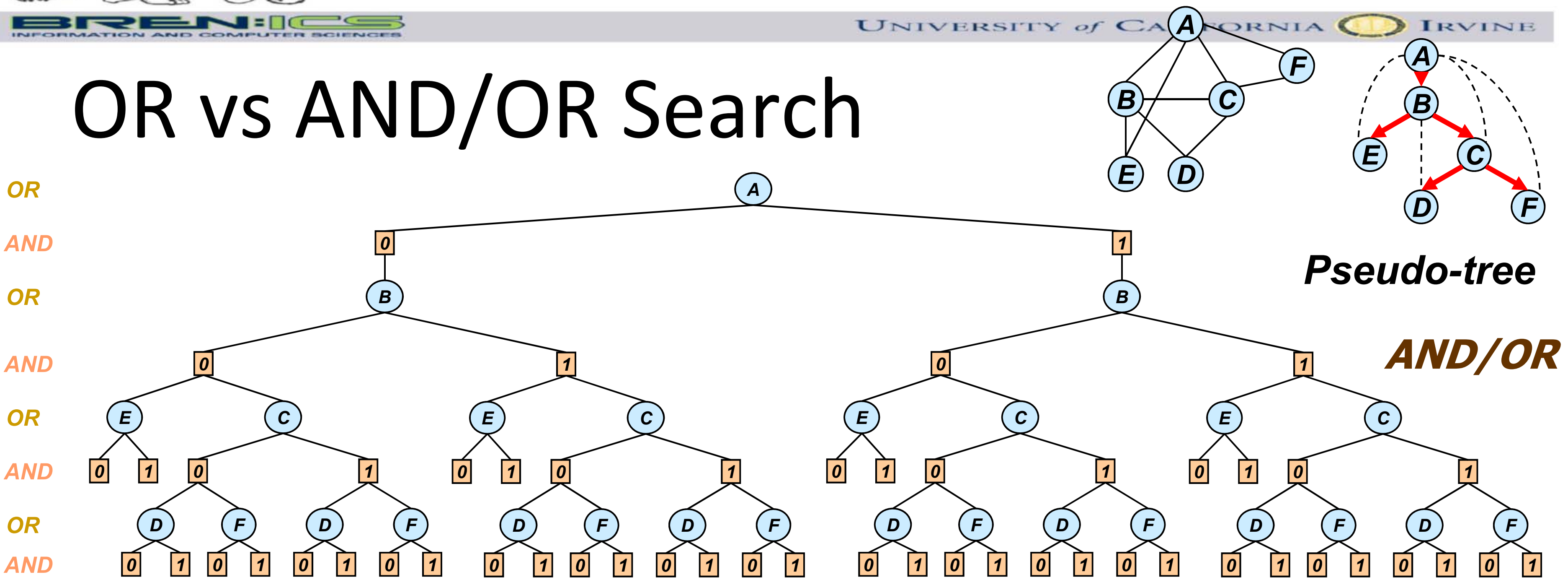
OR vs AND/OR Search



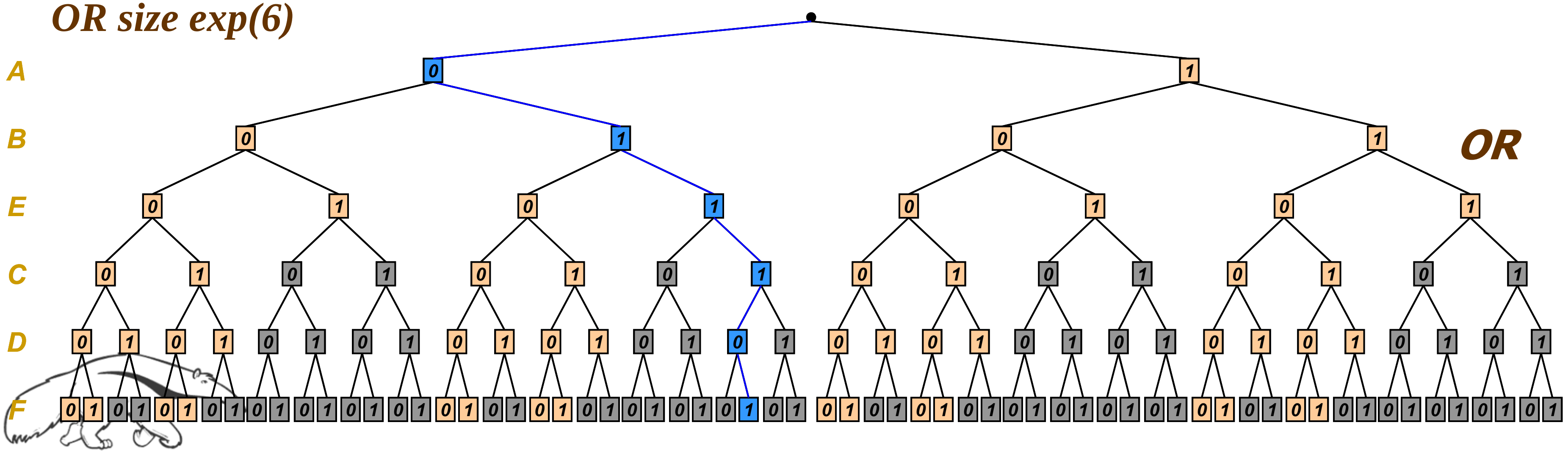
AND/OR size: exp(4),
OR size exp(6)



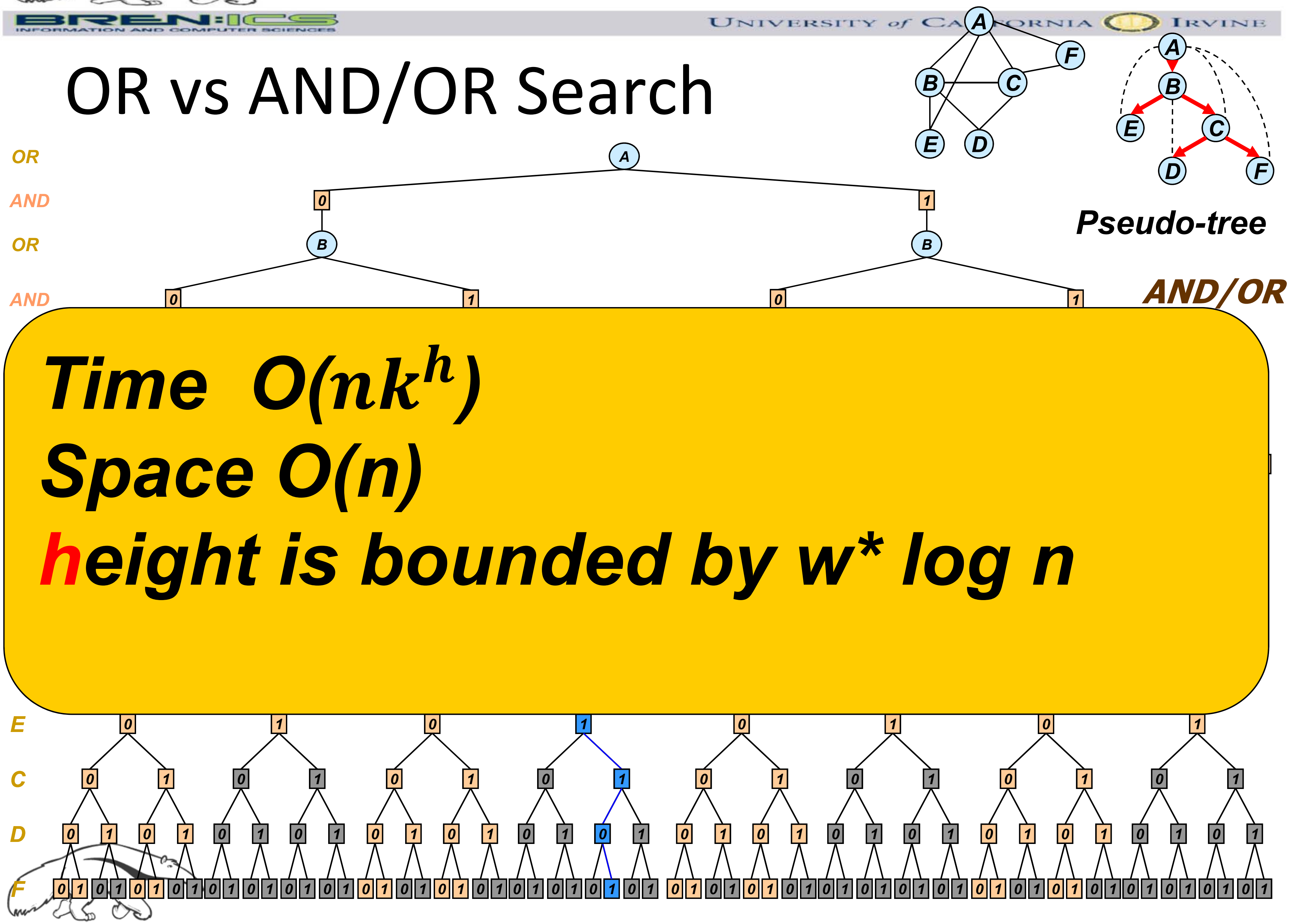
OR vs AND/OR Search



AND/OR size: exp(4),
OR size exp(6)



OR vs AND/OR Search

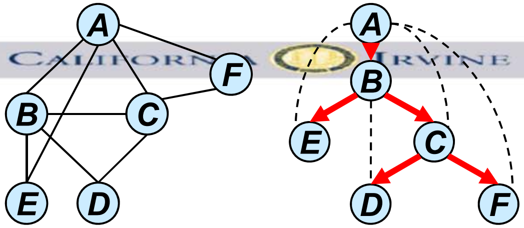


Time $O(nk^h)$

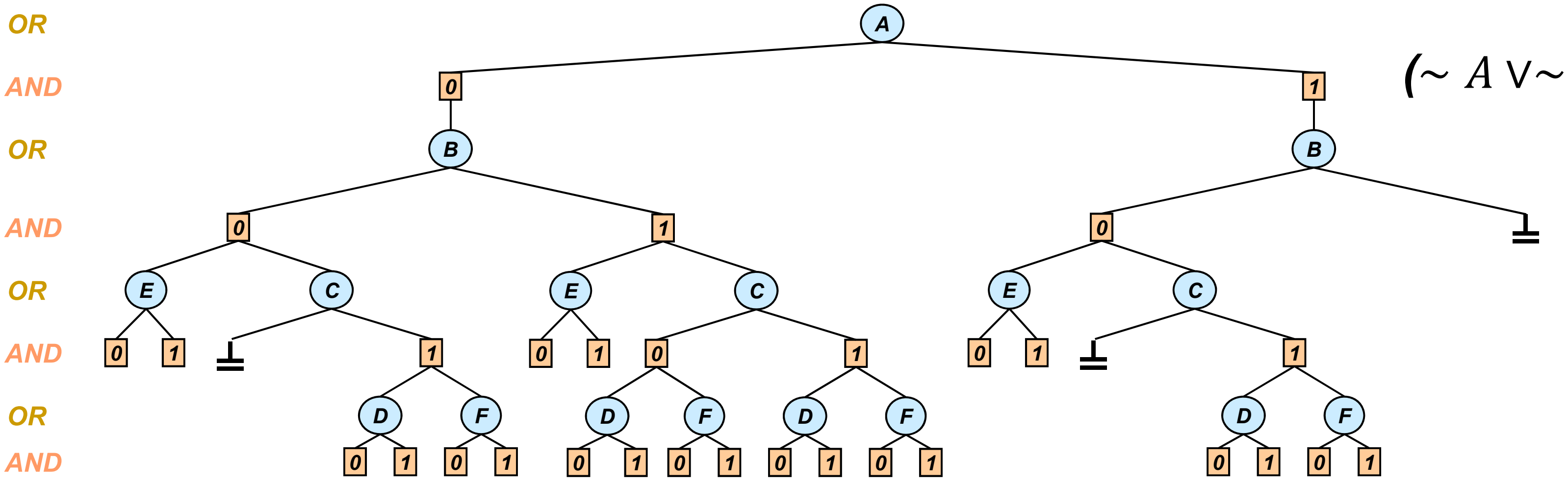
Space O(n)

height is bounded by w* log n

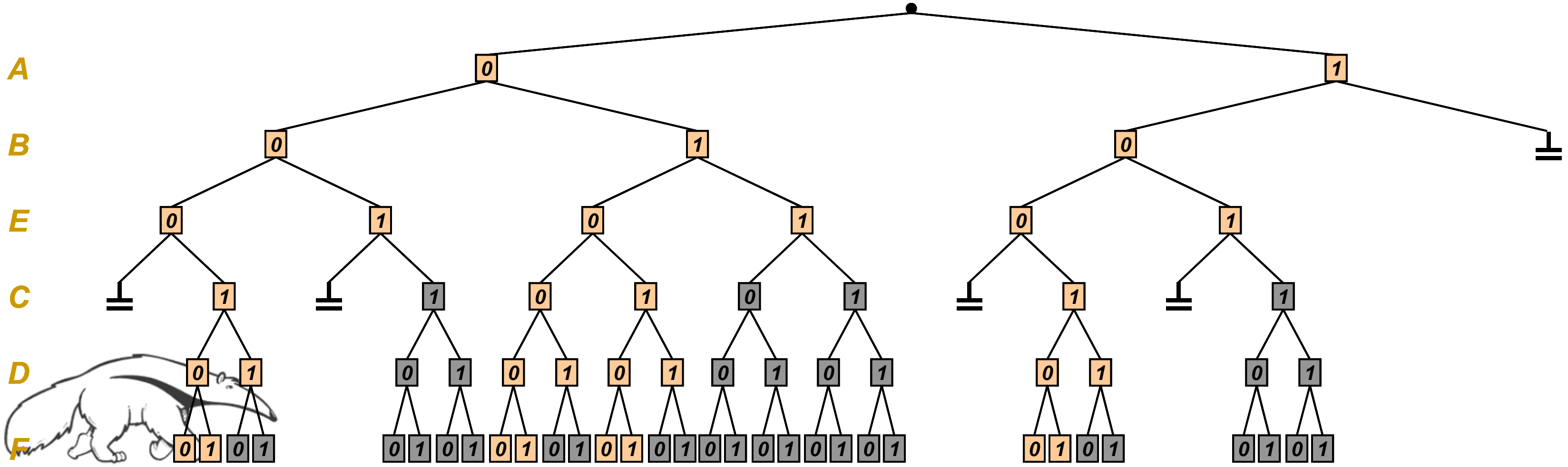
AND/OR vs. OR with Constraints



$(\sim A \vee \sim B) \wedge (B \vee C)$



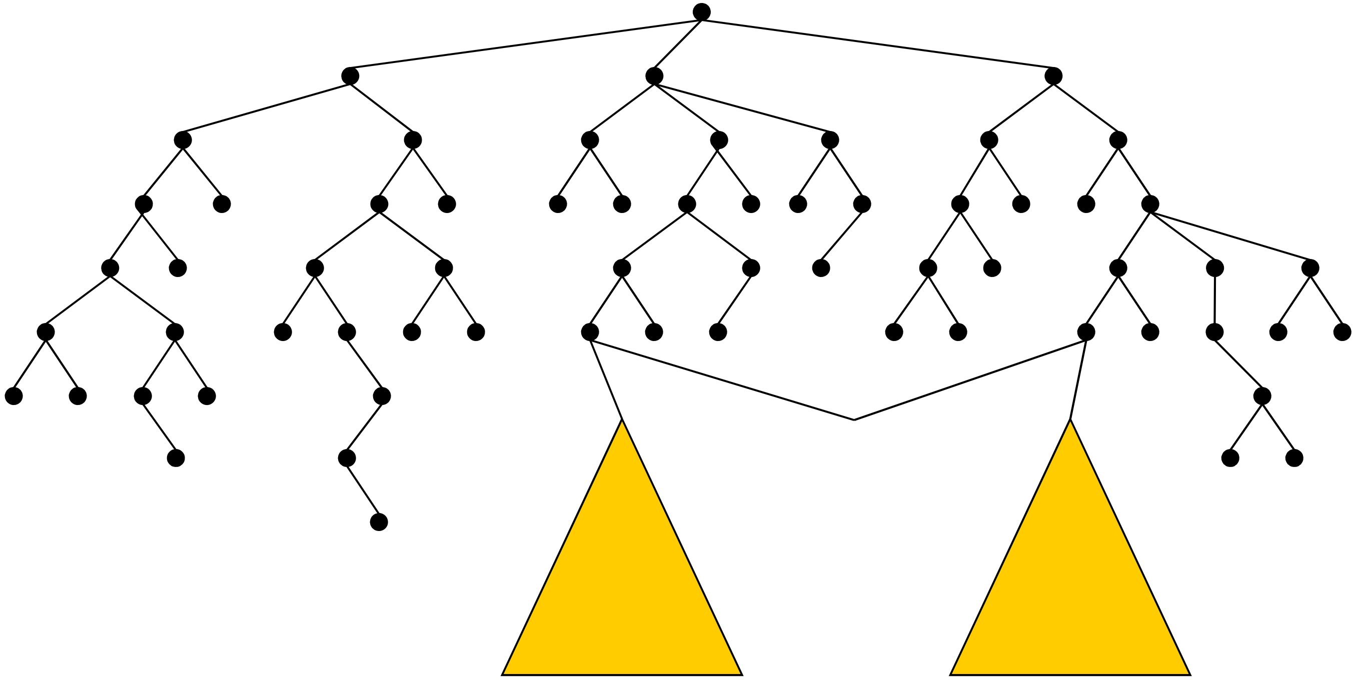
AND/OR



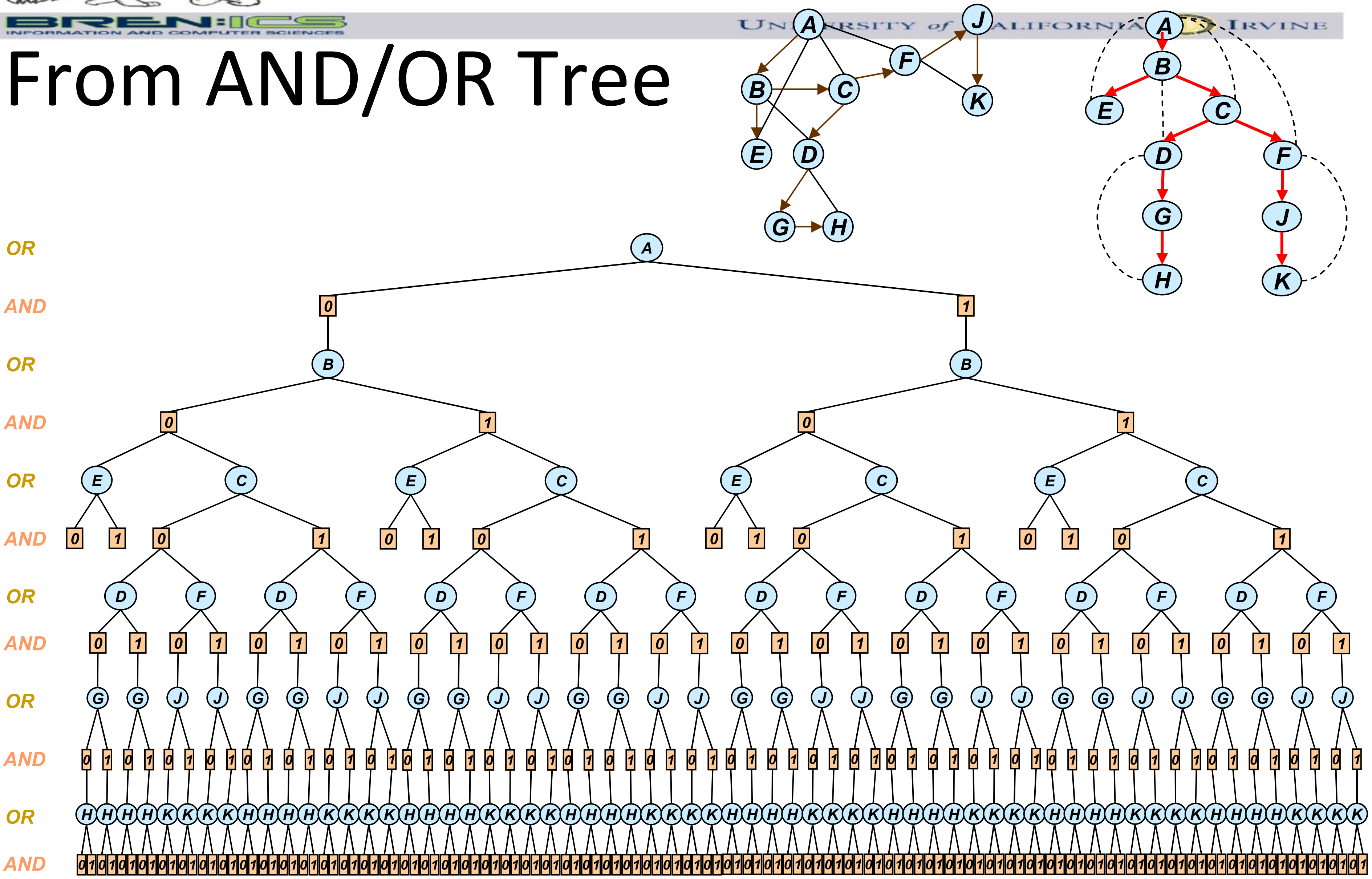
OR

From Search Trees to Search Graphs

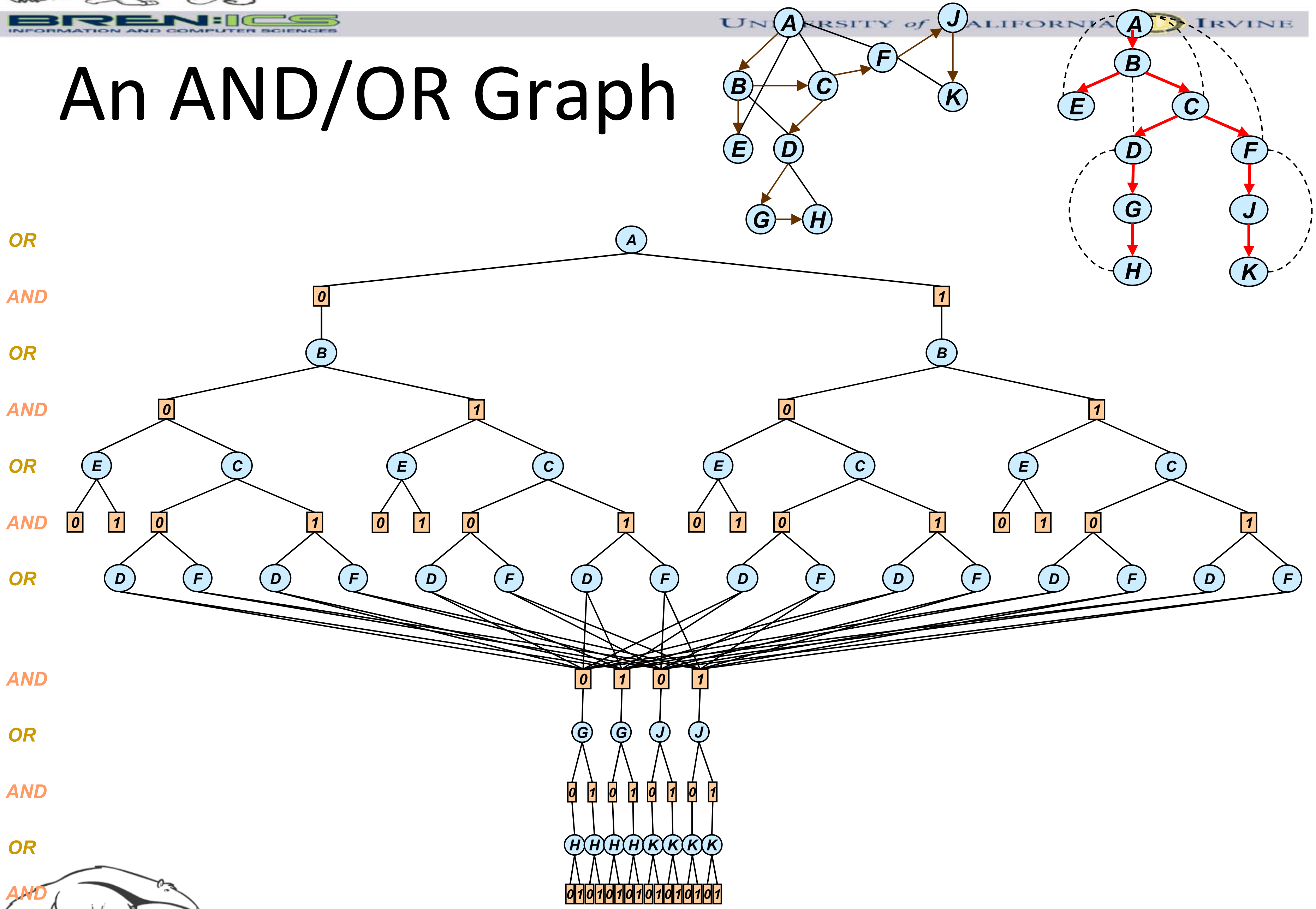
- Any two nodes that root identical sub-trees or sub-graphs can be merged



From AND/OR Tree



An AND/OR Graph

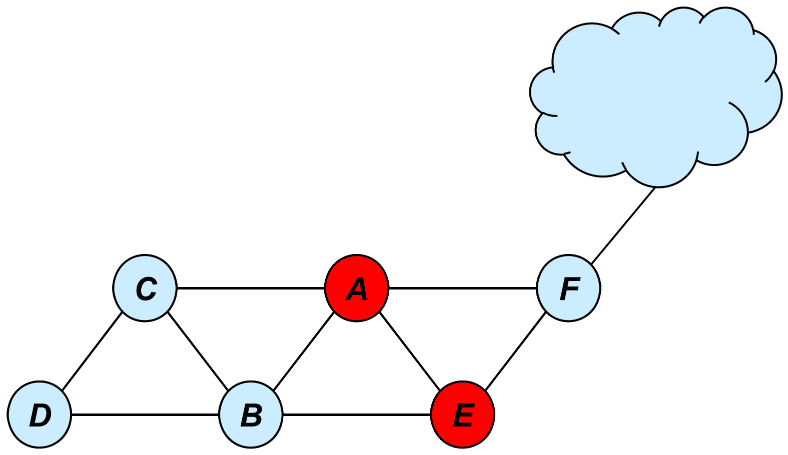


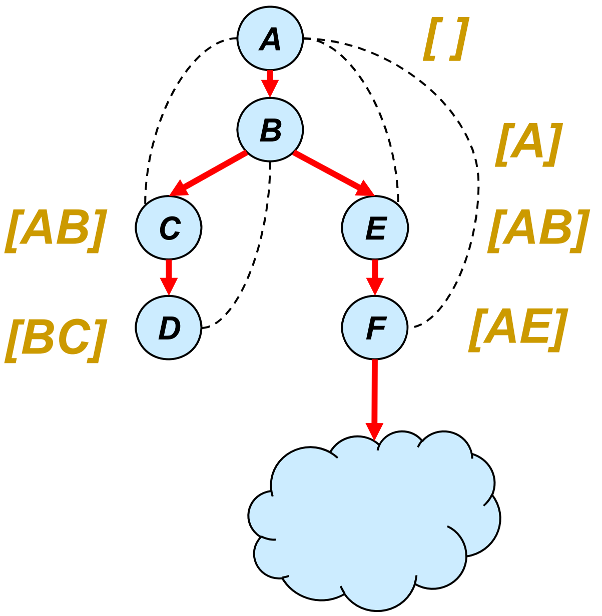
Merging based on context

context (X) = ancestors of X connected to

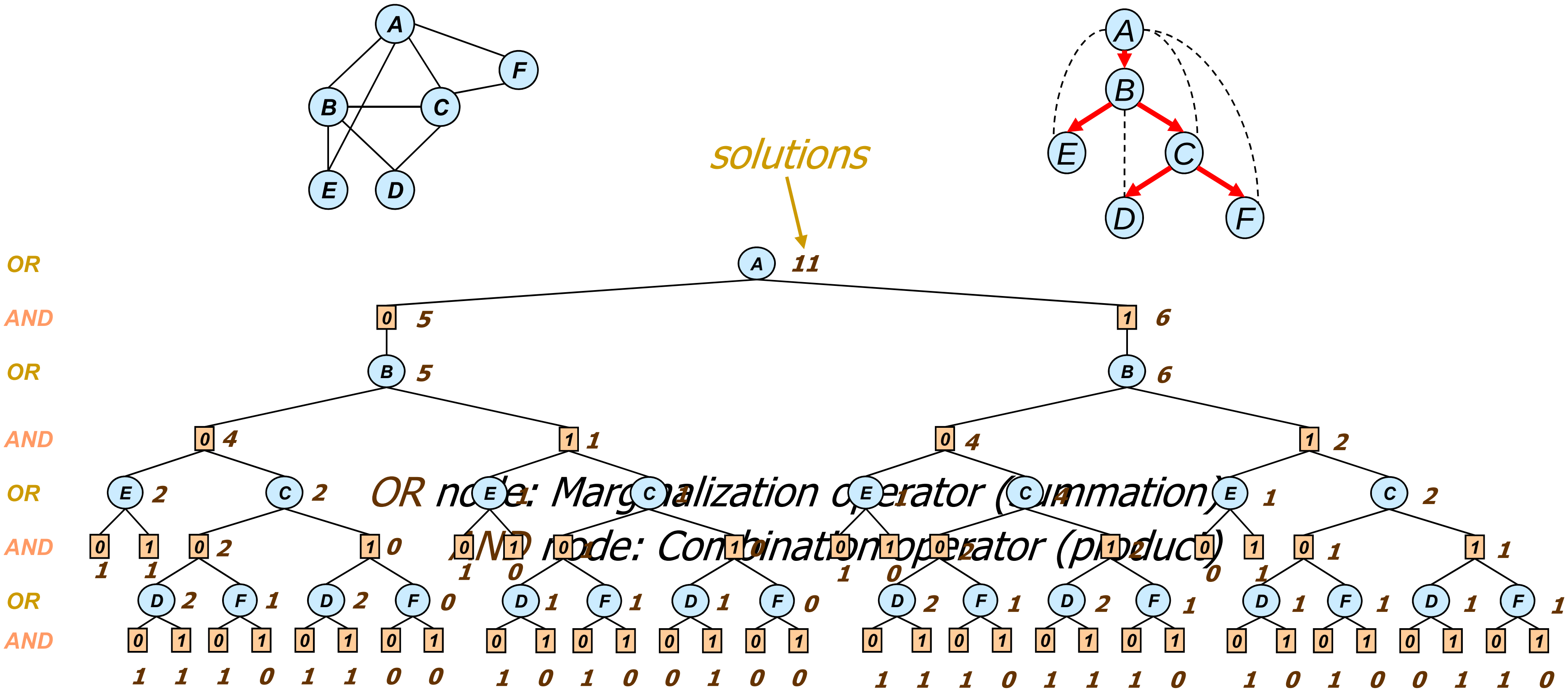
X

descendants of X





Counting Solutions: Sum-Product Networks

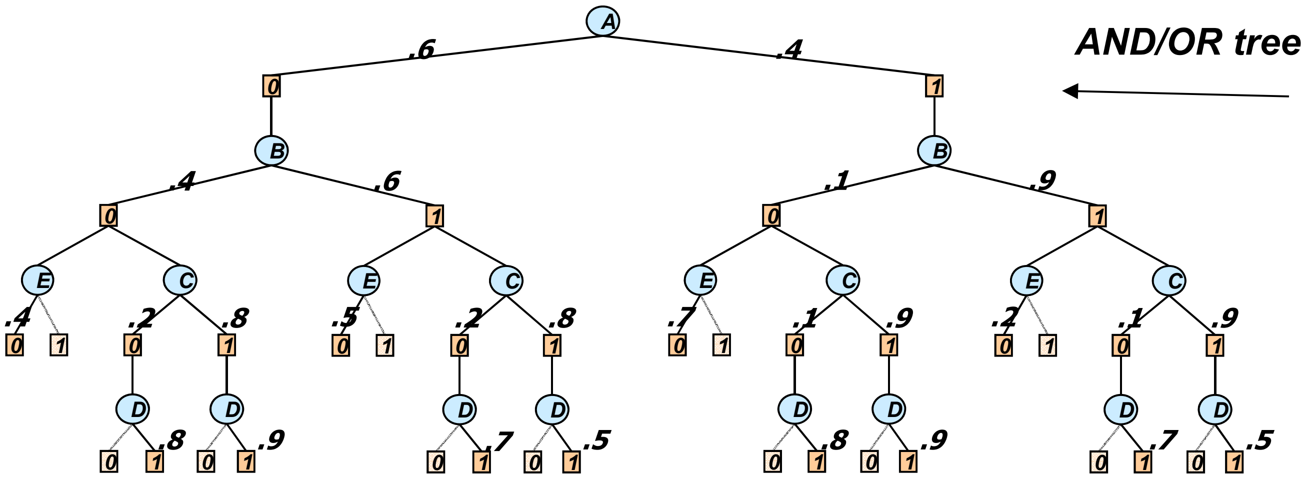


OR node: Marginalization operator (summation)

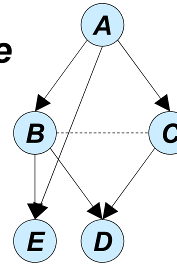
AND node: Combination operator (product)

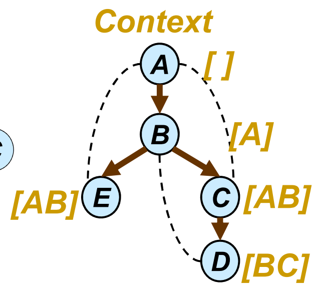
Value of node = number of solutions below it

Weighted AND/OR Search Space



AND/OR tree





$P(B|A)$

A	B=0	B=1
0	.4	.6
1	.1	.9

$P(E|A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

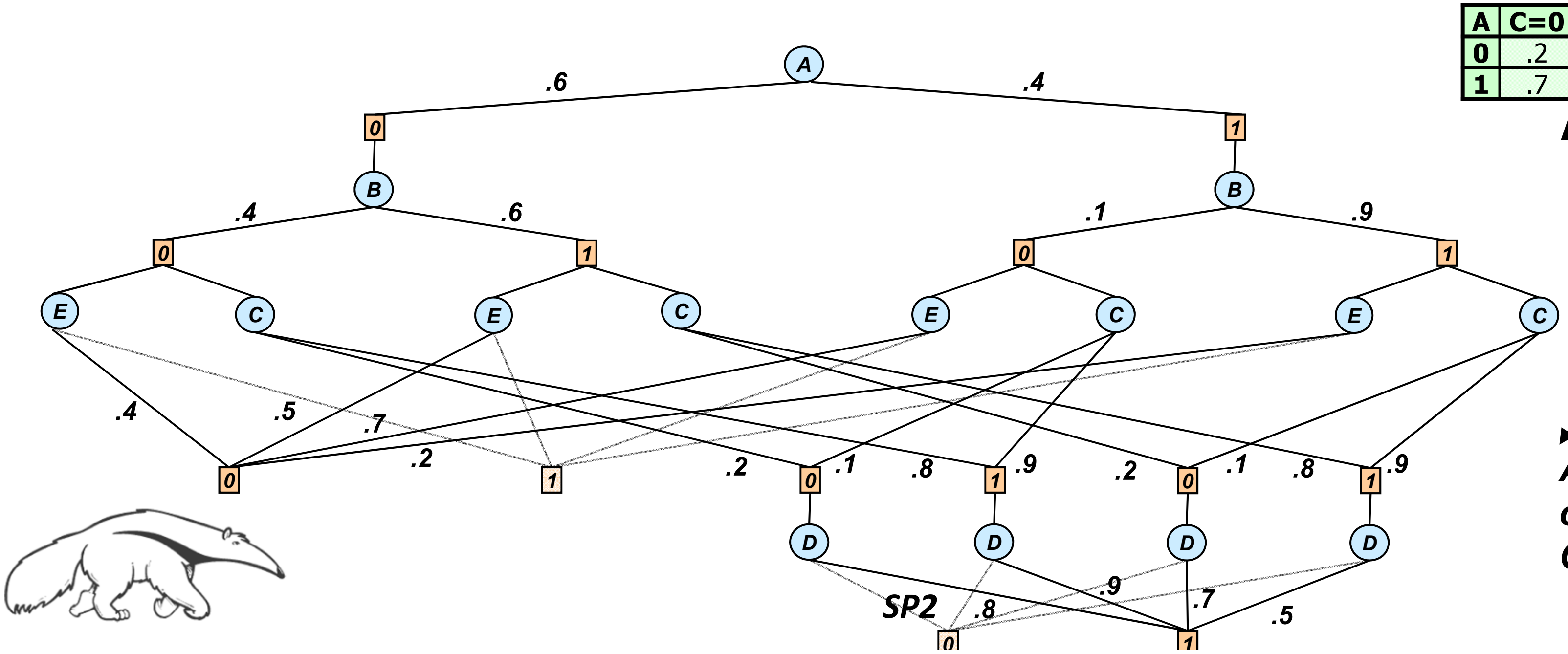
$P(C|A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

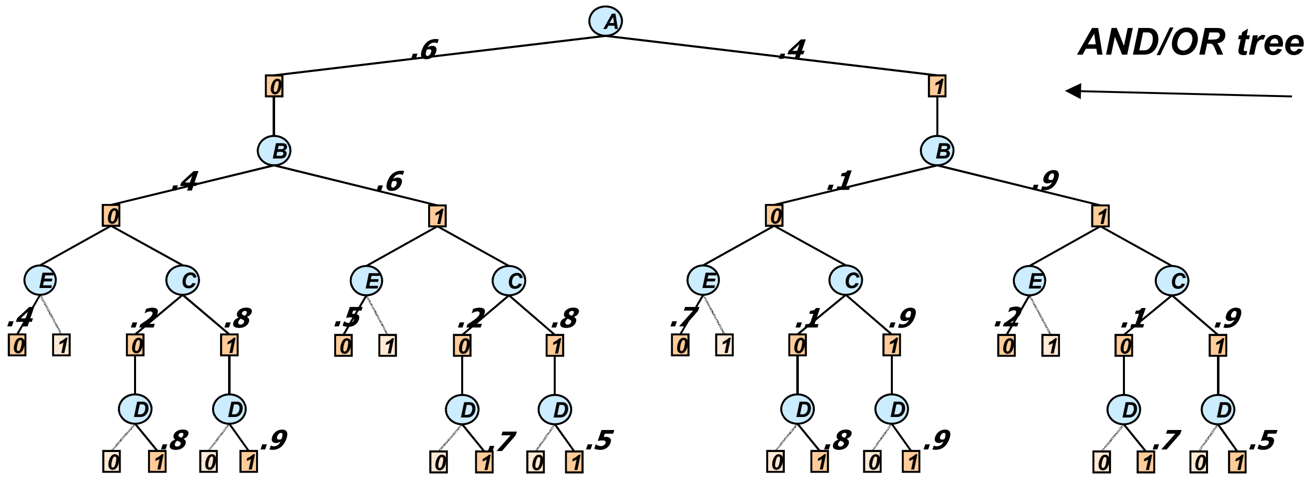
A	P(A)
0	.6
1	.4

Evidence: E=0

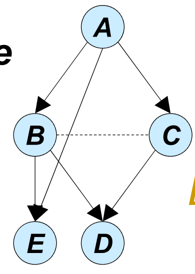


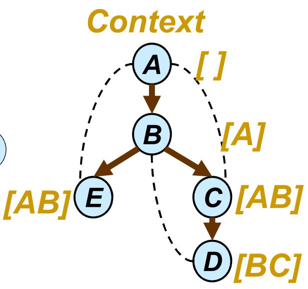
AND/OR context-minimal Graph

Weighted AND/OR Search Space



AND/OR tree





$P(B|A)$

A	B=0	B=1
0	.4	.6
1	.1	.9

$P(E|A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

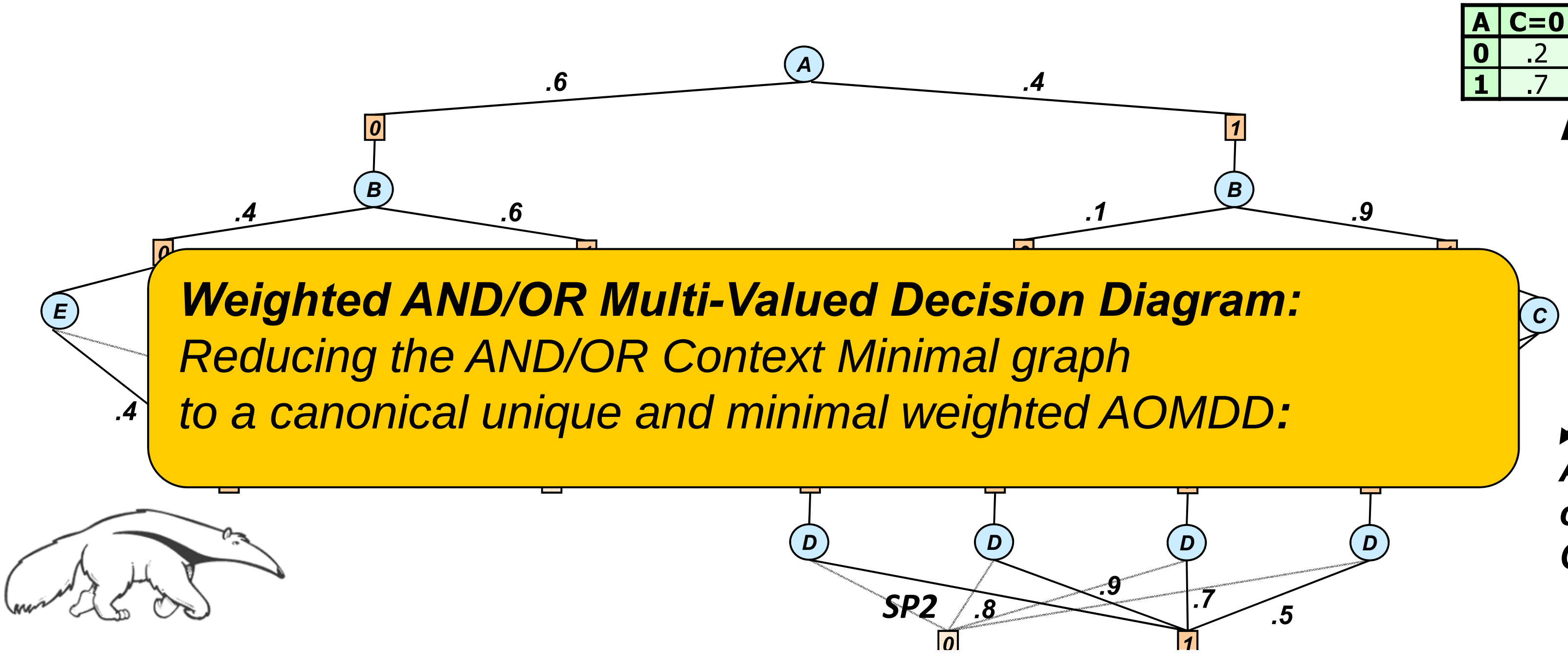
$P(C|A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4

Evidence: E=0



Weighted AND/OR Multi-Valued Decision Diagram:

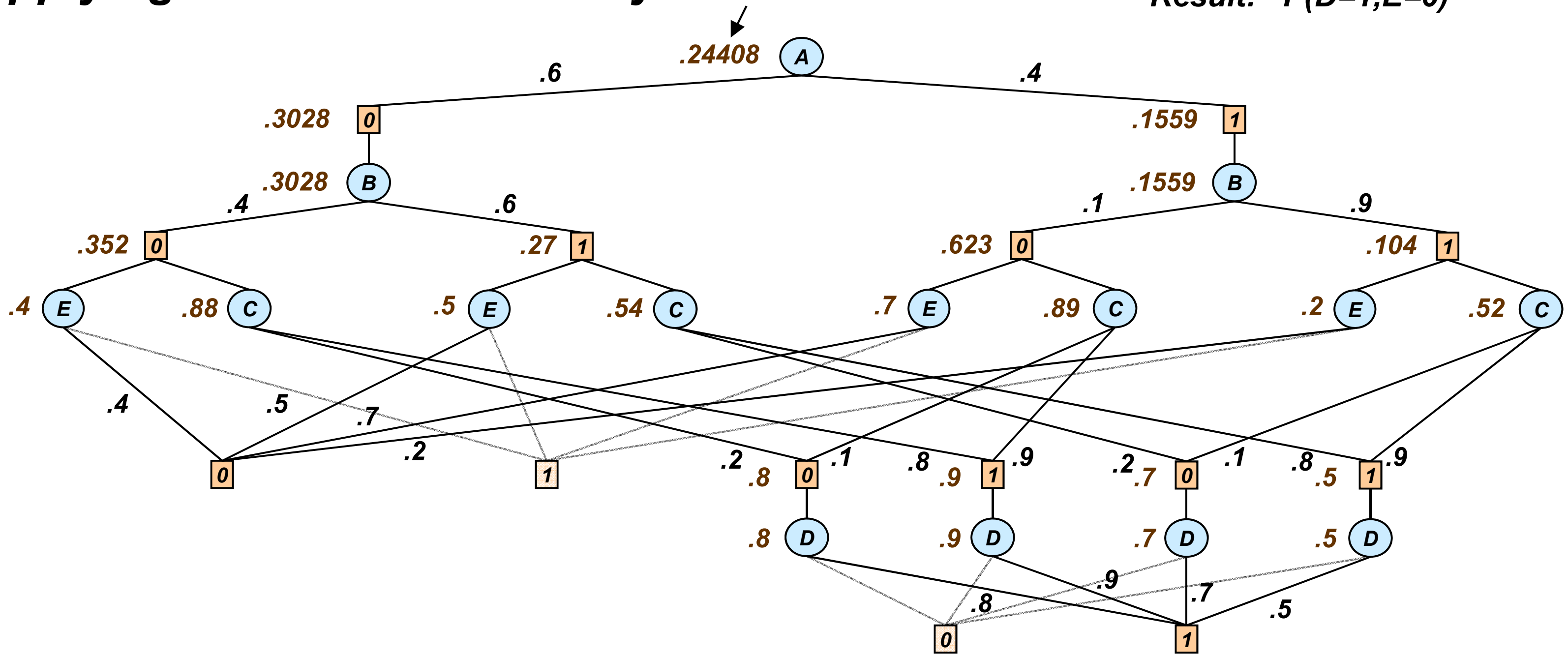
Reducing the AND/OR Context Minimal graph
to a canonical unique and minimal weighted AOMDD:

AND/OR context-minimal Graph

Answering Queries: Sum-Product

Applying Sum-Product is easy

Result: P(D=1,E=0)



$P(B|A)$

A	B=0	B=1
0	.4	.6
1	.1	.9

$P(E|A,B)$

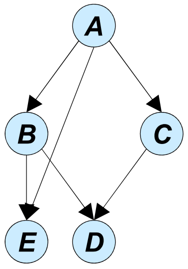
A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

$P(C|A)$

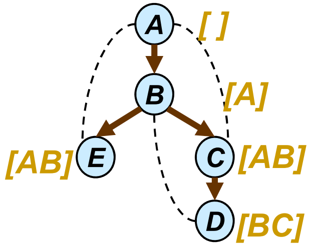
A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4



Context



Answering Queries: Sum-Product(Belief Updating)

$P(E \mid A,B)$

A	B	E=0	E=1
0	0	.4	.6
0	1	.5	.5
1	0	.7	.3
1	1	.2	.8

Evidence: E=0

$P(B \mid A)$

A	B=0	B=1
0	.4	.6
1	.1	.9

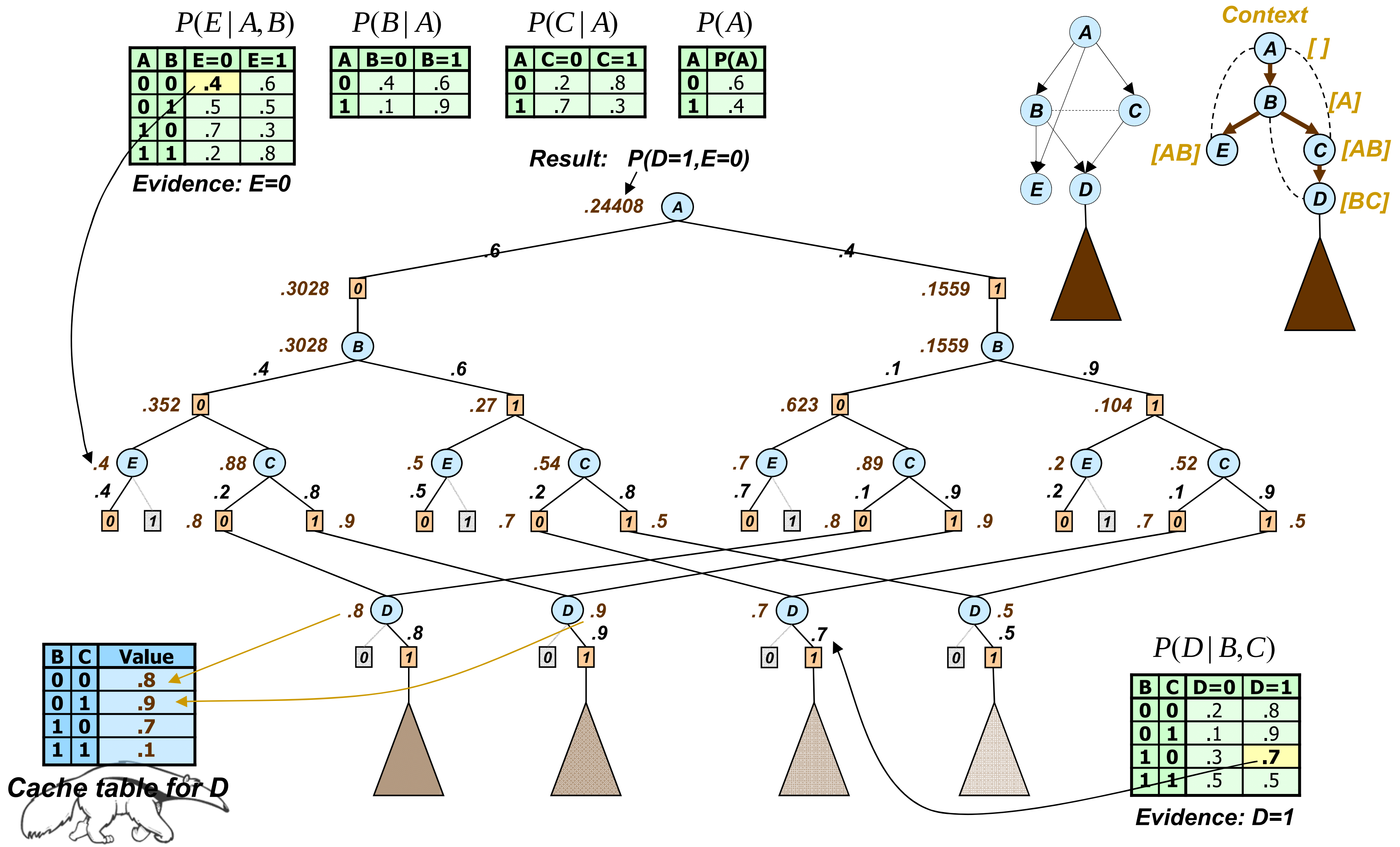
$P(C \mid A)$

A	C=0	C=1
0	.2	.8
1	.7	.3

$P(A)$

A	P(A)
0	.6
1	.4

Result: P(D=1,E=0)



B	C	Value
0	0	.8
0	1	.9
1	0	.7
1	1	.1

Cache table for D

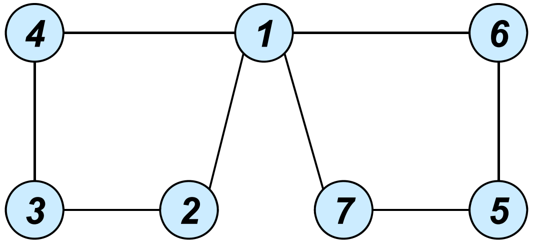
$P(D \mid B,C)$

B	C	D=0	D=1
0	0	.2	.8
0	1	.1	.9
1	0	.3	.7
1	1	.5	.5

Evidence: D=1

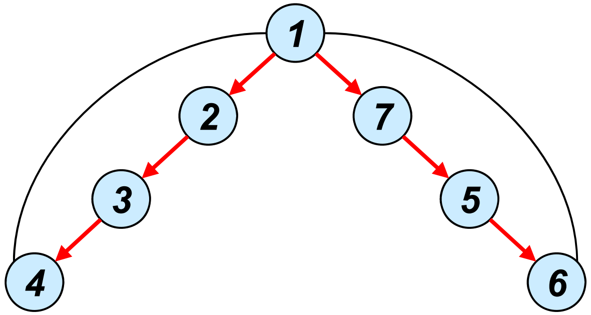
Pseudo-Trees

(Freuder 85, Bayardo 95, Bodlaender and Gilbert, 91)

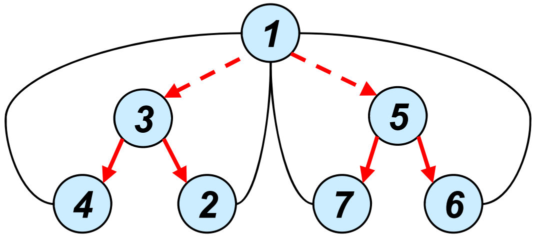


(a) Graph

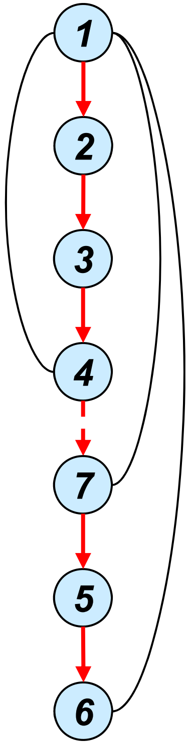
h <= w* log n



(b) DFS tree
depth=3

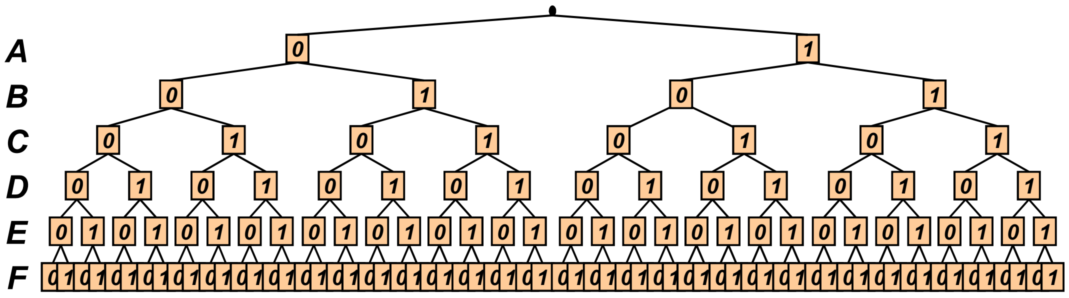


(c) pseudo- tree
depth=2



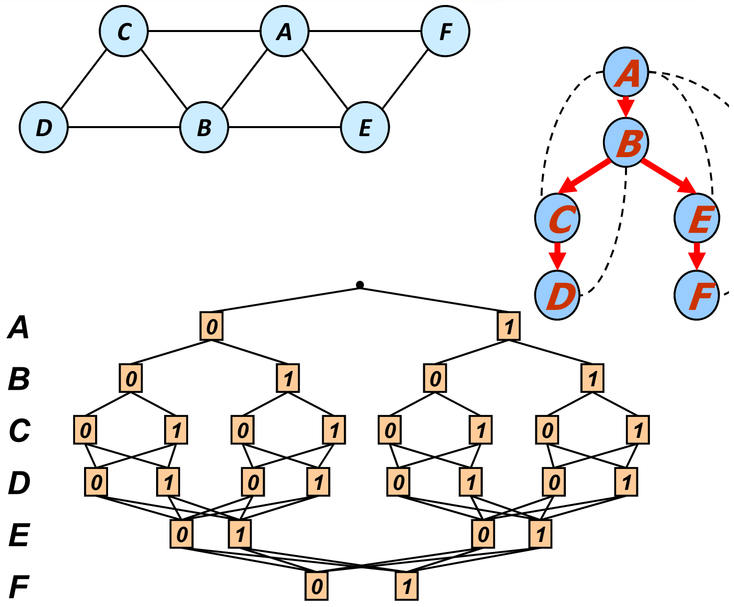
(d) Chain
depth=6

All Four Search Spaces



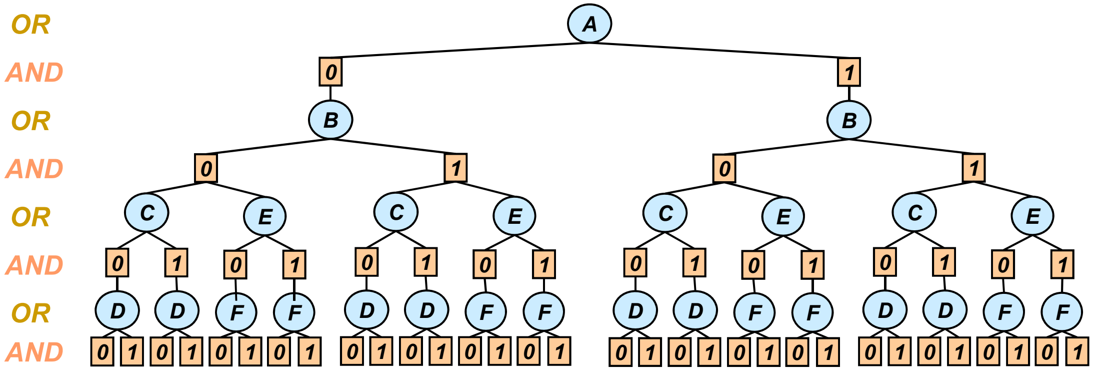
Full OR search tree

126 nodes



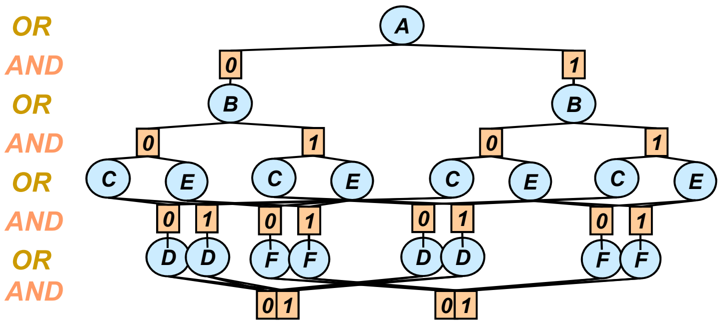
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes

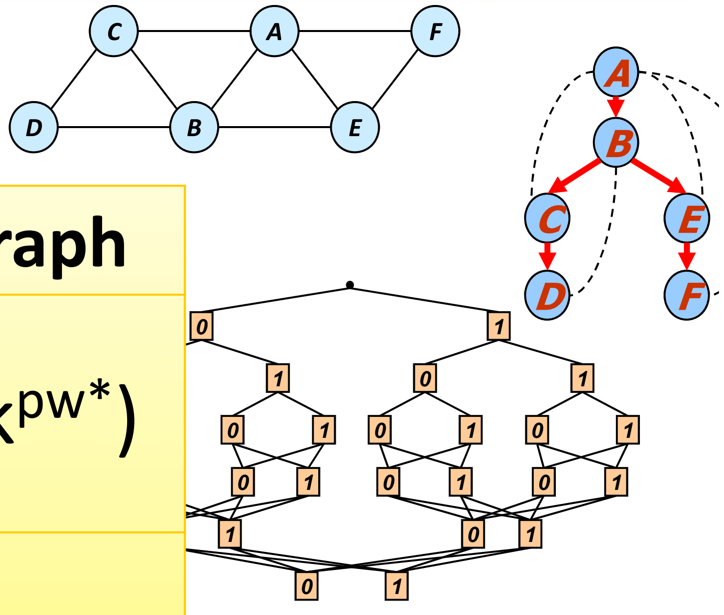


Context minimal AND/OR search graph

18 AND nodes

Any query is best computed Over the c-minimal AO space

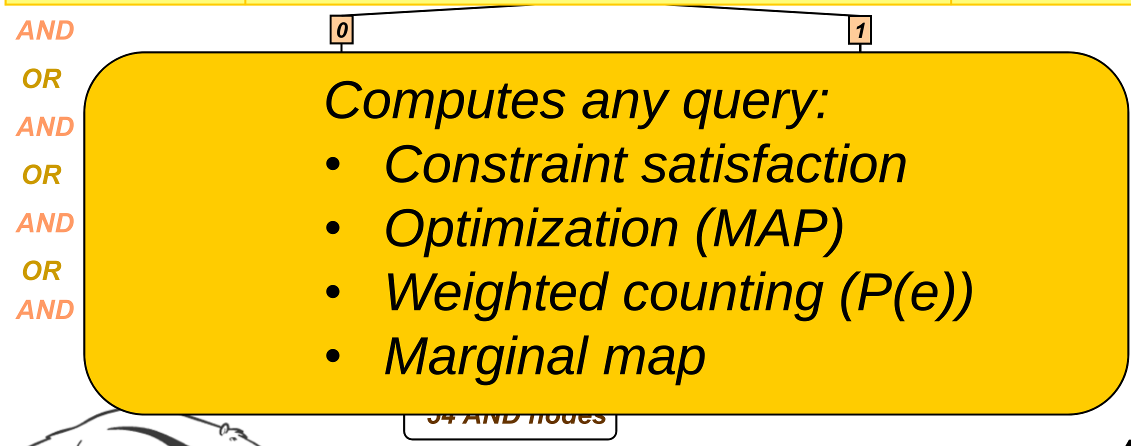
All Four Search Spaces



xt minimal OR search graph

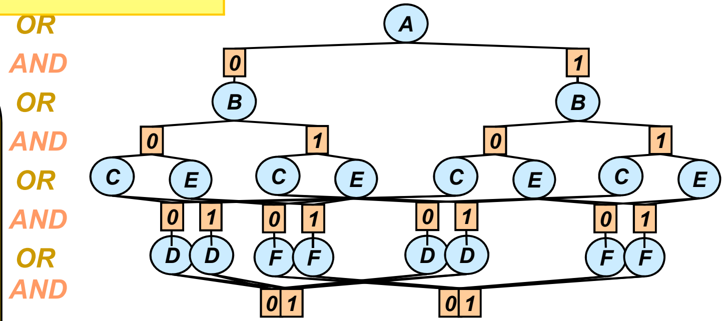
28 nodes

	AND/OR graph	OR graph
Space	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$
Time	$O(n\ k^{w^*})$	$O(n\ k^{pw^*})$



Computes any query:

- *Constraint satisfaction*
- *Optimization (MAP)*
- *Weighted counting (P(e))*
- *Marginal map*



Context minimal AND/OR search graph

18 AND nodes

Any query is best computed Over the c-minimal AO space

MAP: AND/OR BnB search

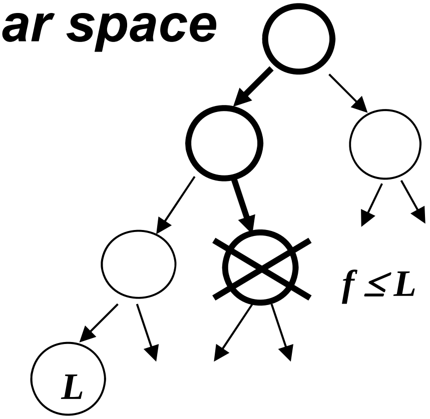
Basic Heuristic Search Schemes

Heuristic function $f(x^p)$ computes a lower bound on the best extension of x^p and can be used to guide a heuristic search algorithm. We focus on:

1. Branch-and-Bound

Use heuristic function $f(x^p)$ to prune the depth-first search tree

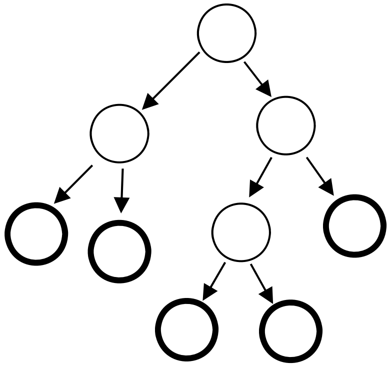
Linear space



2. Best-First Search

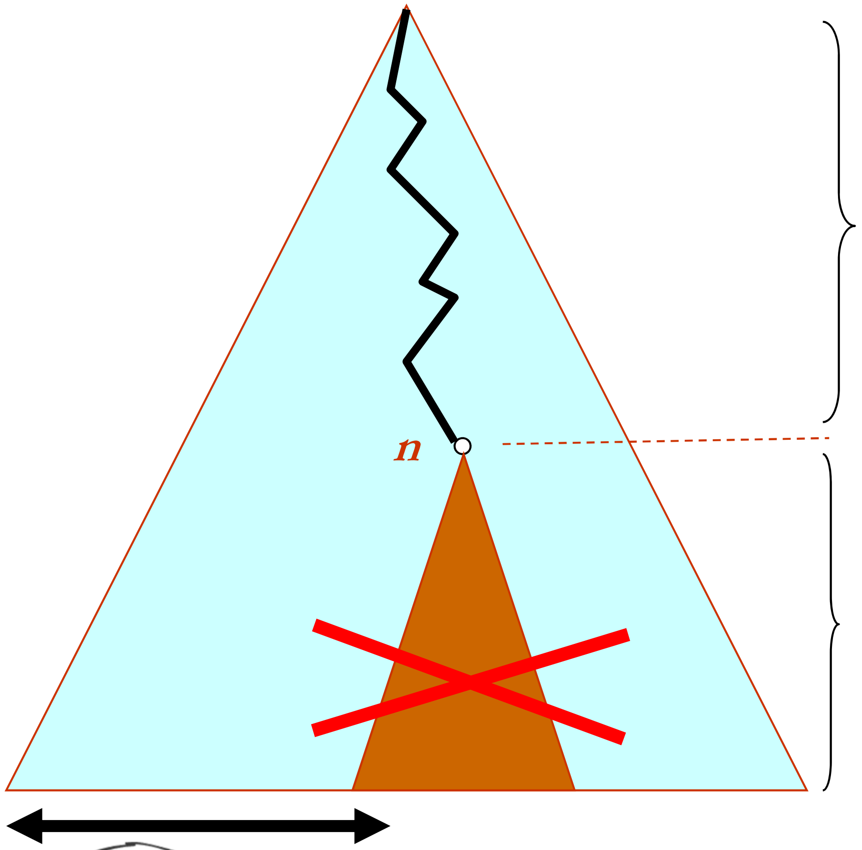
Always expand the node with the highest heuristic value

$f(x^p)$ needs lots of memory



Classic Branch-and-Bound

Each node is a COP subproblem (defined by current conditioning)



g(n)

f(n) = g(n) + h(n)
f(n) = lower bound

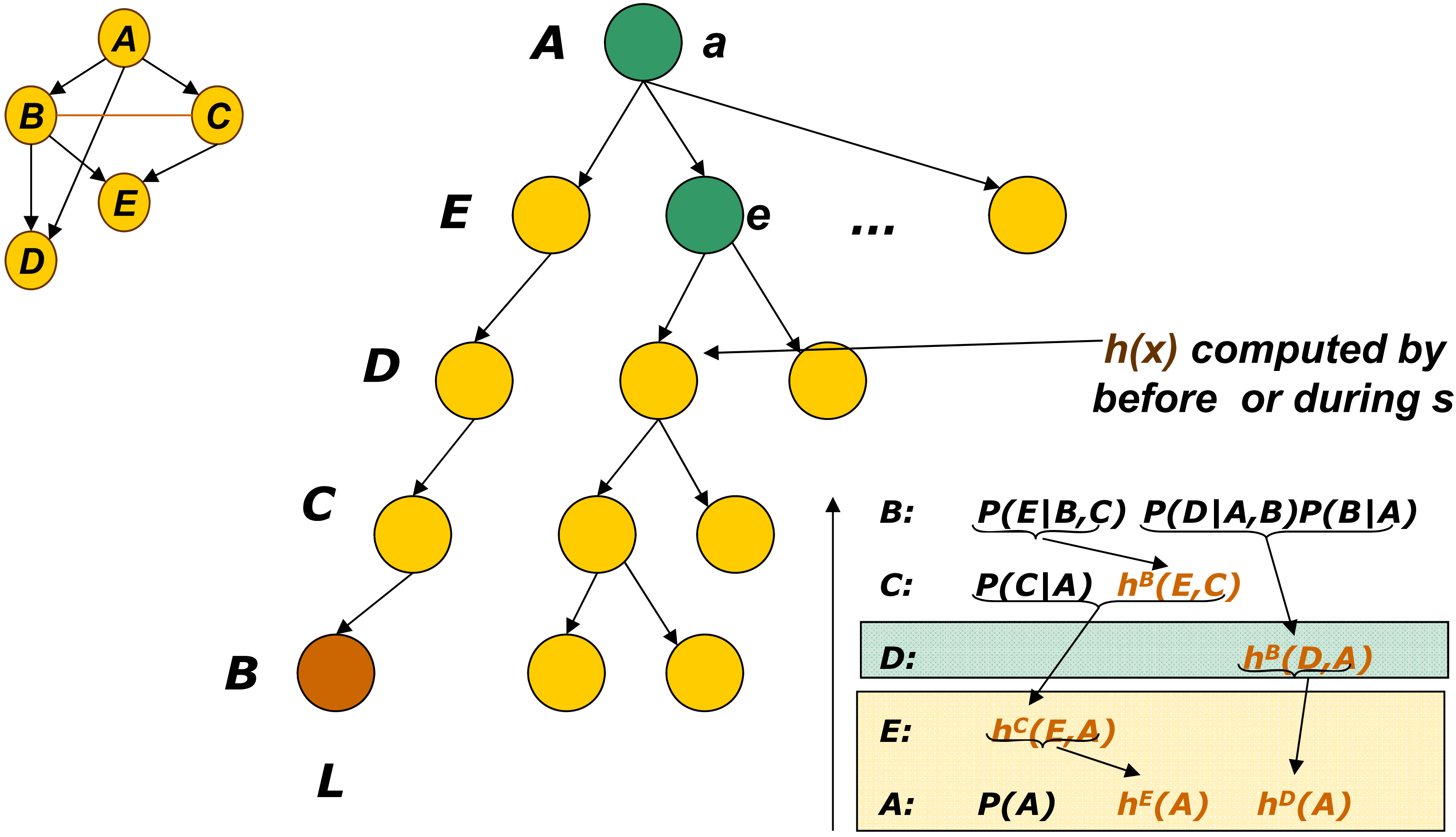
Prune if f(n) ≥ UB

h(n) - under-estimates Optimal cost below n

(UB) Upper Bound = best solution so far

Mini-bucket Heuristics for BB search

(Kask and dechterAIJ, 2001, Kask, Dechter and Marinescu 2004, 2005, 2009, Otten 2012)



h(x) computed by MB(i) before or during search

B: $P(E|B,C)$ $P(D|A,B)P(B|A)$

C: $P(C|A)$ $h^B(E,C)$

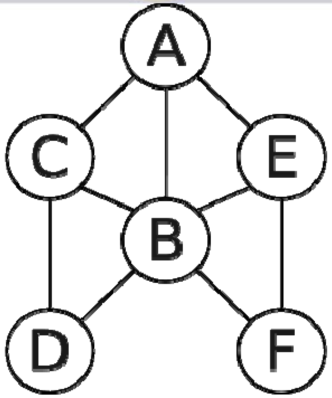
D: $h^B(D,A)$

E: $h^C(E,A)$

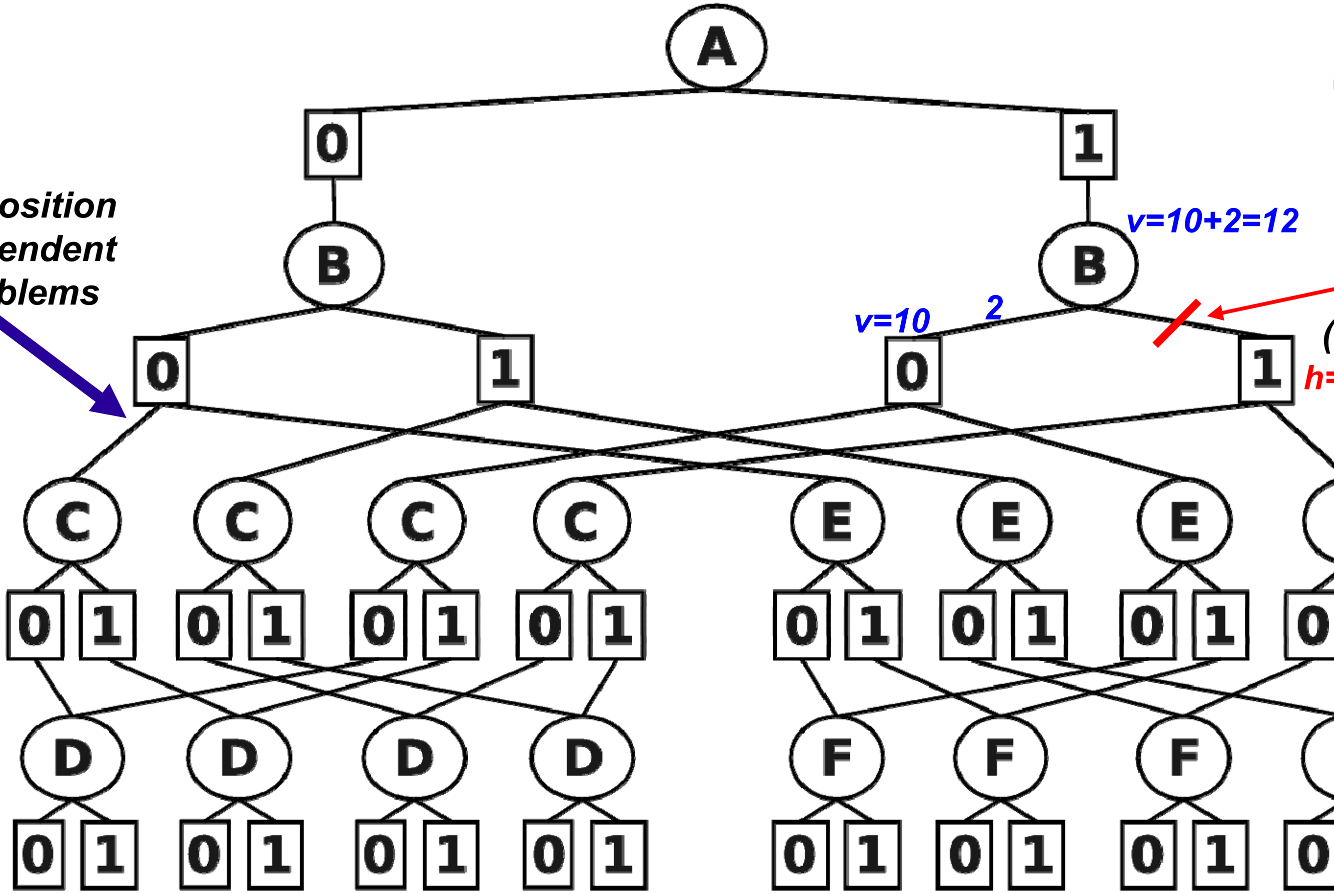
A: $P(A)$ $h^E(A)$ $h^D(A)$

$$f(a,e,D) = P(a)\cdot h^B(D,a)\cdot h^C(e,a)$$

AND/OR Branch-and-Bound



Decomposition of independent subproblems



Prune based on current best solution and heuristic estimate (mini-bucket heuristic).

Cache table for F (independent of A)

B E	cost
0 0	10
0 1	6
1 0	...
1 1	...

[Marinescu & Dechter, AIJ'09]

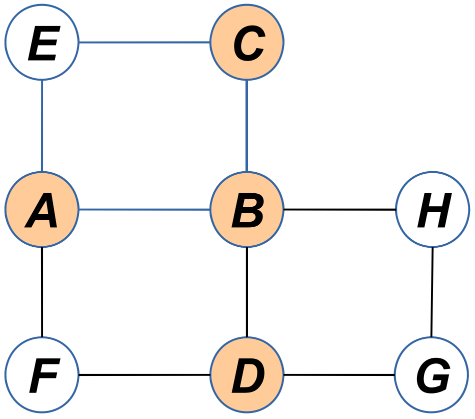
MAP: Anytime, BnB

- Best-First, Recursive Best-First
- Anytime:
  - Breadth-Rotate AND/OR BnB
  - Weighted heuristic AND/OR search
- Finding m-best solutions

Marginal Map: AND/OR BnB Search

AND/OR BnB over the appropriate search space
Guided by weighted mini-bucket +cost-shiifting

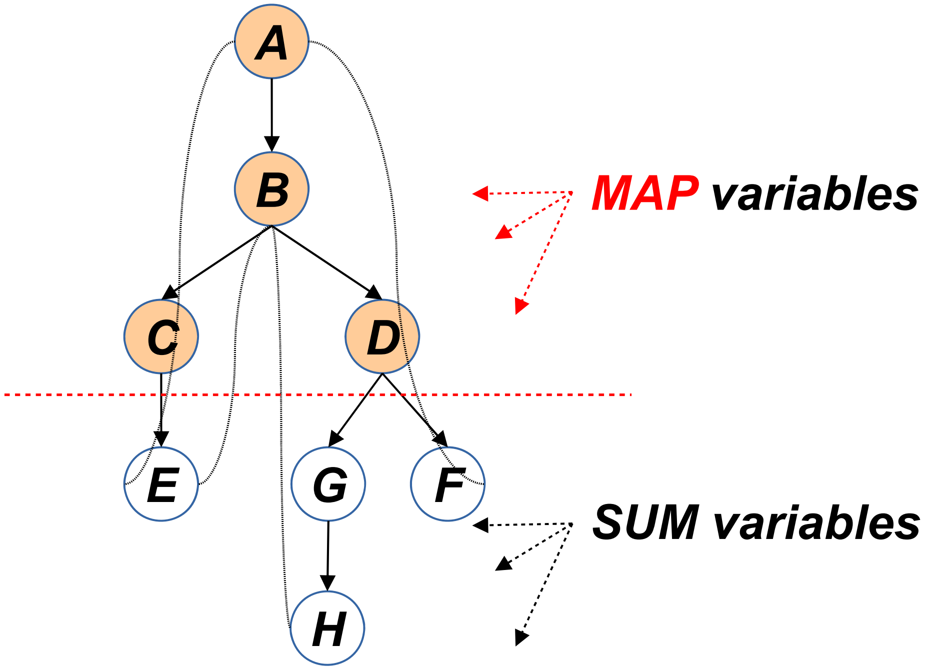
AND/OR Search Spaces for MMAP



primal graph

$$X_M = \{A, B, C, D\}$$

$$X_S = \{E, F, G, H\}$$



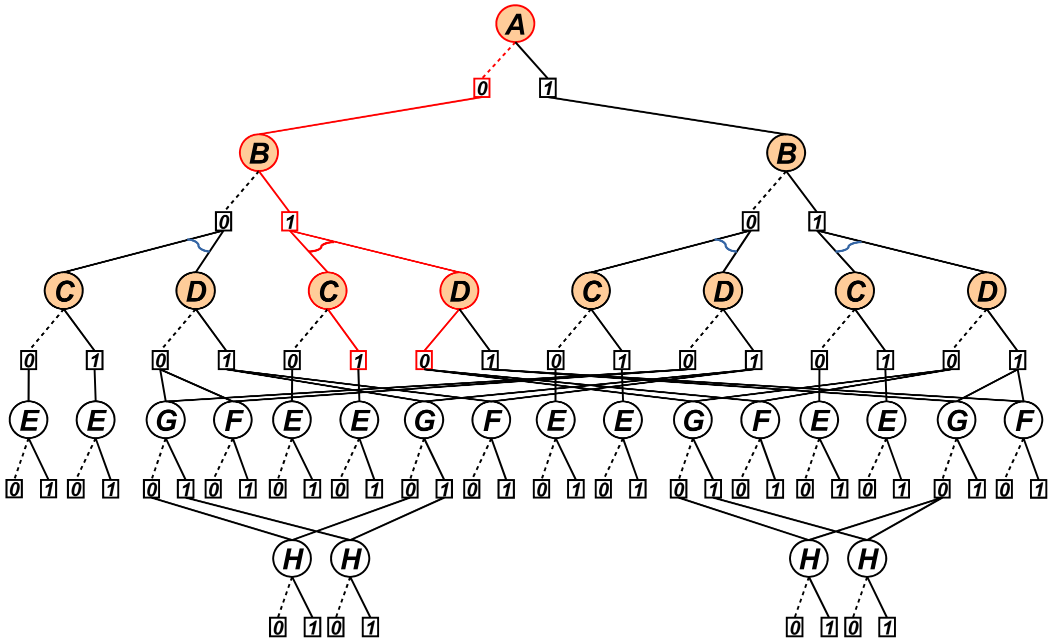
***MAP* variables**

SUM variables

constrained pseudo tree

[Marinescu, Dechter and Ihler, 2014]

AND/OR Search Spaces for MMAP



Node types

- OR (MAP): max
- OR (SUM): sum
- AND: multiplication

Arc weights

- Derived from input F

Problem decomposition over MAP variables

AND/OR Search for Marginal MAP

AOBB: Depth-First AND/OR Branch and Bound

- ***Depth-first traversal of the AND/OR search graph***
- ***Prune only at OR nodes that correspond to MAP variables***
- ***Cost of MAP assignment obtained by searching the SUM sub-problem***

AOBF: Best First AND/OR Search

- ***Best-first (AO*) traversal of the AND/OR space corresponding to the MAP variables***
- ***SUM subproblem solved exactly***

RBFAOO: Recursive Best-First AND/OR Search

- ***Recursive best-first traversal of the AND/OR graph***
- ***For SUM subproblems, the threshold is set to ∞ (equivalent to depth-first search)***

AND/OR Search for Marginal MAP

AOBB: Depth-First AND/OR Branch and Bound

Depth-first traversal of the AND/OR search graph

Prune only at OR nodes that correspond to MAP

Also Anytime Marginal Map solvers

corresponding to the MAP variables

SUM subproblem solved exactly

RBFAOO: Recursive Best-First AND/OR Search

Recursive best-first traversal of the AND/OR graph

For SUM subproblems, the threshold is set to ∞ (equivalent to depth-first search)

Outline

- Graphical models, Queries
- Inference Algorithms
- Bounding Inference schemes (mini-bucket, re-parameterization)
- AND/OR search
- **Evaluation, Software**
- Conclusions and recent work: Parallelism, m-best, weighted best-first, marginal map, tree-SLS)

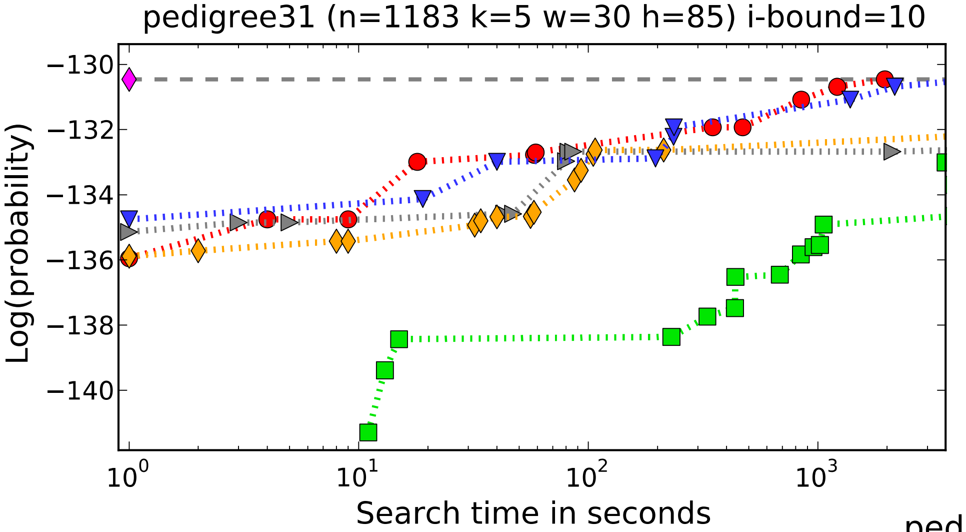
Software

- **aolib**
  - http://graphmod.ics.uci.edu/group/Software (standalone AOBB, AOBF solvers)
- **daoopt**
  - https://github.com/lotten/daoopt (distributed and standalone AOBB solver)

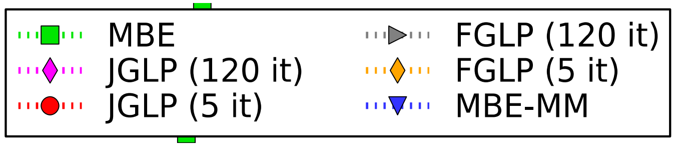
UAI Probabilistic Inference Competitions

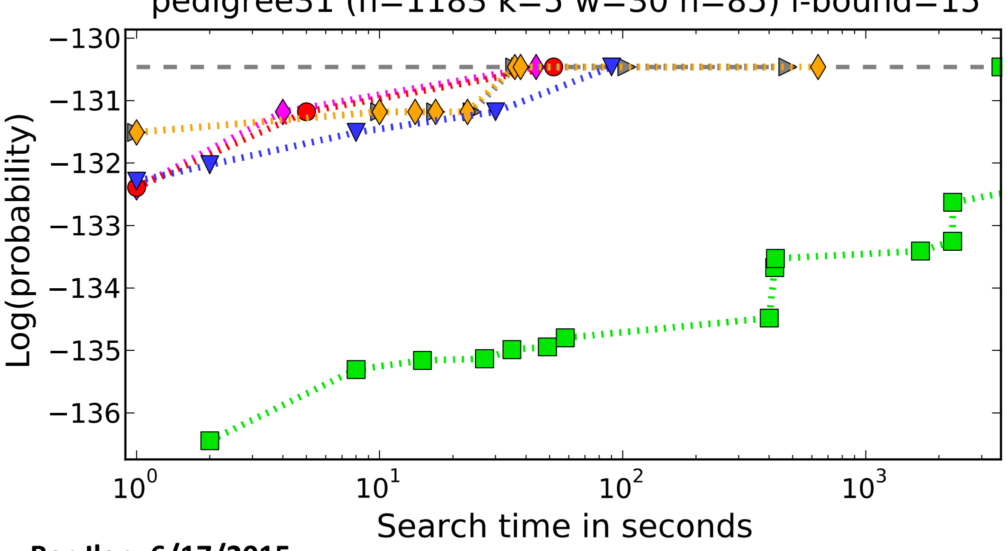
2006	🥇 (aolib)	
2008	🥈 (aolib)	
2011	🥇 (daoopt)	
2014	🥈 (daoopt)	🥇 (daoopt) 🥈 (merlin)
	MPE/MAP	**MMAP**

Empirical Evaluation: Haplotype problems

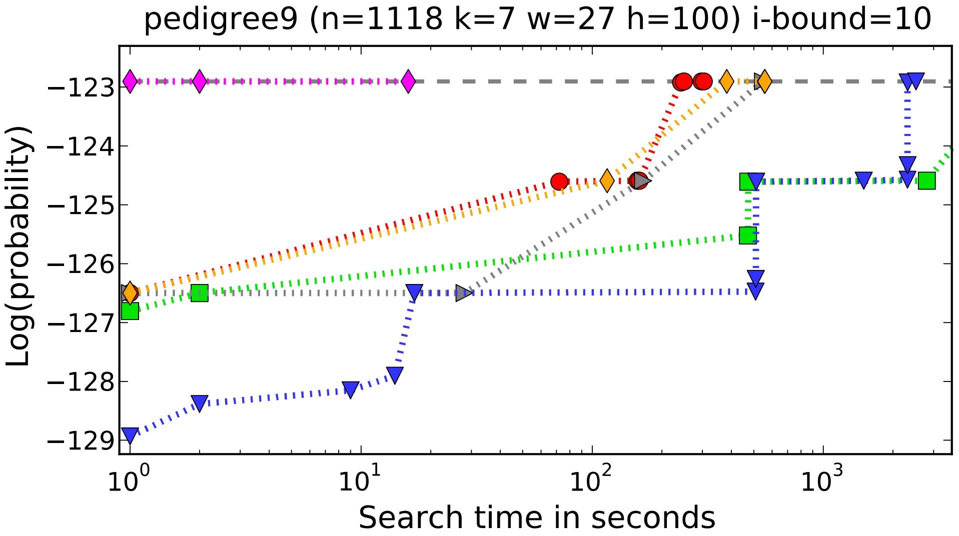


Time bound – 24 h

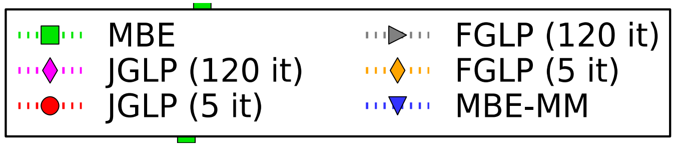


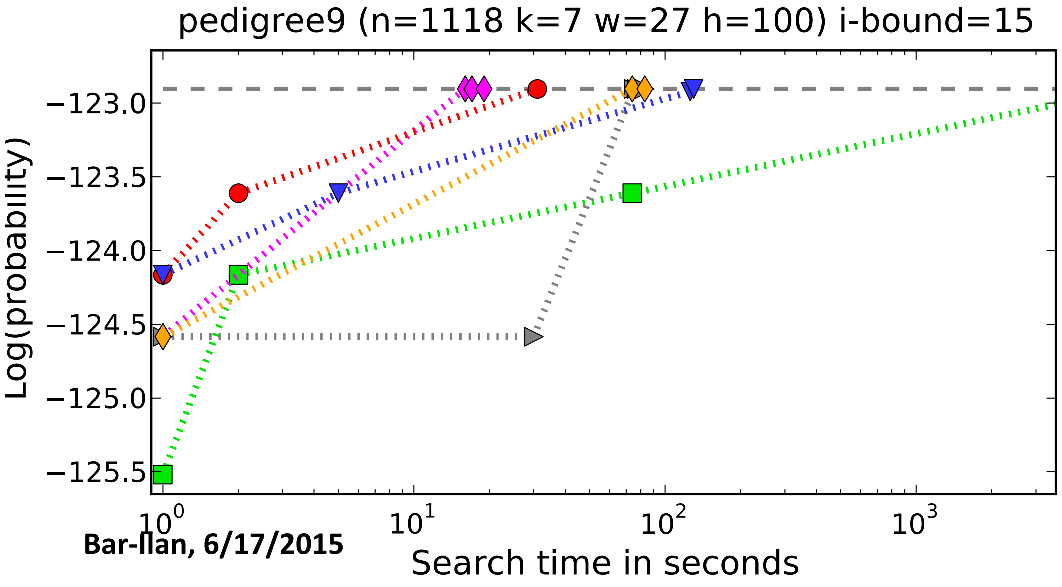


Empirical Evaluation: Haplotype problems



Time bound – 24 h





Marginal Map results

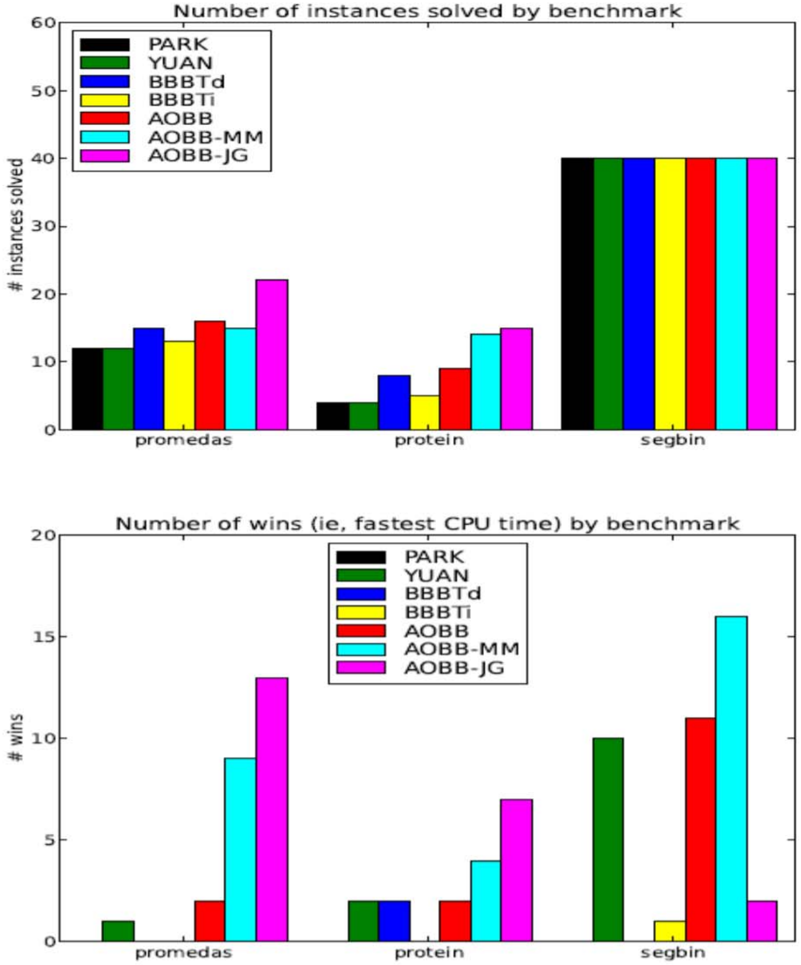


Figure 5: Number of instances solved (top) and number of wins (bottom) by benchmark.

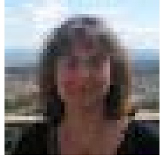
Summary and Future work

- I have shown the primal graph of a graphical model can:
  - □ suggest effective relaxation and heuristics
  - □ Suggest a compact search graph
  - □ Can accommodate vibrational schemes
- Yields one of the best solvers
- *ILP and Boolean methods can be included.*
- *Current work: anytime scheme minimizing the upper-lower gap at termination anytime*
- *Anytime summation solvers*

Thanks You!

For publication see:

http://www.ics.uci.edu/~dechter/publications.html



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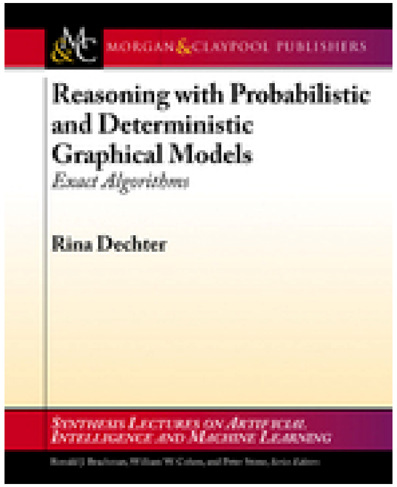

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