Advances in Search and Inference for Graphical Models

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Outline

- Introduction
- Inference
- Search
- Compilation: AND/OR Decision Diagrams
- Software

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- Introduction
  - Graphical models
  - Solution Techniques
- Inference
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Constraint Networks

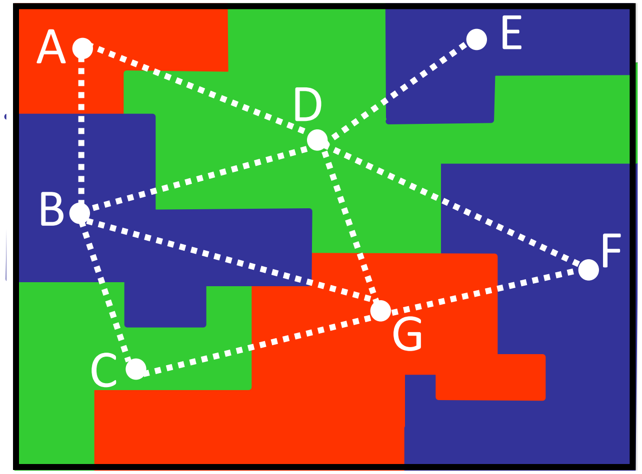
Map coloring

Variables: countries (A B C etc.)

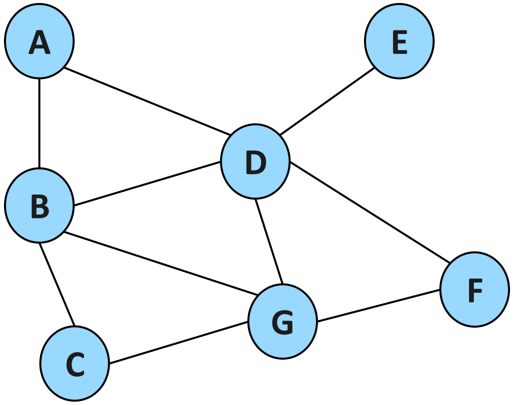
Values: colors (red green blue)

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, …

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red

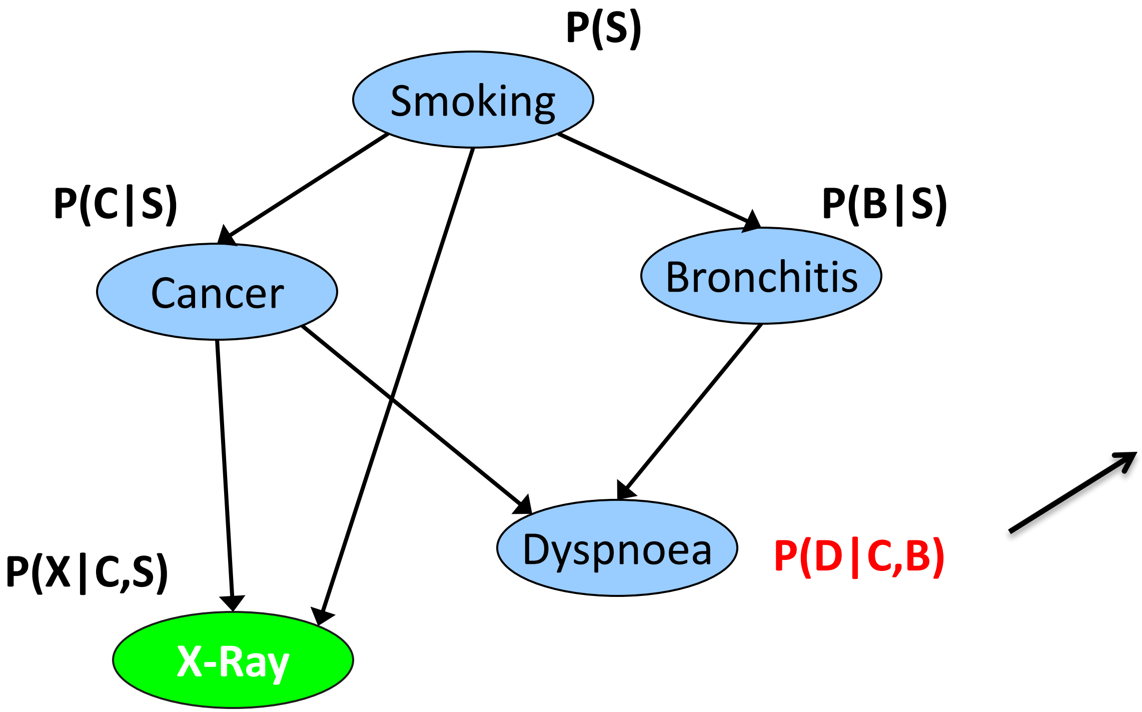


Constraint graph



Bayesian Networks

BN = (X,D,G,P)



P(D|C,B)

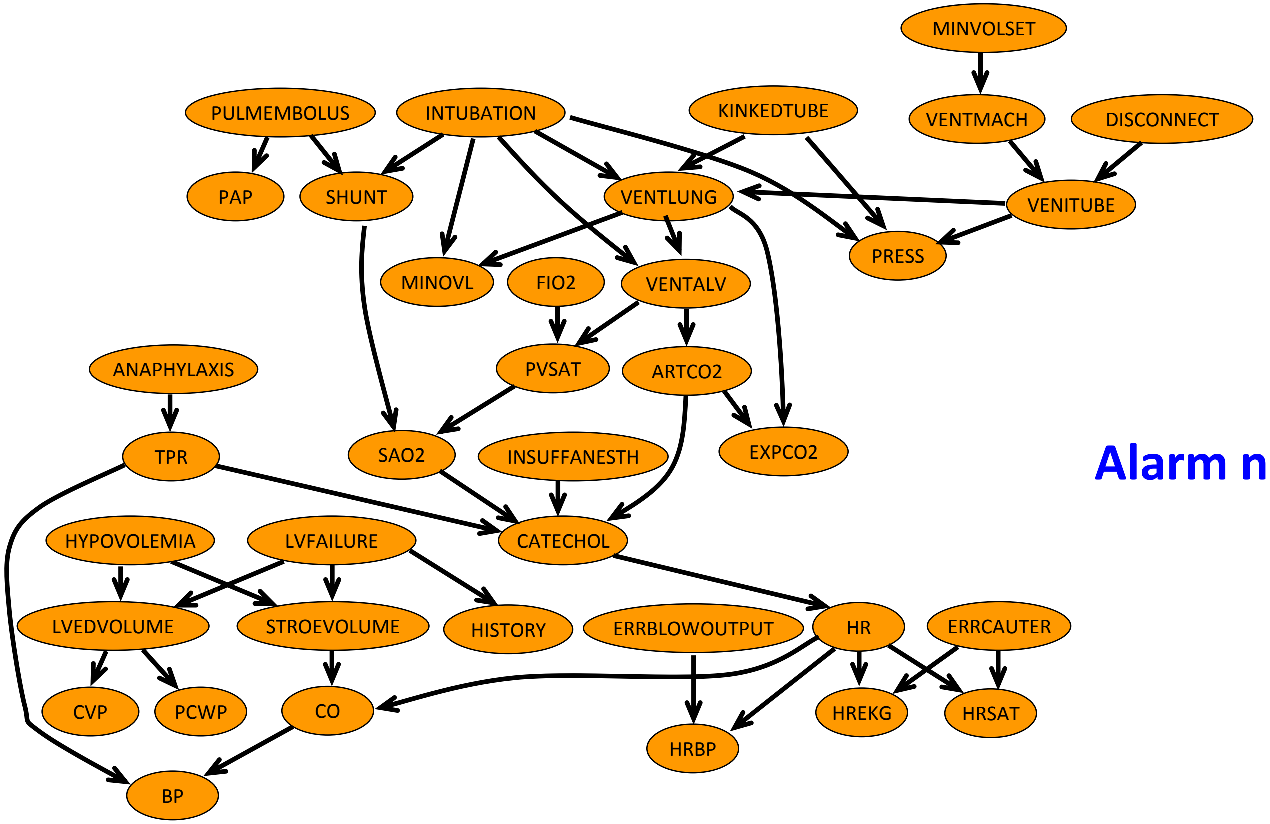
C	B	D=0	D=1
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

P(S,C,B,X,D) = P(S)· P(C|S)· P(B|S)· P(X|C,S)· P(D|C,B)

MPE = Find a maximum probability assignment, given evidence

MPE = find argmax P(S)· P(C|S)· P(B|S)· P(X|C,S)· P(D|C,B)

Monitoring Intensive-Care Patients



Alarm network

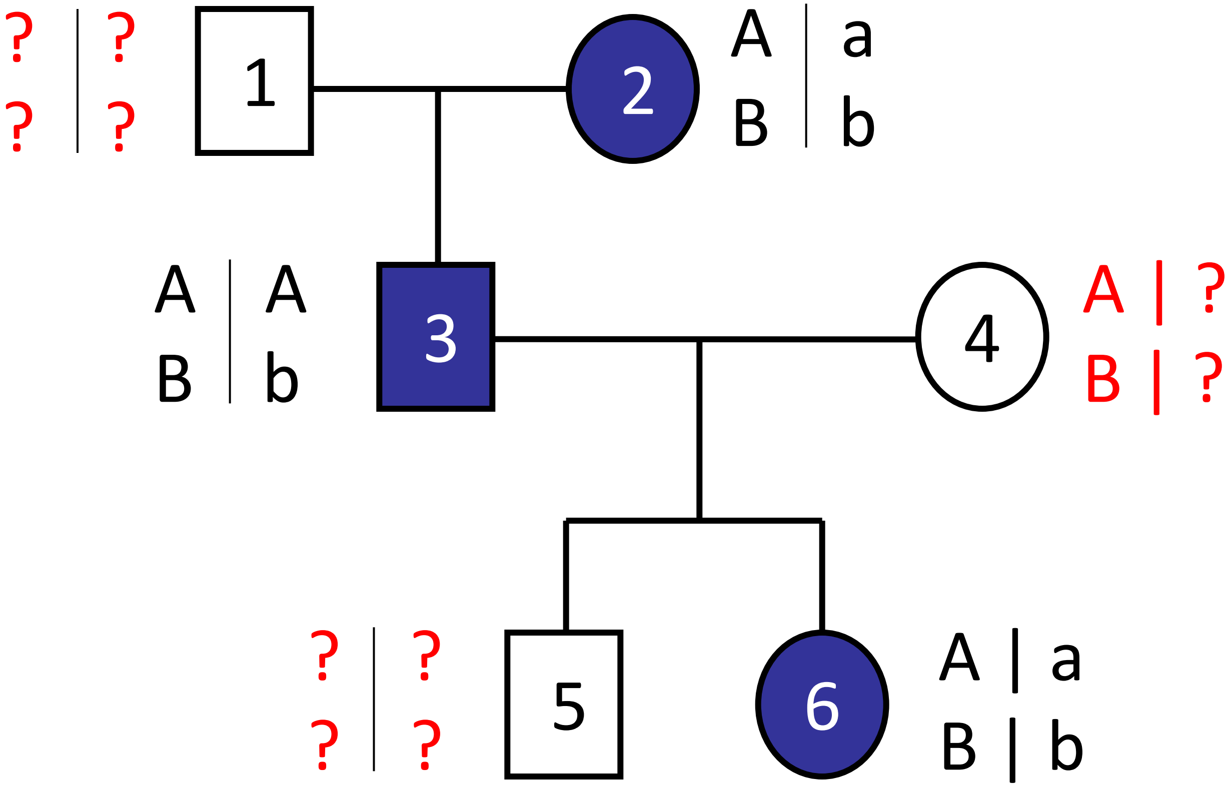
37 variables

509 parameters

<<

2^{37}

Linkage Analysis



? | ?
? | ?
1

2
A | a
B | b

A | A
B | b
3

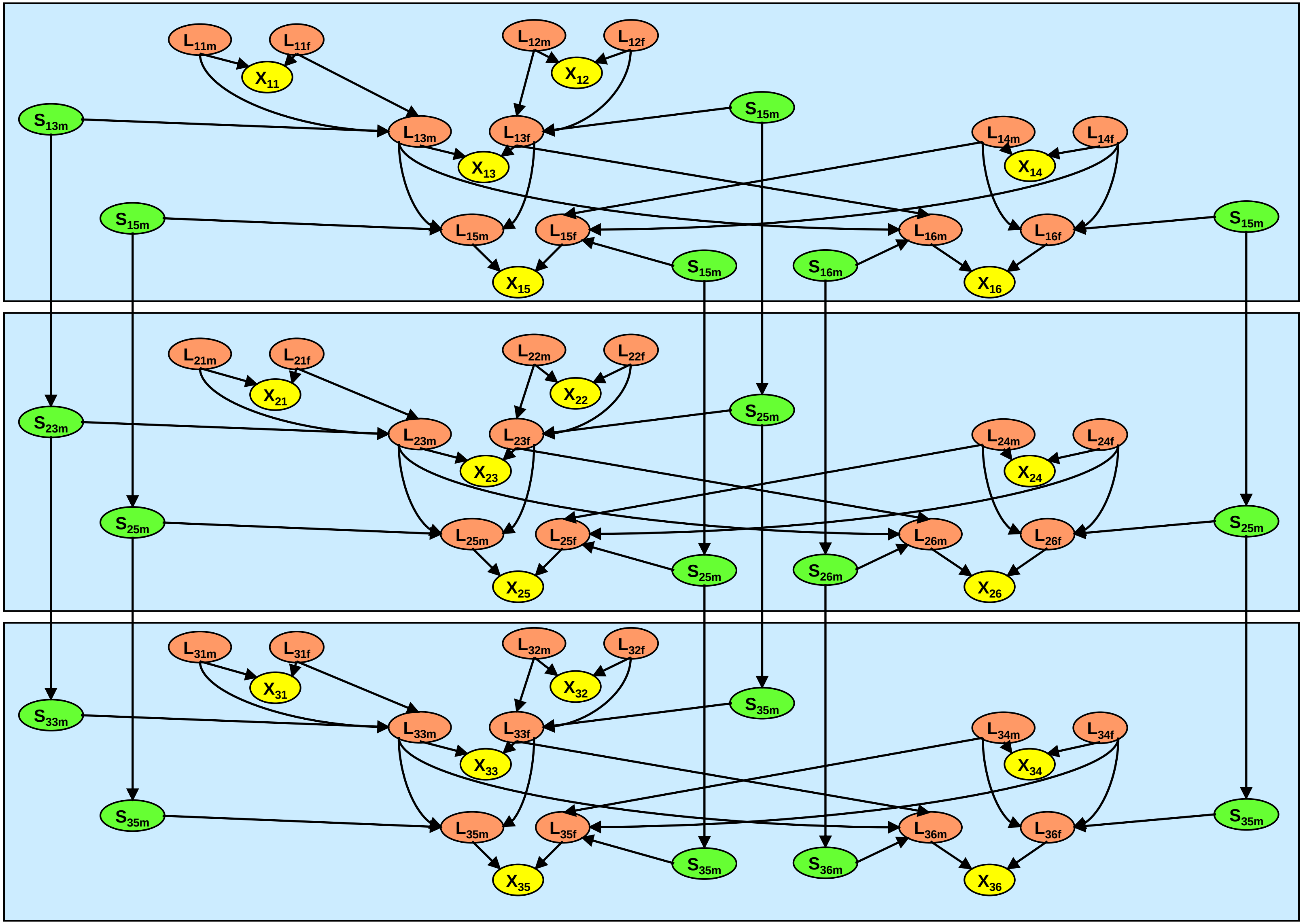
4
A | ?
B | ?

? | ?
? | ?
5

6
A | a
B | b

- 6 individuals
- Haplotype: {2, 3}
- Genotype: {6}
- Unknown

Pedigree: 6 people, 3 markers



Constraint Optimization Problems

for Graphical Models

A *finite COP* is a triple $R = \langle X, D, F \rangle$ where :

$X = \{X_1, \ldots, X_n\}$ - variables

$D = \{D_1, \ldots, D_n\}$ - domains

$F = \{f_1, \ldots, f_m\}$ - cost functions

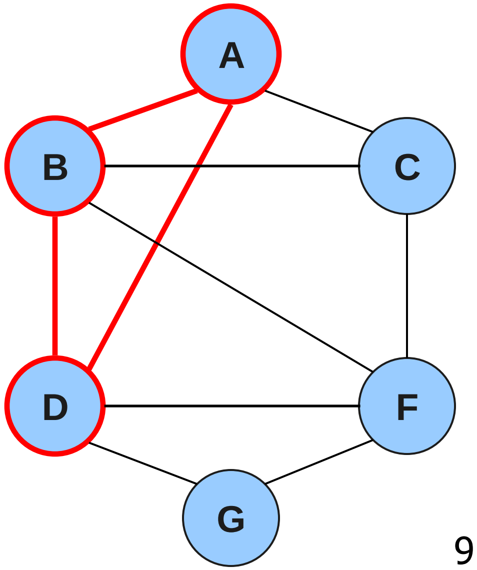
f(A,B,D) has scope {A,B,D}

A	B	D	Cost
1	2	3	3
1	3	2	2
2	1	3	∞
2	3	1	0
3	1	2	5
3	2	1	0

Primal graph =

Variables --> nodes

Functions, Constraints -→ arcs



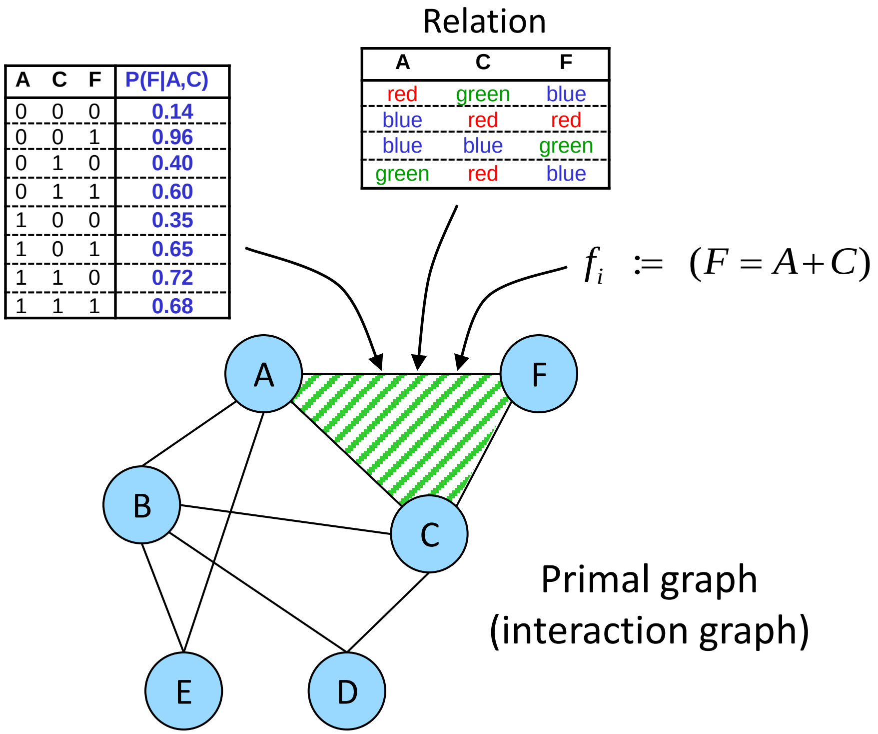
$$F(a,b,c,d,f,g) = f_1(a,b,d) + f_2(d,f,g) + f_3(b,c,f)$$

Global Cost Function

$$F(X) = \sum_{i=1}^{m} f_i(X)$$

Graphical Models

- A graphical model **(X,D,F)**:
  - **X** = $\{X_1,\ldots X_n\}$ variables
  - **D** = $\{D_1, \ldots D_n\}$ domains
  - **F** = $\{f_1,\ldots,f_m\}$ functions
- Operators:
  - combination
  - elimination (projection)
- Tasks:
  - **Belief updating**: $\Sigma_{X-y} \prod_j P_i$
  - **MPE**: $\max_X \prod_j P_j$
  - **CSP**: $\prod_X \times_j C_j$
  - **Max-CSP**: $\min_X \Sigma_j f_j$



A	C	F	P(F|A,C)
0	0	0	0.14
0	0	1	0.96
0	1	0	0.40
0	1	1	0.60
1	0	0	0.35
1	0	1	0.65
1	1	0	0.72
1	1	1	0.68

Relation

A	C	F
red	green	blue
blue	red	red
blue	blue	green
green	red	blue

$f_i := (F = A + C)$

Primal graph (interaction graph)

- All these tasks are NP-hard
  - exploit problem structure
  - identify special cases
  - approximate

Sample Domains for Graphical Models

- Web Pages and Link Analysis
- Communication Networks (Cell phone Fraud Detection)
- Natural Language Processing (e.g. Information Extraction and Semantic Parsing)
- Battle-space Awareness
- Epidemiological Studies
- Citation Networks
- Intelligence Analysis (Terrorist Networks)
- Financial Transactions (Money Laundering)
- Computational Biology
- Object Recognition and Scene Analysis

…

Type of constrained optimization:

- Weighted CSPs, Max-CSPs, Max-SAT
- Most Probable Explanation (MPE)
- Linear Integer Programs

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Solution Techniques

AND/OR search

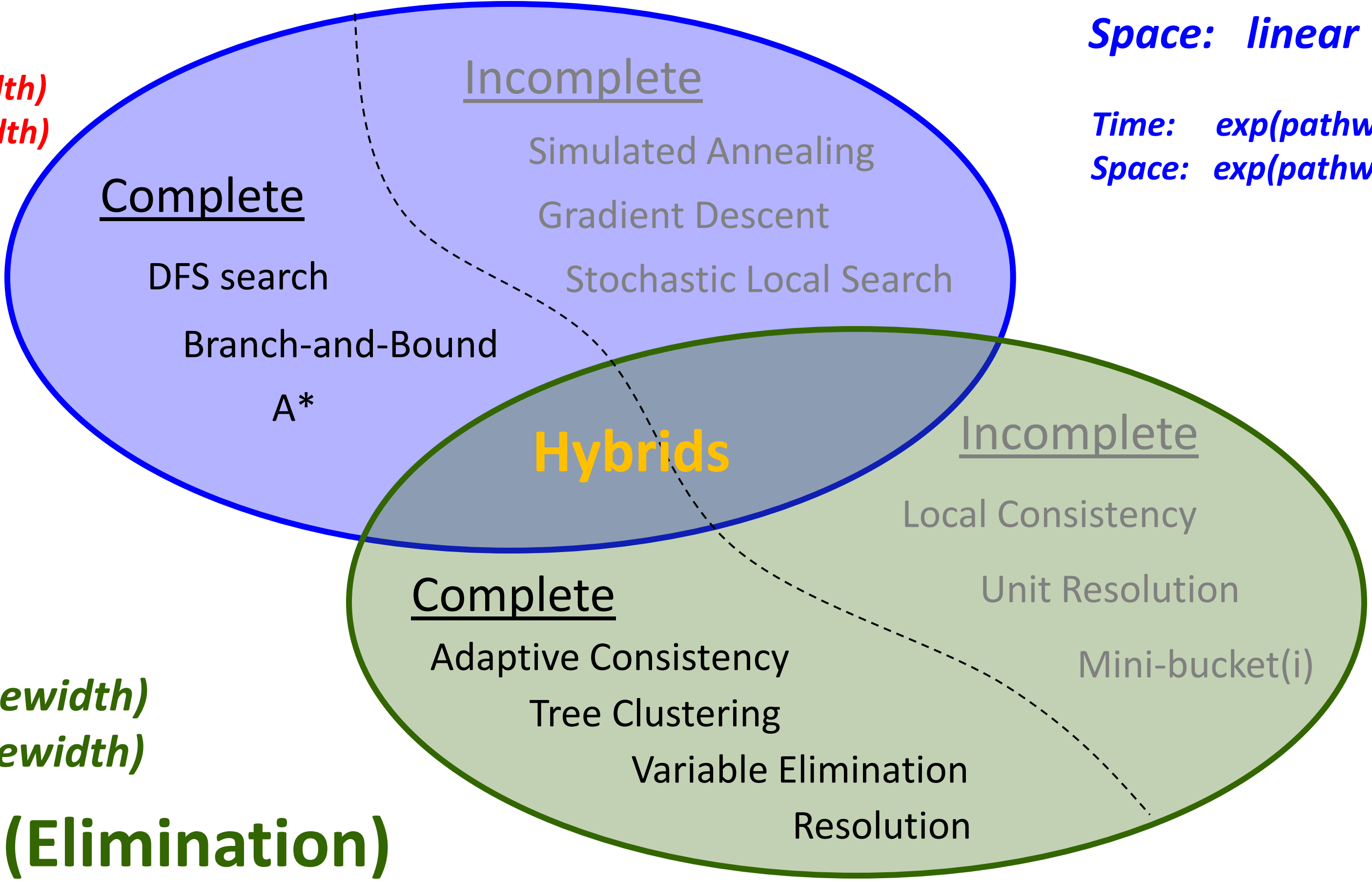
*Time: exp(treewidth * log n)*
Space: linear

Space: exp(treewidth)
Time: exp(treewidth)

Search (Conditioning)

Time: exp(n)
Space: linear

Time: exp(pathwidth)
Space: exp(pathwidth)



Complete

DFS search

Branch-and-Bound

A*

Incomplete

Simulated Annealing

Gradient Descent

Stochastic Local Search

Hybrids

Incomplete

Local Consistency

Unit Resolution

Mini-bucket(i)

Complete

Adaptive Consistency

Tree Clustering

Variable Elimination

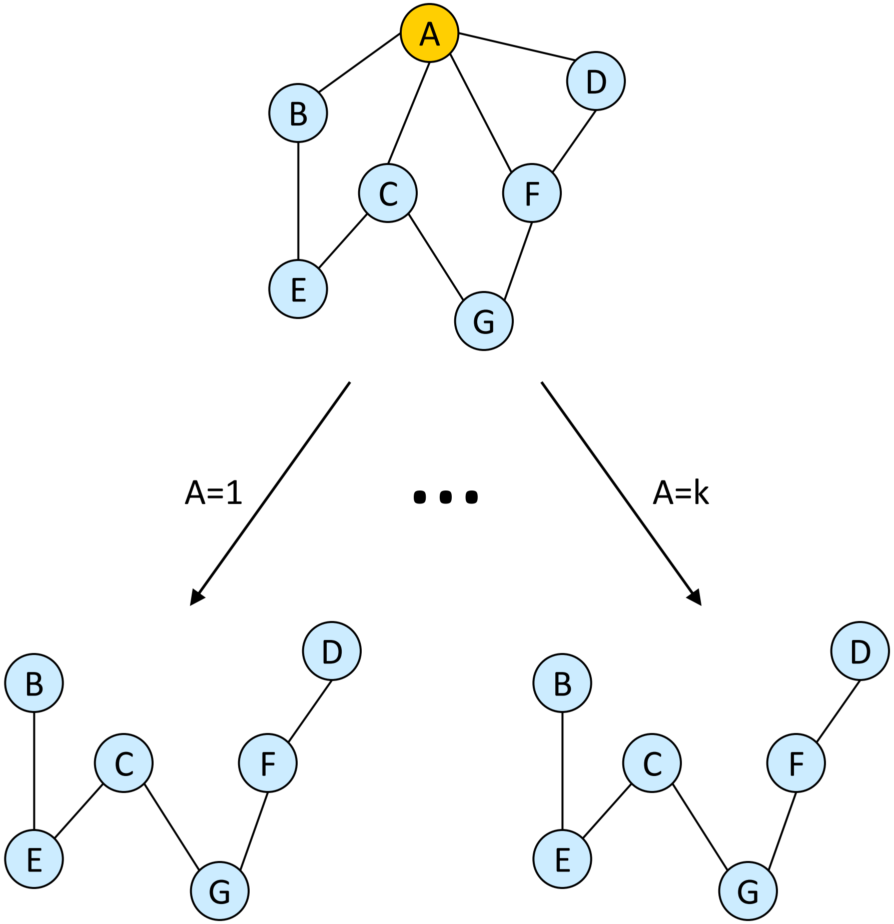
Resolution

Time: exp(treewidth)
Space: exp(treewidth)

Inference (Elimination)

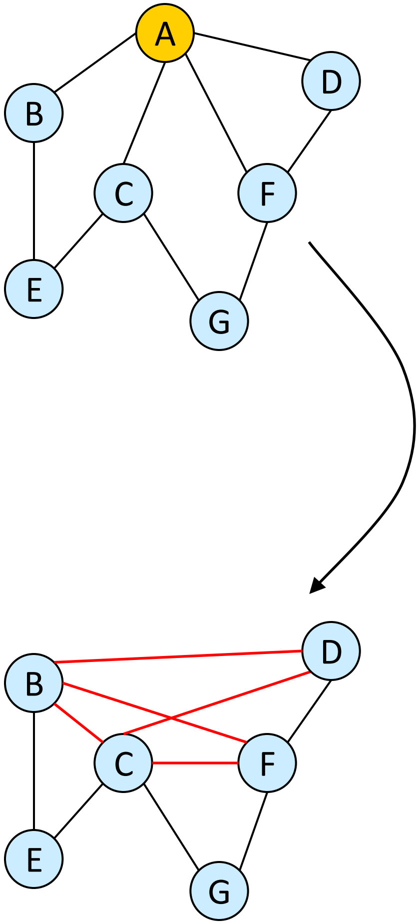
Search vs. Inference

Search (conditioning)



k “sparser” problems

Inference (elimination)



1 “denser” problem

Combination of Cost Functions

A	B	f(A,B)
b	b	6
b	g	0
g	b	0
g	g	6

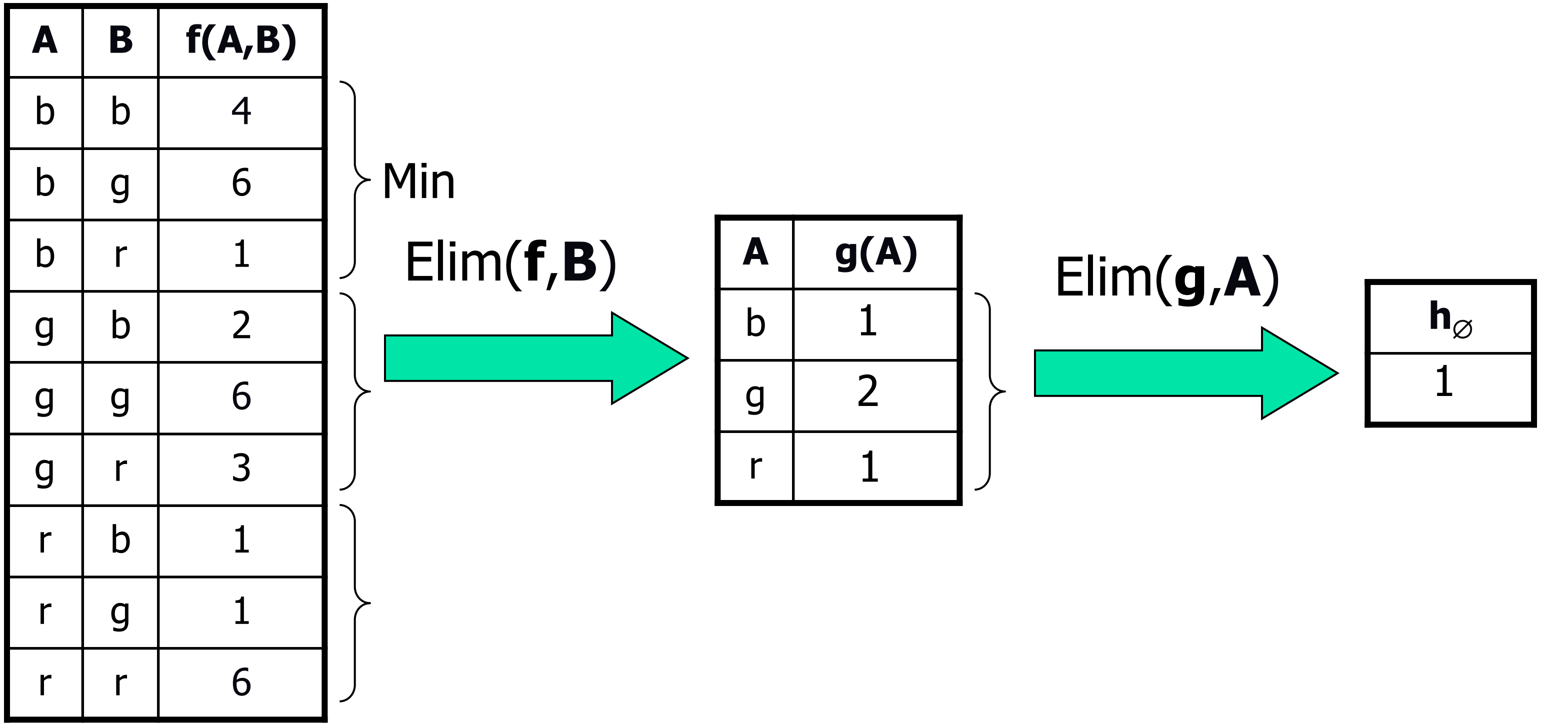
+

B	C	f(B,C)
b	b	6
b	g	0
g	b	0
g	g	6

A	B	C	f(A,B,C)
b	b	b	12
b	b	g	6
b	g	b	0
b	g	g	6
g	b	b	6
g	b	g	0
g	g	b	6
g	g	g	12

= 0 + 6

Elimination in a Cost Function

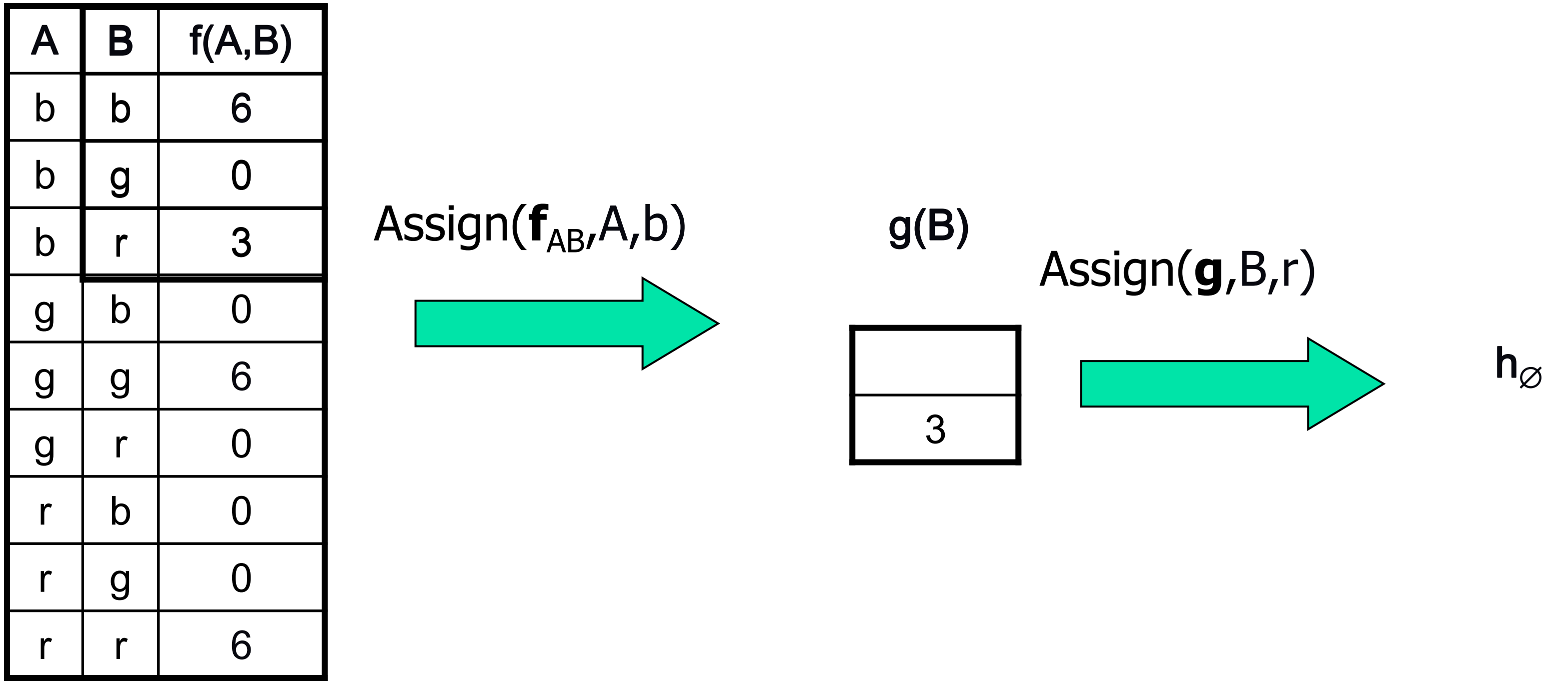


A	**B**	**f(A,B)**
b	b	4
b	g	6
b	r	1
g	b	2
g	g	6
g	r	3
r	b	1
r	g	1
r	r	6

A	**g(A)**
b	1
g	2
r	1

$\mathbf{h}_{\emptyset}$
1

Conditioning a Cost Function



A	B	f(A,B)
b	b	6
b	g	0
b	r	3
g	b	0
g	g	6
g	r	0
r	b	0
r	g	0
r	r	6

Assign($\mathbf{f}_{AB}$,A,b)

g(B)

3

Assign(**g**,B,r)

$\mathbf{h}_{\emptyset}$

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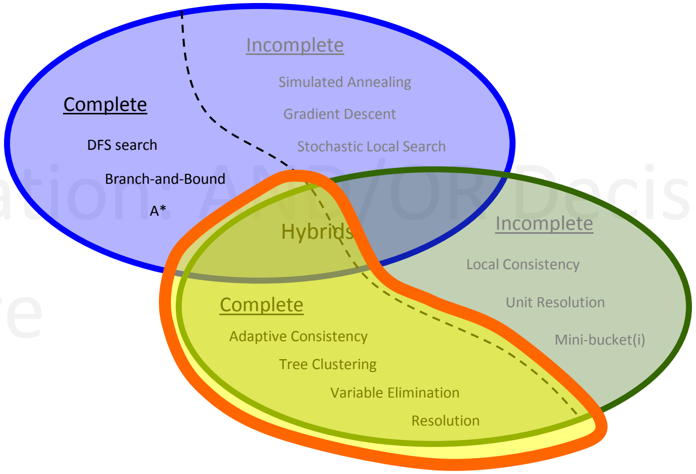
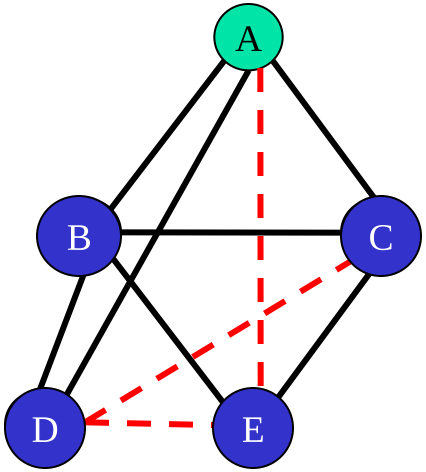


Computing the Optimal Cost Solution



Constraint graph

$$\mathbf{OPT} = \min_{e=0,d,c,b} \underbrace{f(a,b)}_{}+f(a,c)+f(a,d)+f\underbrace{(b,c)+f(b,d)+f(b,e)+f}_{}(c,e)$$

Combination

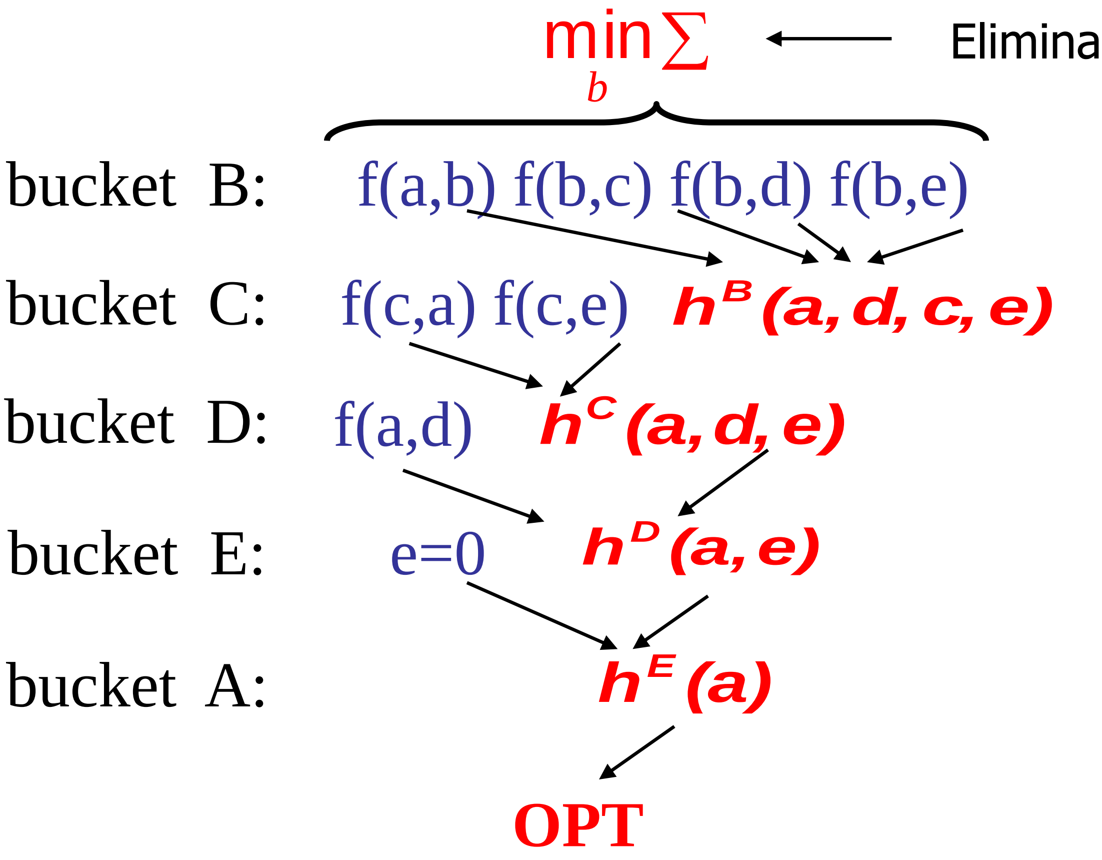
$$\min_{e=0}\min_{d} f(a,d) + \min_{c} f(a,c)+f(c,e) + \min_{b} \underbrace{f(a,b)+f(b,c)+f(b,d)+f(b,e)}_{h^B(a,d,c,e)}$$

Variable Elimination

Finding $OPT = \min_{X_1,\ldots,X_n} \sum_{j=1}^{r} f_j(X)$

Algorithm **elim-opt** (Dechter, 1996)
Non-serial Dynamic Programming (Bertele and Briochi, 1973)

$$OPT = \min_{a,e,d,c,b} F(a,b) + F(a,c) + F(a,d) + F(b,c) + F(b,d) + F(b,e) + F(c,e)$$



$\min_{b} \sum$ ← Elimination operator

bucket B: $\underbrace{f(a,b)\ f(b,c)\ f(b,d)\ f(b,e)}$

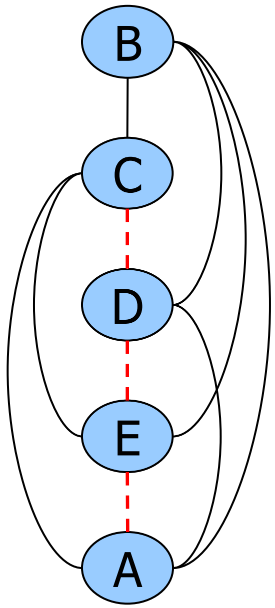
bucket C: $f(c,a)\ f(c,e)$ **$h^B(a,d,c,e)$**

bucket D: $f(a,d)$ **$h^C(a,d,e)$**

bucket E: $e=0$ **$h^D(a,e)$**

bucket A: **$h^E(a)$**

OPT



Generating the Optimal Assignment

5. $b' = \arg\min_{b} f(a',b) + f(b,c') + f(b,d') + f(b,e')$

4. $c' = \arg\min_{c} f(c,a') + f(c,e') + h^{B}(a',d',c,e')$

3. $d' = \arg\min_{d} f(a',d) + h^{C}(a',d,e')$

2. $e' = 0$

1. $a' = \arg\min_{a} h^{E}(a)$

B: $f(a,b)$ $f(b,c)$ $f(b,d)$ $f(b,e)$

C: $f(c,a)$ $f(c,e)$ $h^{B}(a,d,c,e)$

D: $f(a,d)$ $h^{C}(a,d,e)$

E: $e=0$ $h^{D}(a,e)$

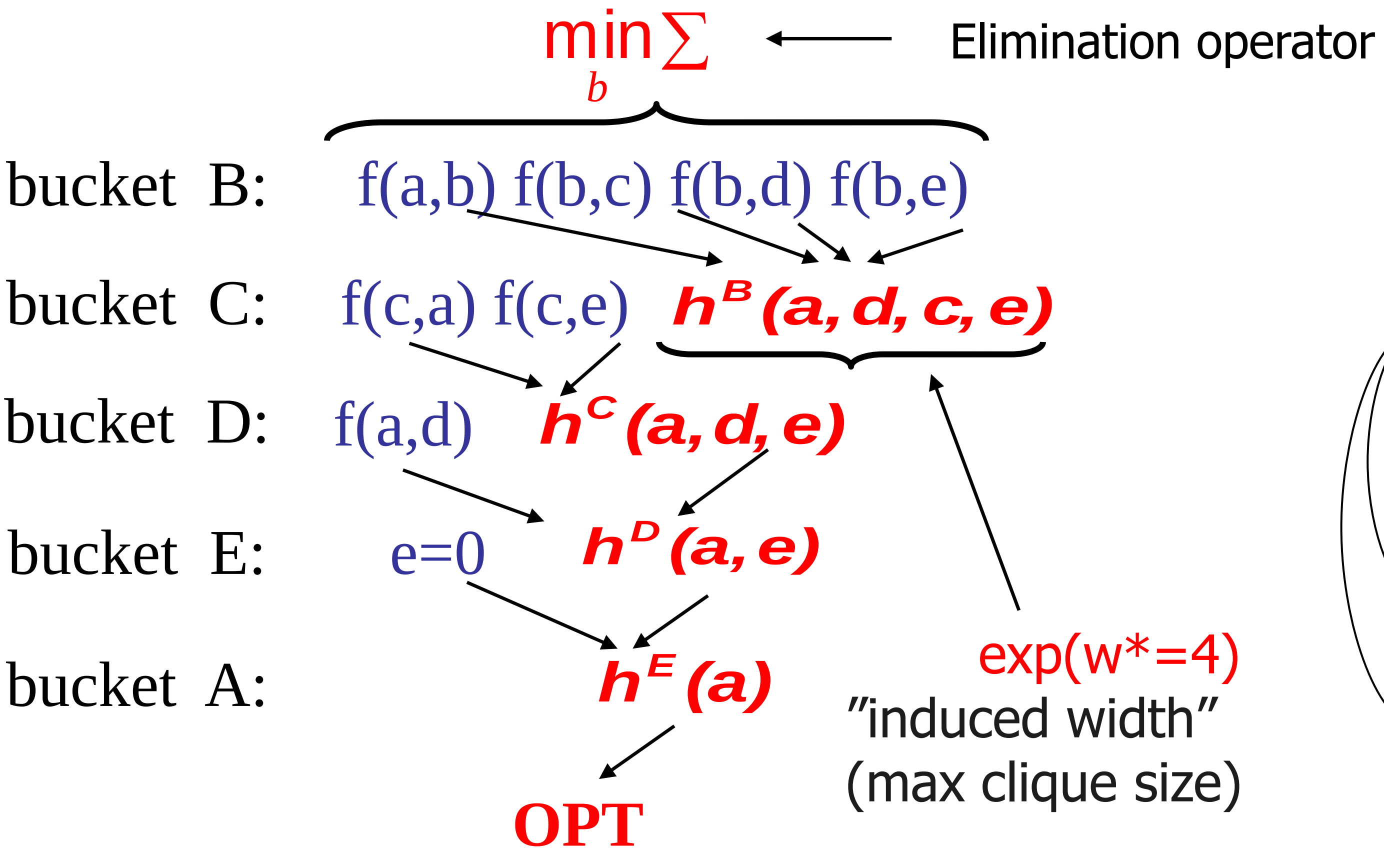
A: $h^{E}(a)$

Return (a',b',c',d',e')

Complexity

Algorithm **elim-opt** (Dechter, 1996)
Non-serial Dynamic Programming (Bertele and Briochi, 1973)

$$OPT = \min_{a,e,d,c,b} F(a,b) + F(a,c) + F(a,d) + F(b,c) + F(b,d) + F(b,e) + F(c,e)$$



$\min_b \sum$ ← Elimination operator

bucket B: f(a,b) f(b,c) f(b,d) f(b,e)

bucket C: f(c,a) f(c,e) $h^B(a,d,c,e)$

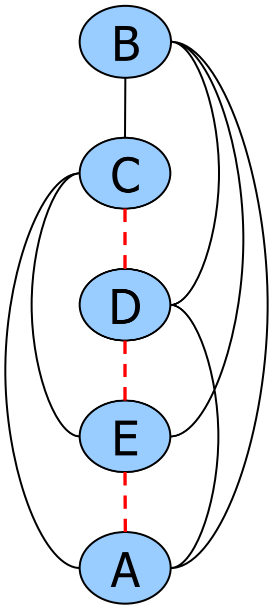
bucket D: f(a,d) $h^C(a,d,e)$

bucket E: e=0 $h^D(a,e)$

bucket A: $h^E(a)$

OPT

exp(w*=4)
"induced width"
(max clique size)



Complexity of Bucket Elimination

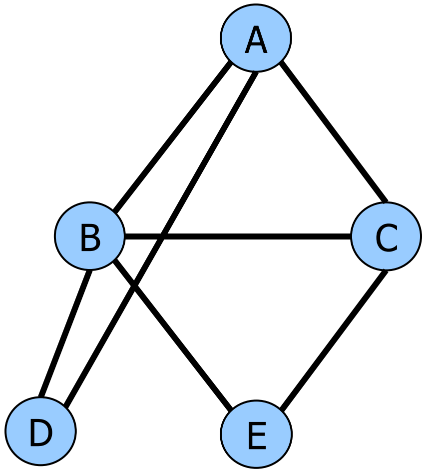
Bucket-Elimination is **time** and **space**

$$O(r \exp(w^*(d)))$$

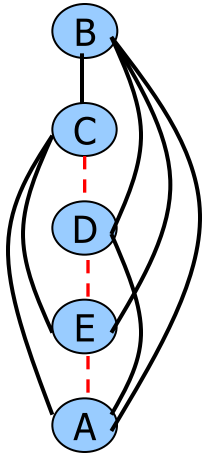
$w^*(d)$ – the induced width of the primal graph along ordering d

r = number of functions

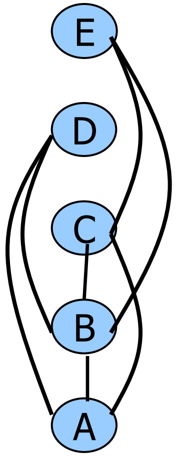
The effect of the ordering:



constraint graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

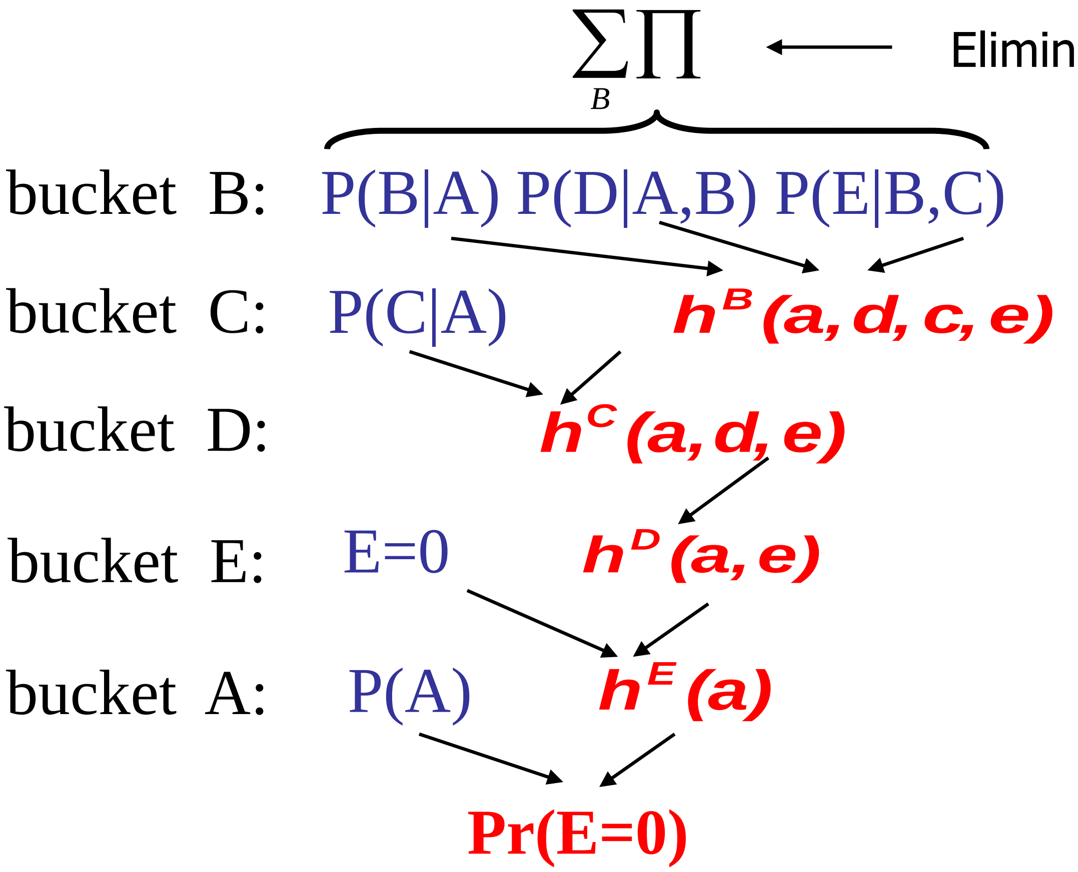
Finding smallest induced-width is hard!

Finding

$$P(evidence) = \sum_{X_1,\ldots,X_n} \prod_{j=1}^{n} P(X_j \mid pa_j)$$

Algorithm **elim-bel** (Dechter, 1996)

$$P(E=0) = \sum_{A,E=0,D,C,B} P(A) \cdot P(B \mid A) \cdot P(C \mid A) \cdot P(D \mid A,B) \cdot P(E \mid B,C)$$



$$\underbrace{\sum_{B}\prod}$$ ← Elimination operator

bucket B: P(B|A) P(D|A,B) P(E|B,C)

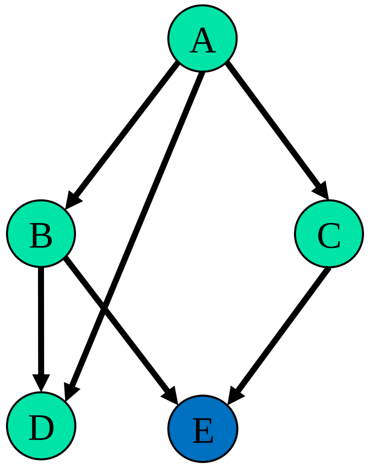
bucket C: P(C|A) $h^B(a,d,c,e)$

bucket D: $h^C(a,d,e)$

bucket E: E=0 $h^D(a,e)$

bucket A: P(A) $h^E(a)$

Pr(E=0)



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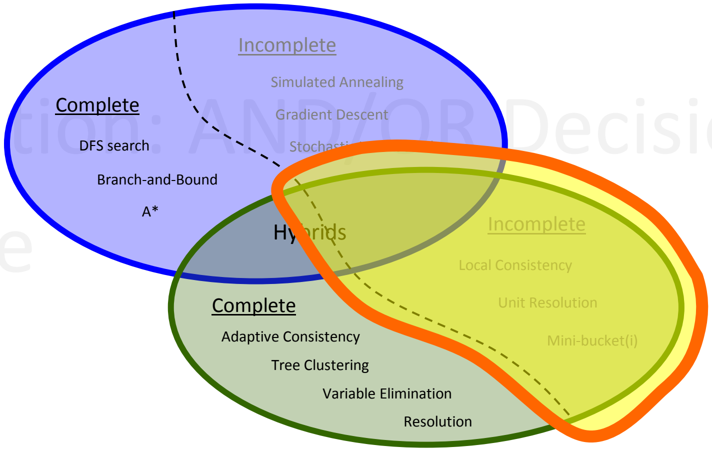


Incomplete
Simulated Annealing
Gradient Descent
Stochasti

Complete
DFS search
Branch-and-Bound
A*

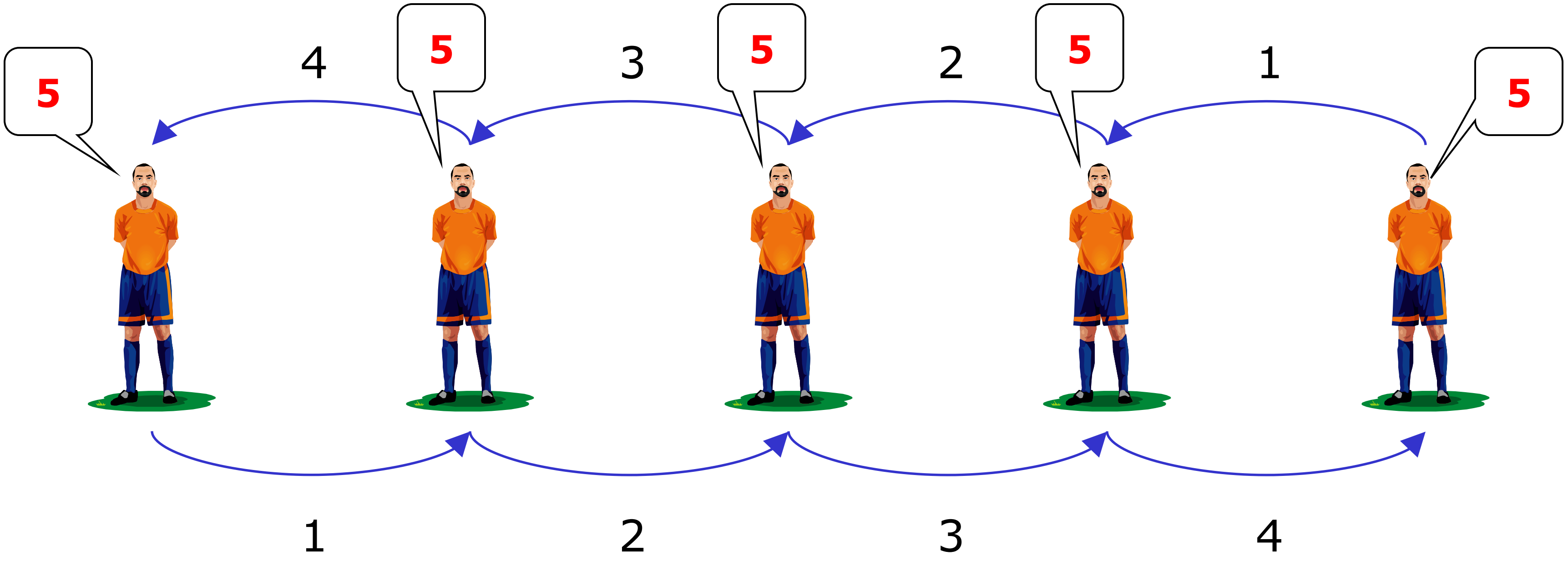
Hybrids

Incomplete
Local Consistency
Unit Resolution
Mini-bucket(i)

Complete
Adaptive Consistency
Tree Clustering
Variable Elimination
Resolution

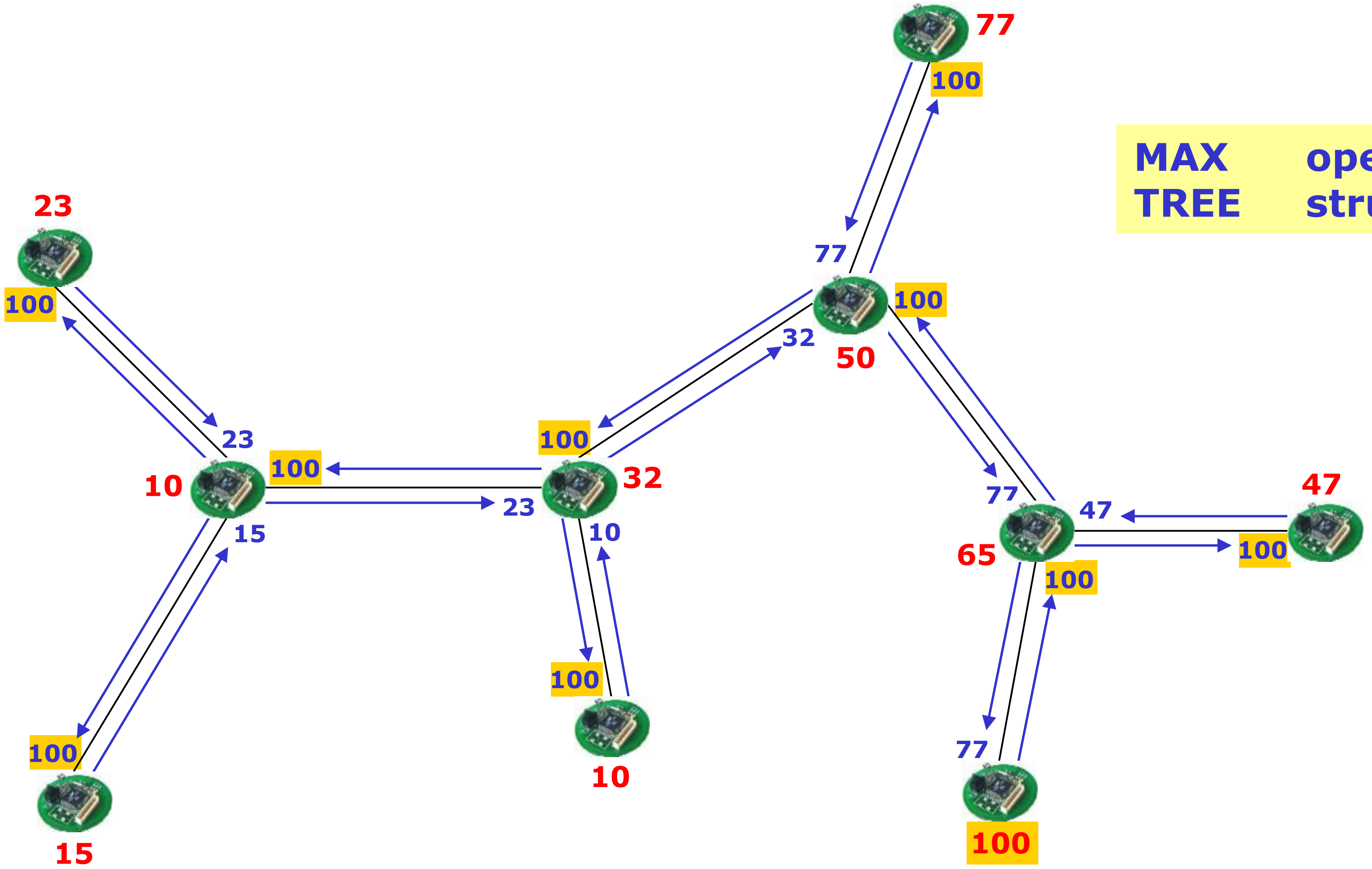
Counting

SUM operator
CHAIN structure



How many people?

Maximization



MAX operator
TREE structure

What is the maximum?

Min-Cost Assignment

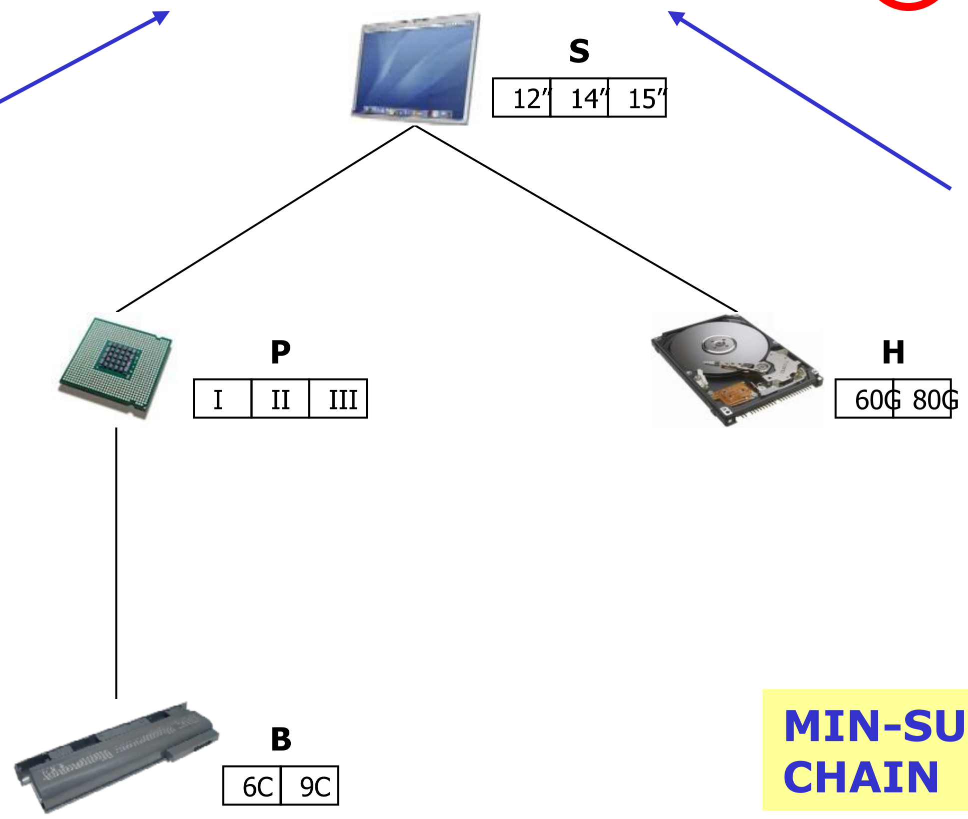
12″	14″	15″
75	80	105

+

12″	14″	15″
30	40	50

=

12″	14″	15″
105	120	155



	I	II	III
12″	**75**	∞	∞
14″	**80**	1600	1730
15″	∞	**105**	1840

+

I	II	III
30	40	60

	6C	9C
I	**30**	50
II	**40**	55
III	∞	**60**

	60G	80G
12″	**30**	50
14″	**40**	45
15″	**50**	∞

MIN-SUM **operators**
CHAIN **structure**

What is minimum cost configuration?

Belief Updating

H	P(H)
0	.9
1	.1

F	P(F)
0	.99
1	.01

*

F	h_3(F)
0	.1245
1	.73175

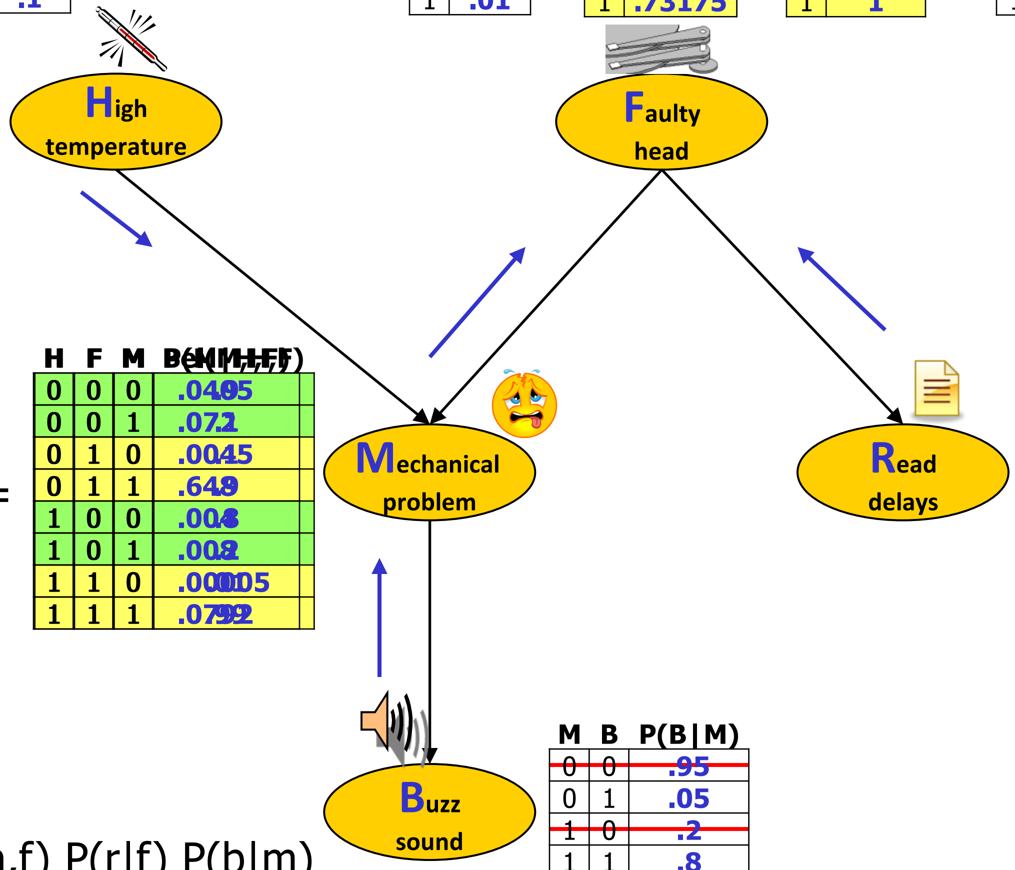
*

F	h_4(F)
0	1
1	1

=

F	P(F,B=1)
0	.123255
1	.073175

SUM-PROD operators
POLY-TREE structure



H	F	M	P(M|H,F)
0	0	0	.9
0	0	1	.1
0	1	0	.1
0	1	1	.9
1	0	0	.8
1	0	1	.2
1	1	0	.01
1	1	1	.99

*

M	h_1(M)
0	.05
1	.8

*

H	h_2(H)
0	.9
1	.1

=

H	F	M	Bel(M,H,F)
0	0	0	.0405
0	0	1	.072
0	1	0	.0045
0	1	1	.648
1	0	0	.008
1	0	1	.002
1	1	0	.00005
1	1	1	.0792

F	R	P(R|F)
0	0	.8
0	1	.2
1	0	.3
0	1	.7

M	B	P(B|M)
~~0~~	~~0~~	~~.95~~
0	1	.05
~~1~~	~~0~~	~~.2~~
1	1	.8

P(h,f,r,m,b) = P(h) P(f) P(m|h,f) P(r|f) P(b|m)

P(F | B=1) = ?

P(B=1) = .19643

Probability of evidence

P(F=1|B=1) = .3725

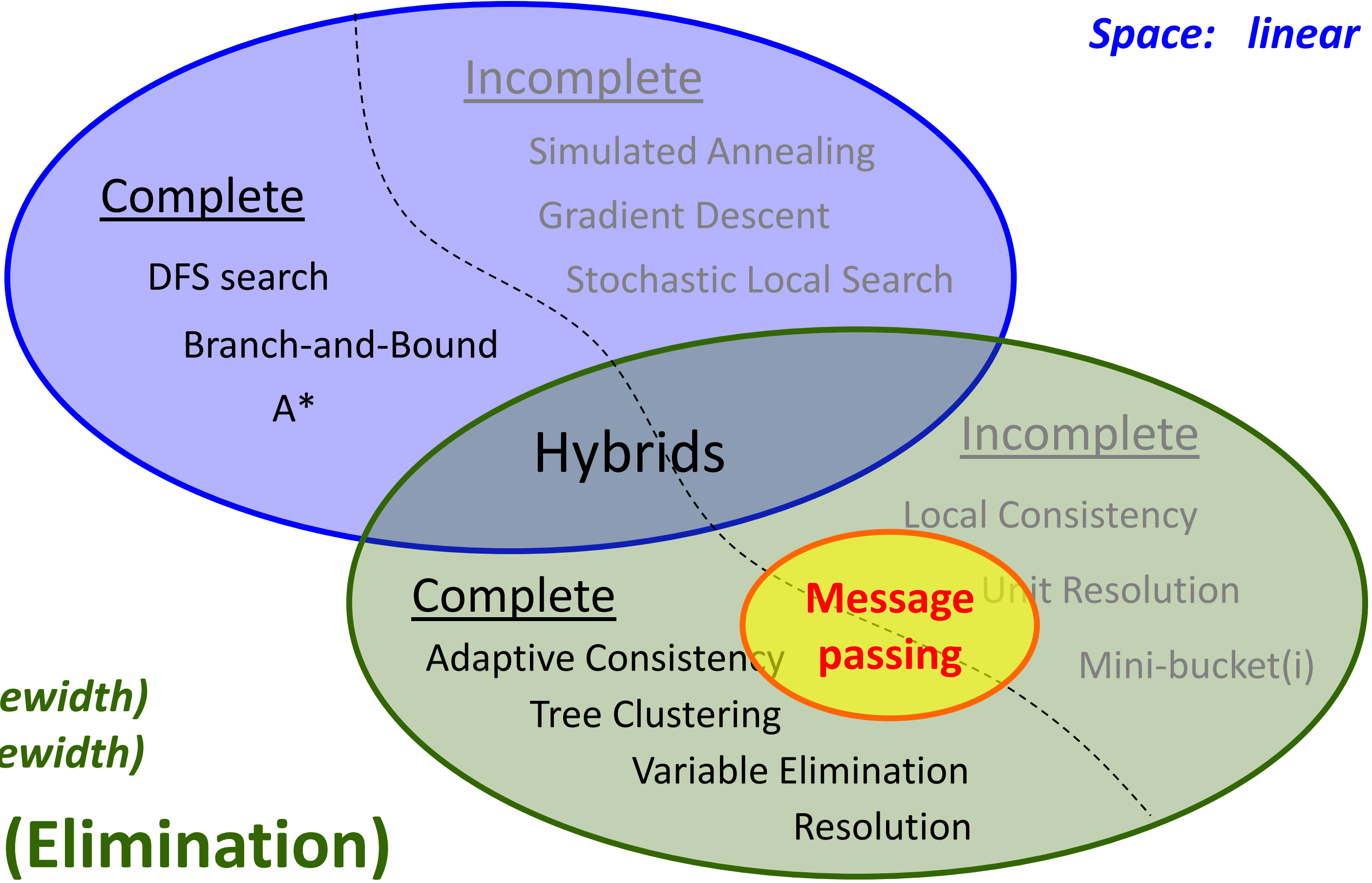
Updated belief

Belief Propagation

(Pearl, 1988)

- Instances of tree message passing algorithm
- Exact for trees
- Linear in the input size
- Importance:
  - One of the first algorithms for inference in Bayesian networks
  - Gives a cognitive dimension to its computations
  - Basis for conditioning algorithms for arbitrary Bayesian network
  - Basis for Loopy Belief Propagation (approximate algorithms)

Solution Techniques



Search (Conditioning)

Time: exp(n)
Space: linear

Incomplete

Simulated Annealing

Gradient Descent

Stochastic Local Search

Complete

DFS search

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Local Consistency

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Mini-bucket(i)

Message passing

Complete

Adaptive Consistency

Tree Clustering

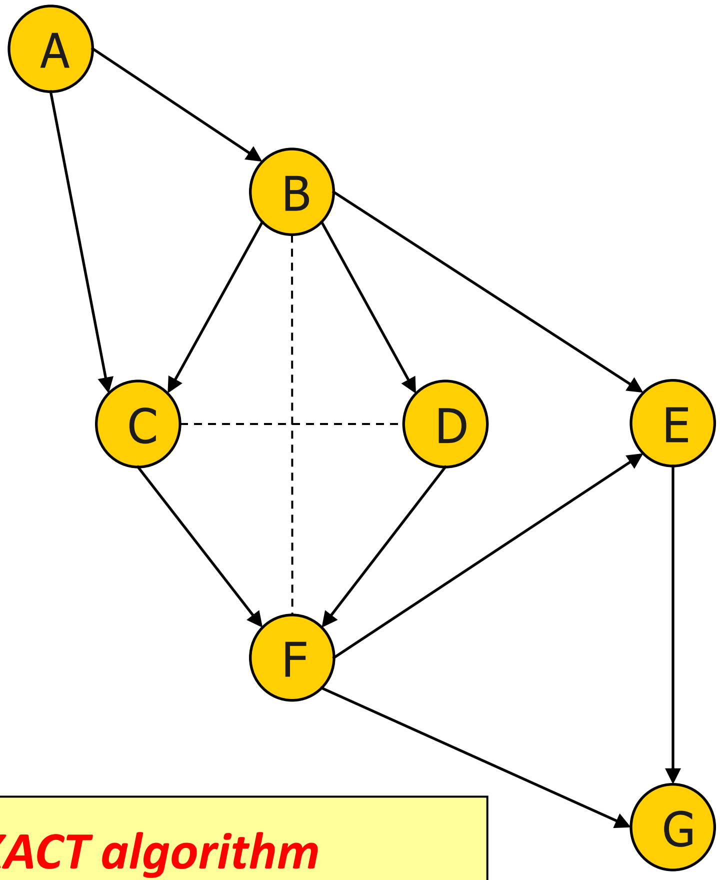
Variable Elimination

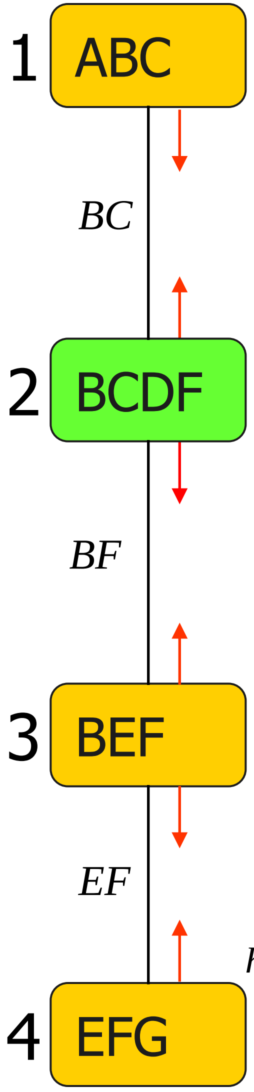
Resolution

Time: exp(treewidth)
Space: exp(treewidth)

Inference (Elimination)

Join-Tree Clustering





$$h_{(1,2)}(b,c) = \sum_{a} p(a)\cdot p(b\,|\,a)\cdot p(c\,|\,a,b)$$

$$h_{(2,1)}(b,c) = \sum_{d,f} p(d\,|\,b)\cdot p(f\,|\,c,d)\cdot h_{(3,2)}(b,f)$$

$$h_{(2,3)}(b,f) = \sum_{c,d} p(d\,|\,b)\cdot p(f\,|\,c,d)\cdot h_{(1,2)}(b,c)$$

$$h_{(3,2)}(b,f) = \sum_{e} p(e\,|\,b,f)\cdot h_{(4,3)}(e,f)$$

$$h_{(3,4)}(e,f) = \sum_{b} p(e\,|\,b,f)\cdot h_{(2,3)}(b,f)$$

$$h_{(4,3)}(e,f) = p(G=g_e\,|\,e,f)$$

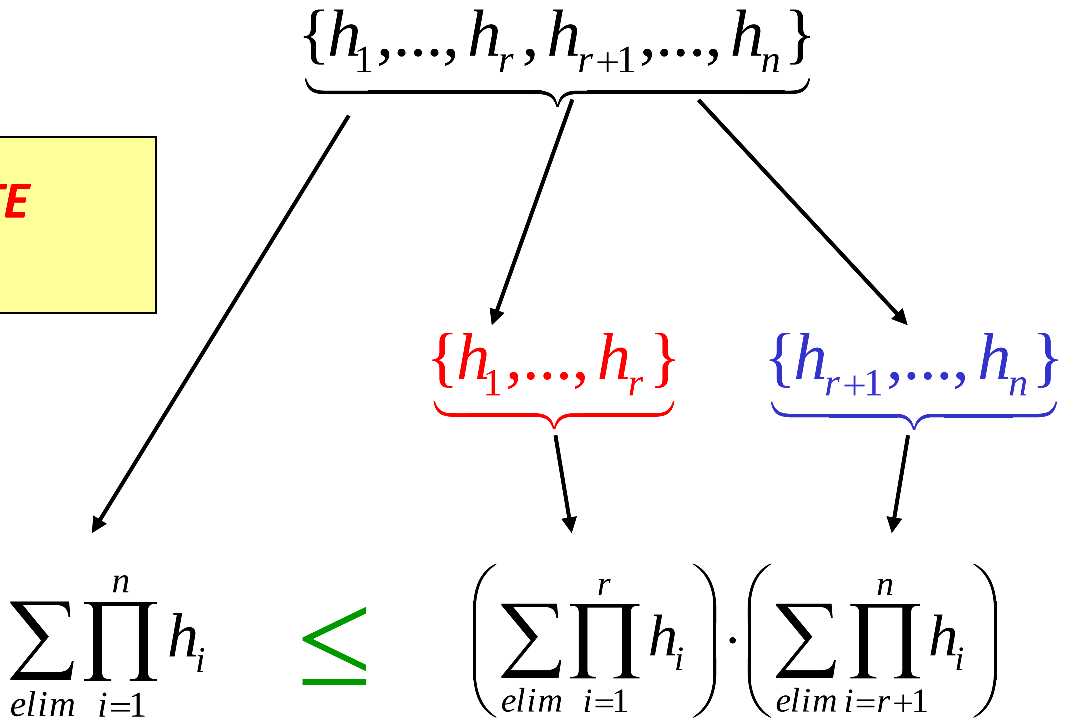
EXACT algorithm

Time and space:
exp(cluster size)=
exp(treewidth)

Mini-Clustering

Split a cluster into mini-clusters => bound complexity

APPROXIMATE algorithm



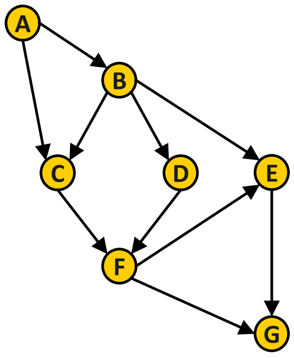
$$\underbrace{\{h_1,\dots,h_r,h_{r+1},\dots,h_n\}}$$

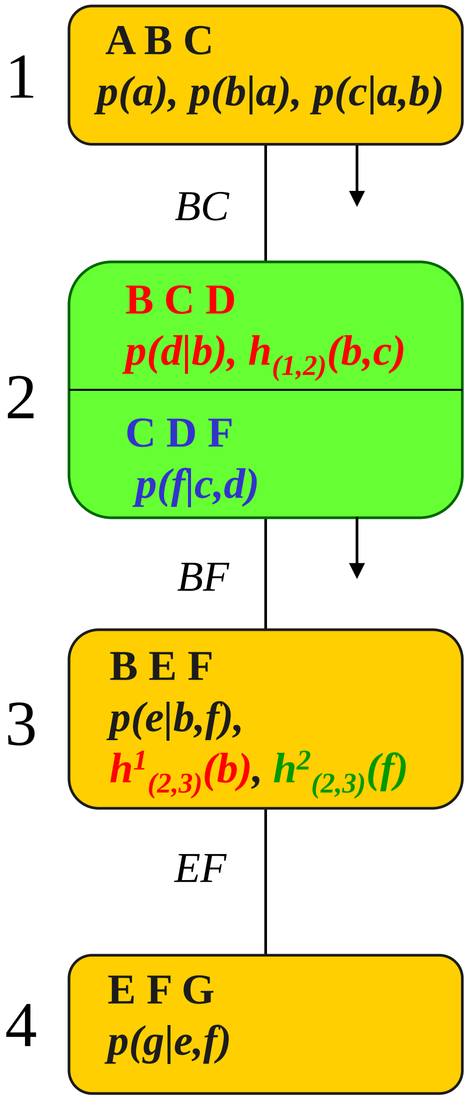
$$\underbrace{\{h_1,\dots,h_r\}} \qquad \underbrace{\{h_{r+1},\dots,h_n\}}$$

$$\sum_{elim}\prod_{i=1}^{n} h_i \le \left(\sum_{elim}\prod_{i=1}^{r} h_i\right)\cdot\left(\sum_{elim}\prod_{i=r+1}^{n} h_i\right)$$

Exponential complexity decrease $O(e^{n}) \rightarrow O(e^{\mathrm{var}(r)}) + O(e^{\mathrm{var}(n-r)})$

Mini-Clustering, i-bound=3





1

A B C

p(a), p(b|a), p(c|a,b)

BC

2

B C D

p(d|b), h $_{(1,2)}$ ***(b,c)***

C D F

p(f|c,d)

BF

3

B E F

p(e|b,f),

$h^1_{(2,3)}(b)$, $h^2_{(2,3)}(f)$

EF

4

E F G

p(g|e,f)

$$h^1_{(1,2)}(b,c) = \sum_a p(a) \cdot p(b|a) \cdot p(c|a,b)$$

$$h^1_{(2,3)}(b) = \sum_{c,d} p(d|b) \cdot h^1_{(1,2)}(b,c)$$

$$h^2_{(2,3)}(f) = \max_{c,d} p(f|c,d)$$

APPROXIMATE algorithm

Time and space:

exp(i-bound)

↓

Number of variables in a mini-cluster

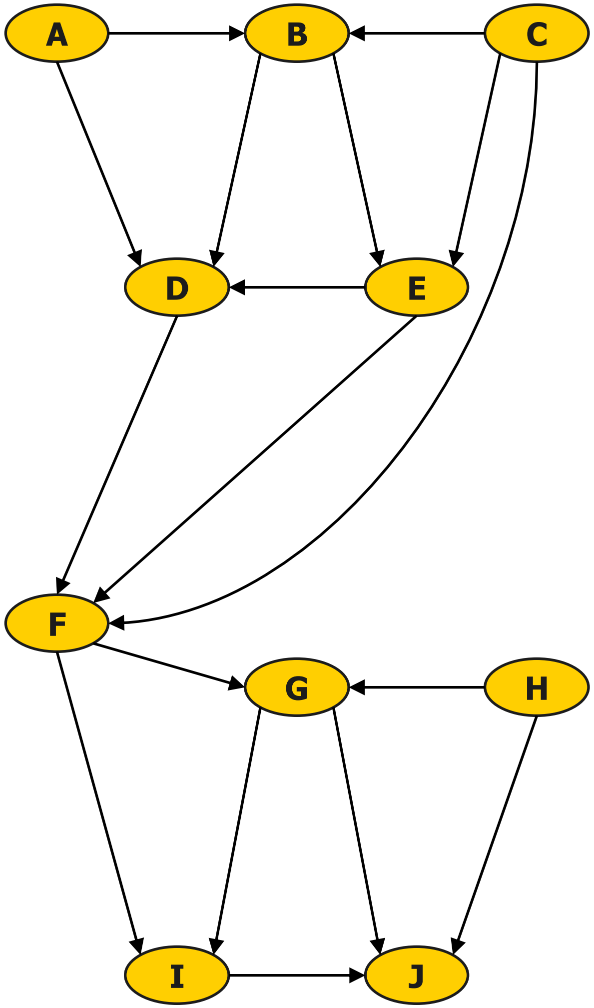
Iterative Join Graph Propagation

- Loopy Belief Propagation
  - Cyclic graphs
  - Iterative
  - Converges fast in practice (no guarantees though)
  - Very good approximations (e.g., turbo decoding, LDPC codes, SAT – survey propagation)

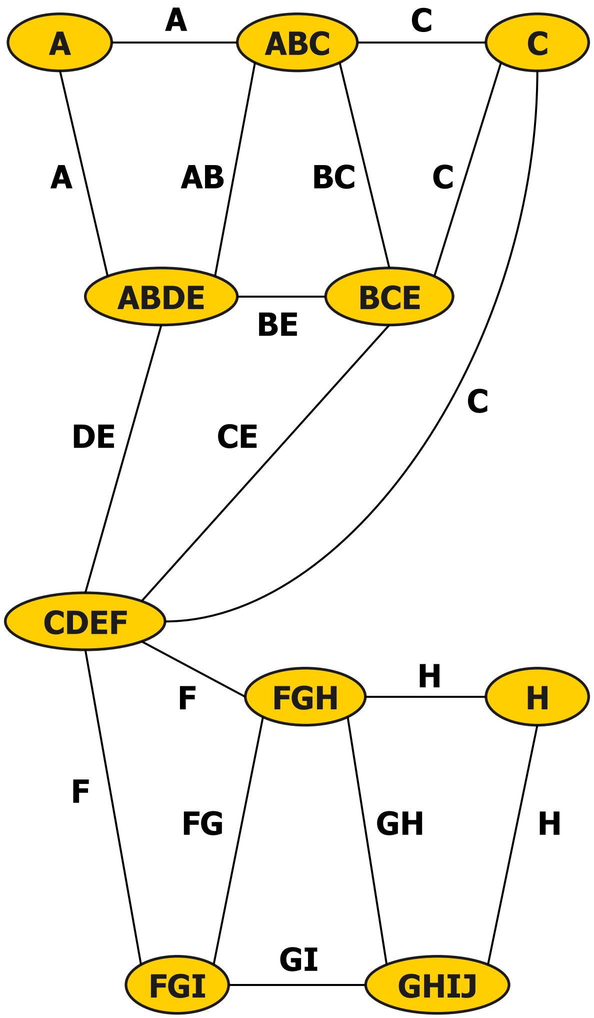
- Mini-Clustering(i)
  - Tree decompositions
  - Only two sets of messages (inward, outward)
  - Anytime behavior – can improve with more time by increasing the i-bound

- We want to combine:
  - Iterative virtues of Loopy BP
  - Anytime behavior of Mini-Clustering(i)

IJGP - Example

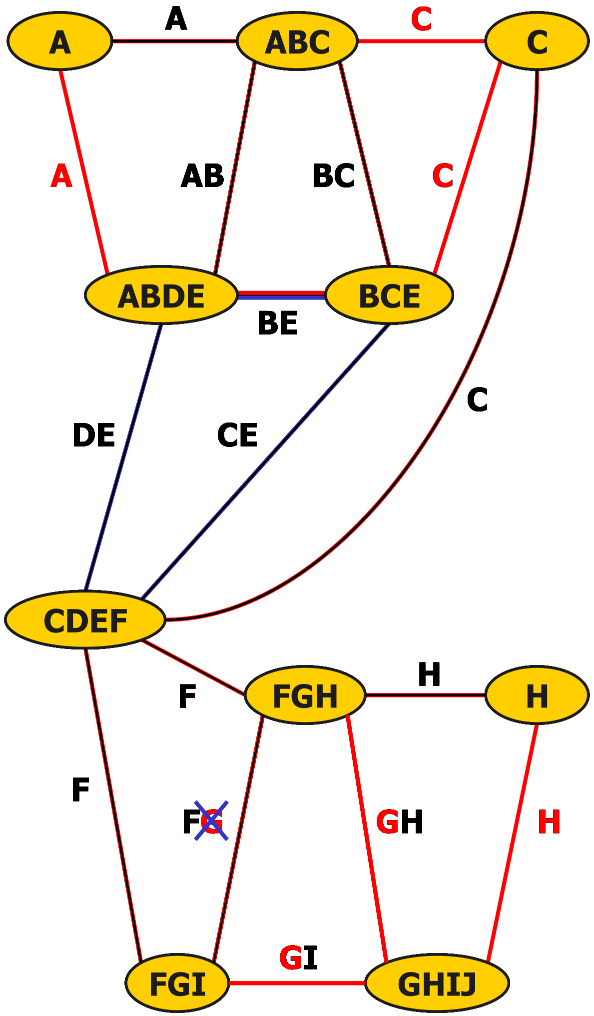


Belief network



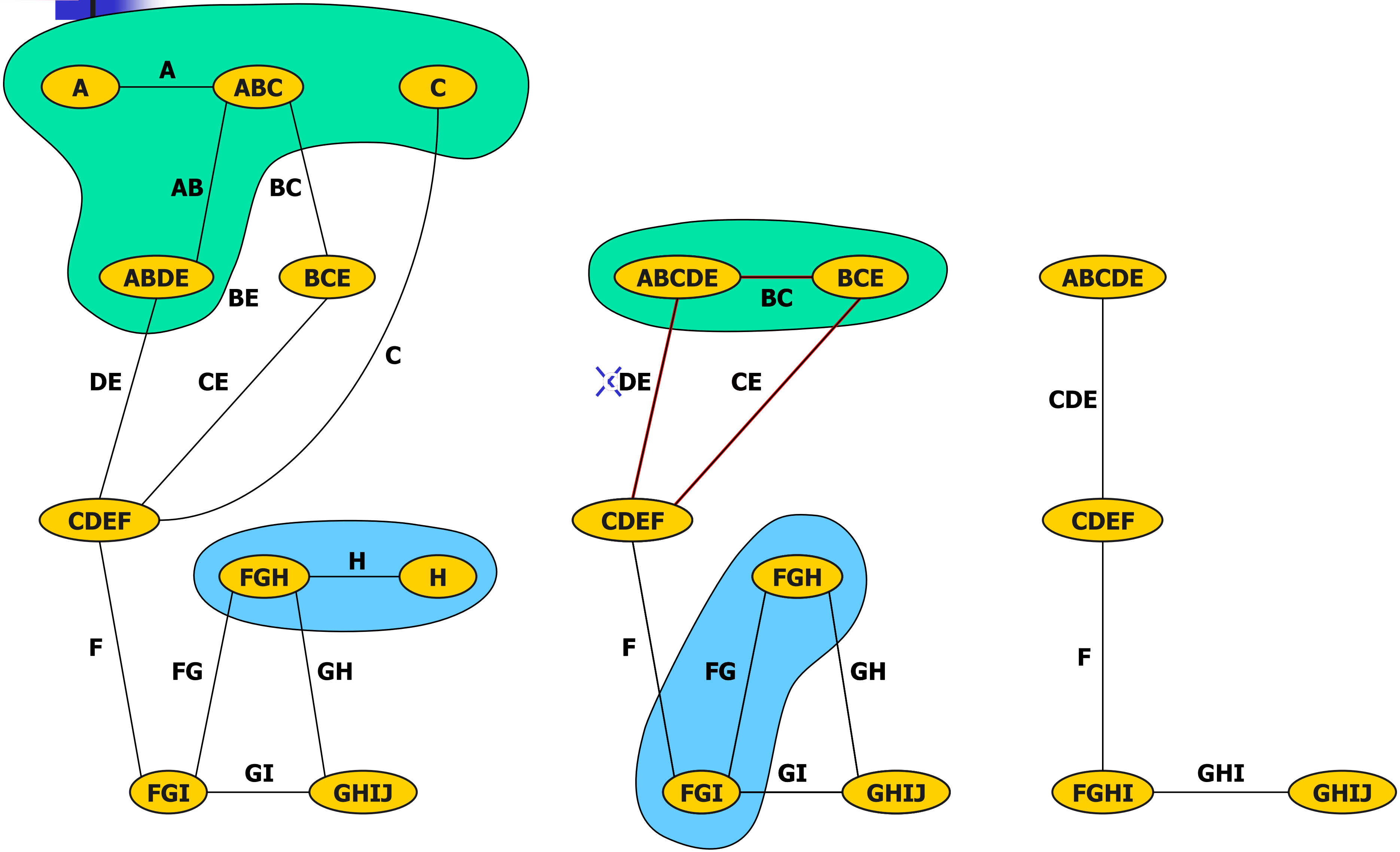
Loopy BP graph

Arc-Minimal Join-Graph

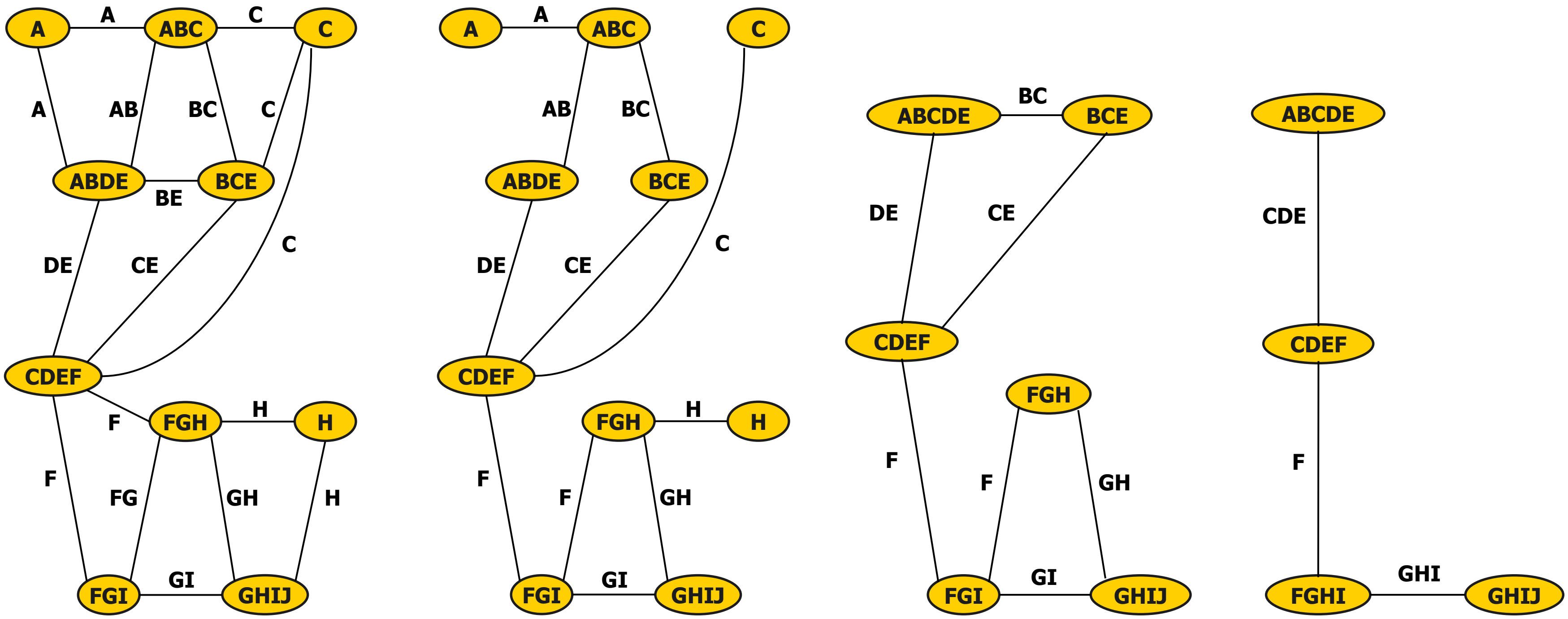


Arcs labeled with any single variable should form a TREE

Collapsing Clusters



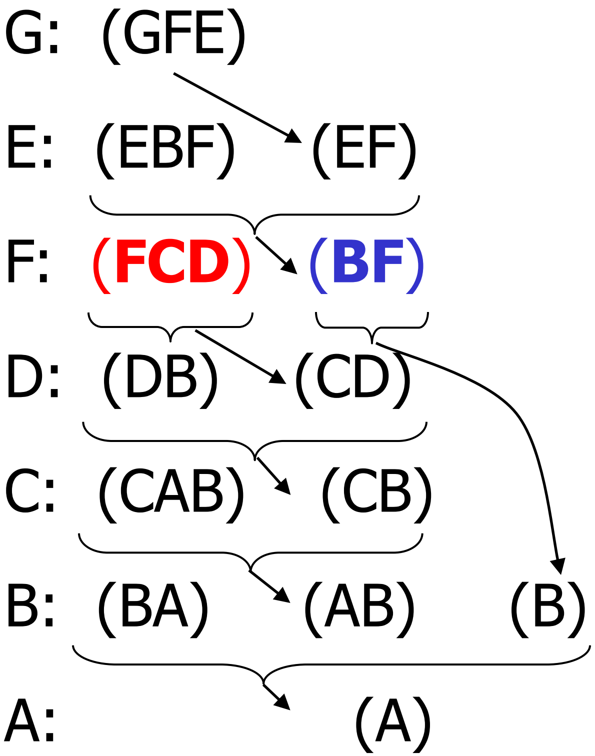
Join-Graphs



more accuracy

less complexity

Constructing Join-Graphs



G: (GFE)

E: (EBF) (EF)

F: **(FCD)** **(BF)**

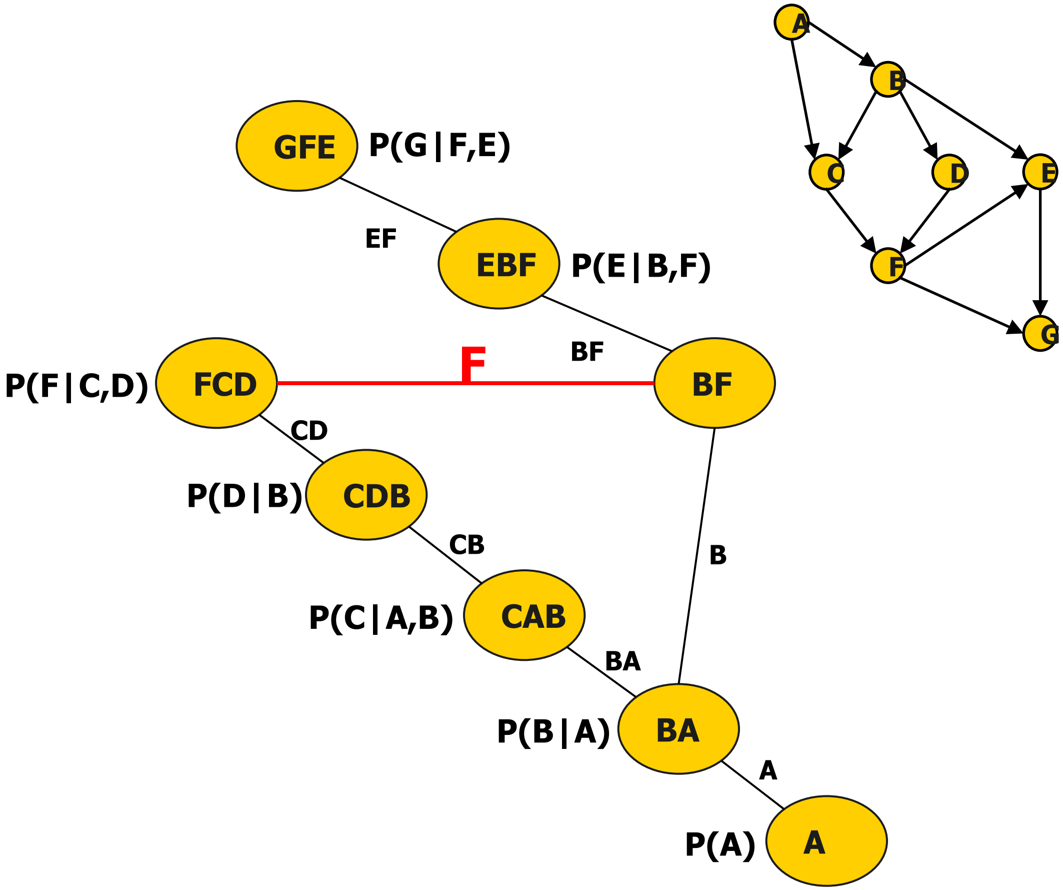
D: (DB) (CD)

C: (CAB) (CB)

B: (BA) (AB) (B)

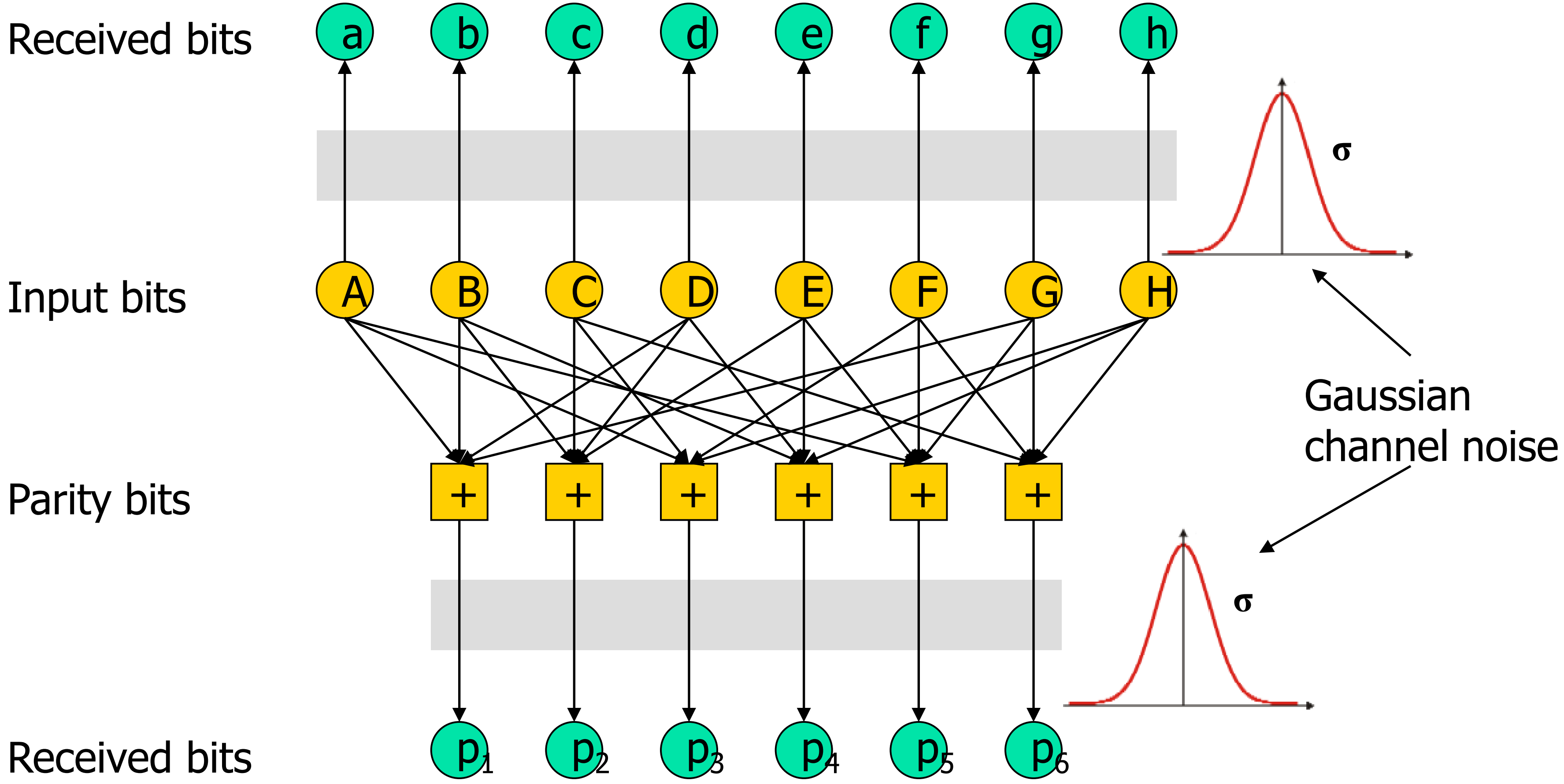
A: (A)

a) schematic mini-bucket(i), i=3



b) arc-labeled join-graph decomposition

Linear Block Codes


Received bits

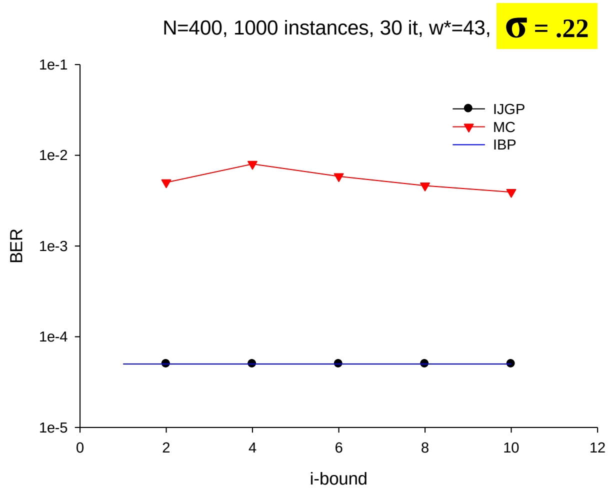
Input bits

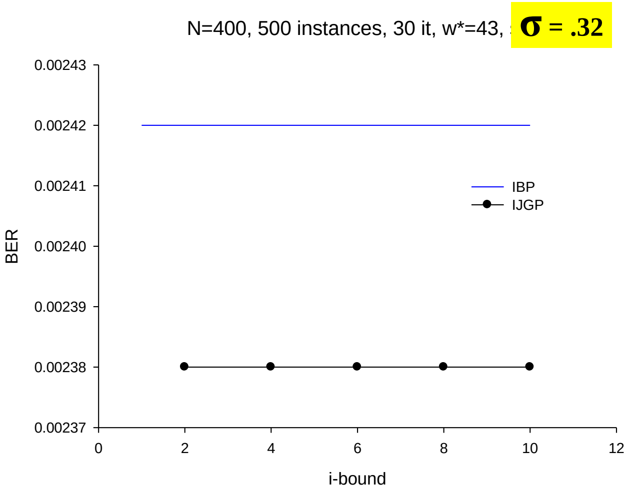
Parity bits

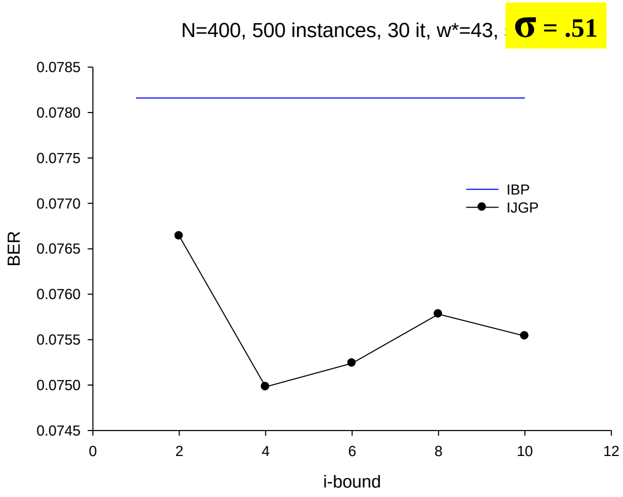
Received bits

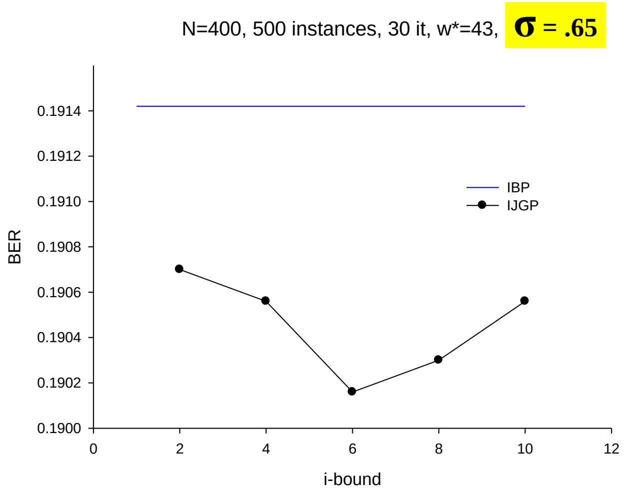
Gaussian channel noise

Coding Networks – Bit Error Rate

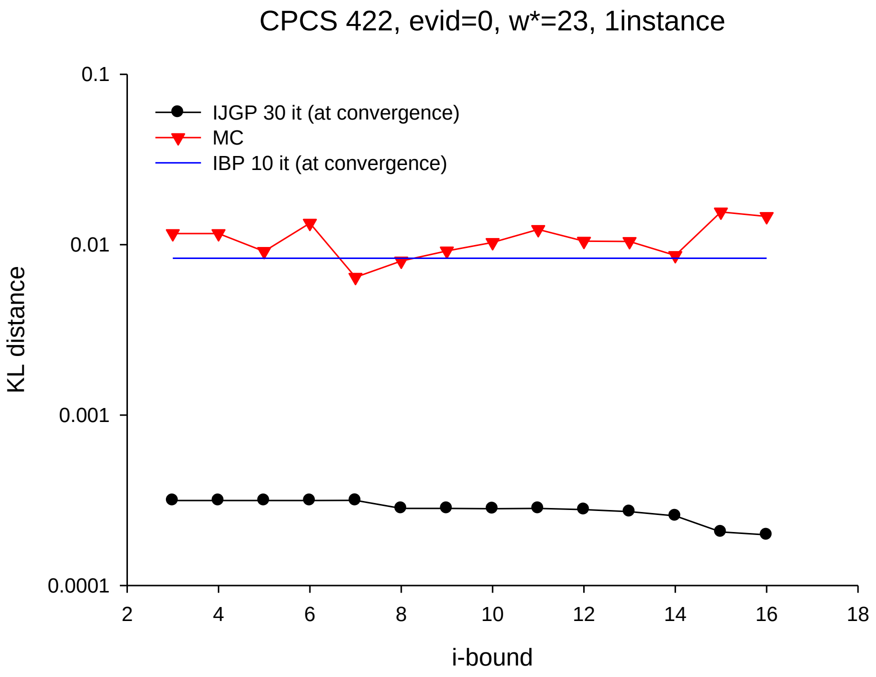




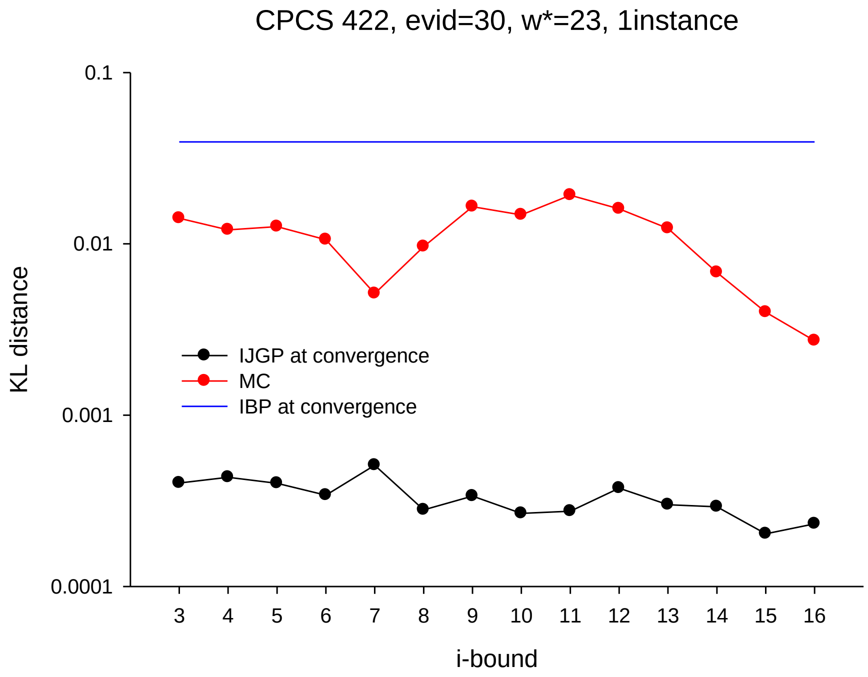




CPCS 422 – KL Distance

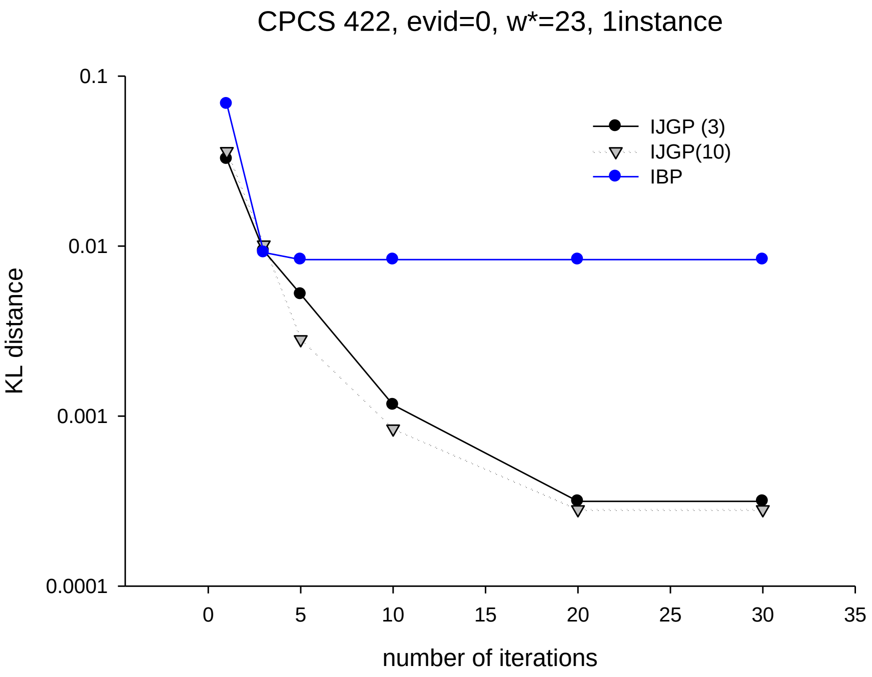


evidence=0

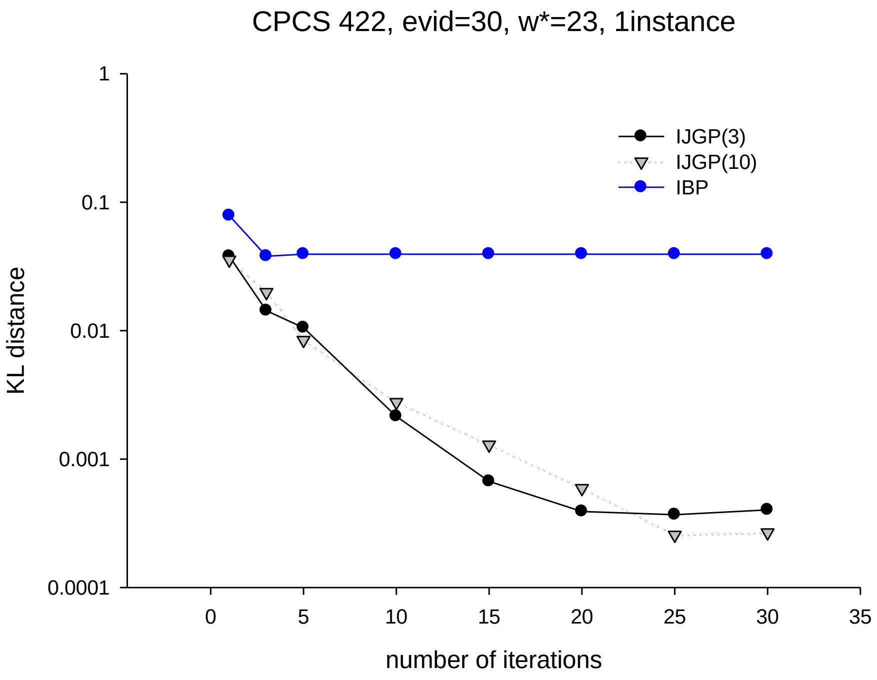


evidence=30

CPCS 422 – KL vs. Iterations



evidence=0



evidence=30

Inference Power of Loopy BP

- Comparison with iterative algorithms in constraint networks
- Zero-beliefs inconsistent assignments

$\Leftrightarrow$

- ε-small beliefs – experimental study

Constraint networks

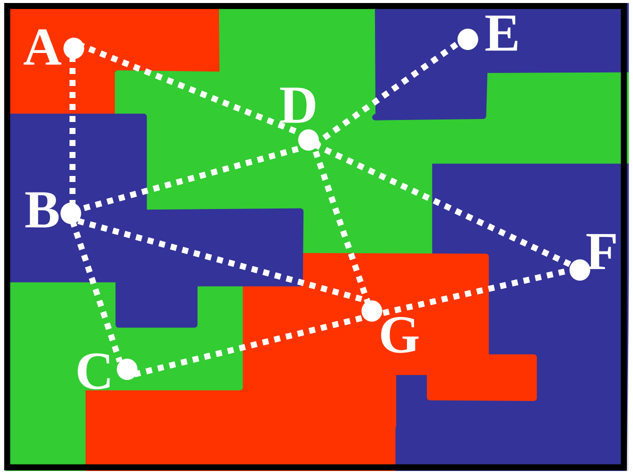
Map coloring

Variables: countries (A B C etc.)

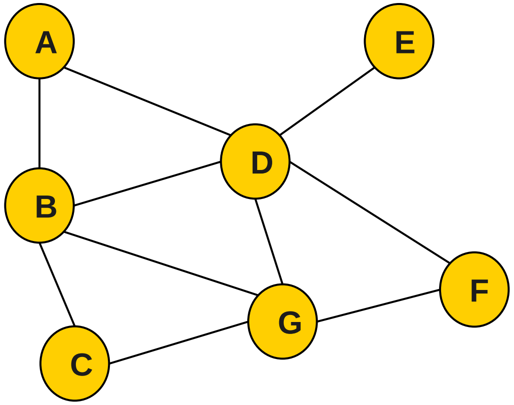
Values: colors (red green blue)

Constraints: **$A \neq B$, $A \neq D$, $D \neq E$**, *etc.*

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red

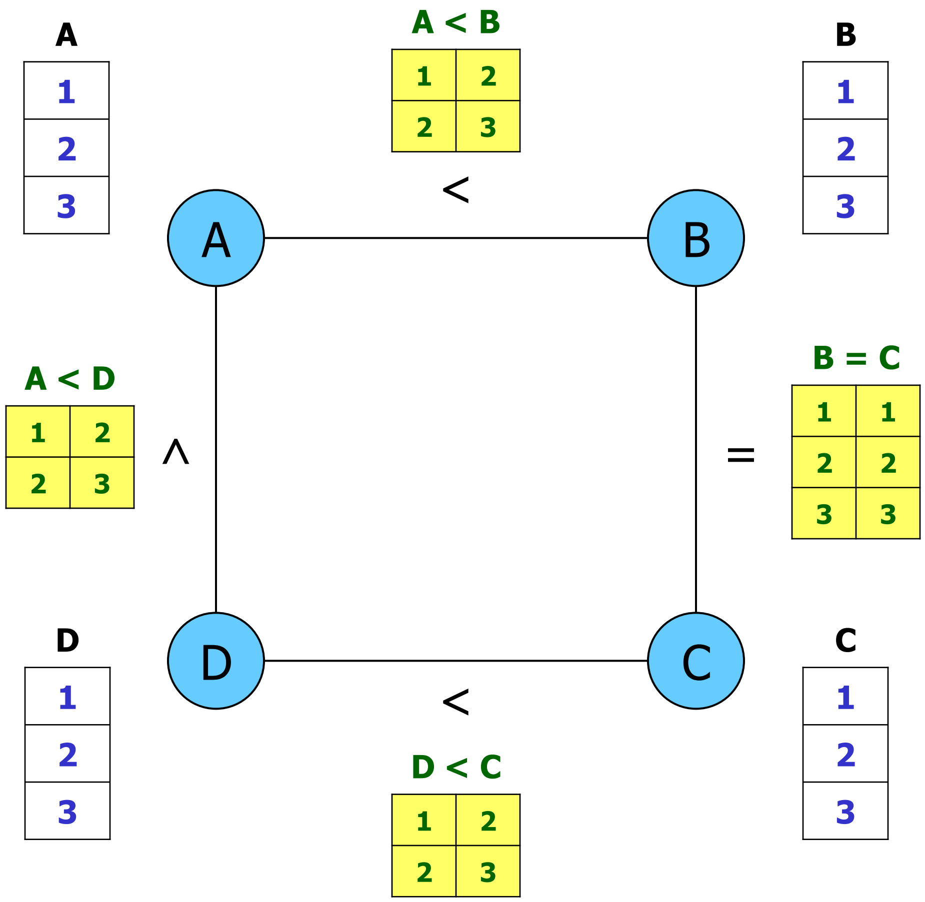


Constraint graph



Arc-consistency

- Sound
- Incomplete
- Always converges (polynomial)



A
1
2
3

B
1
2
3

C
1
2
3

D
1
2
3

A < B

1	2
2	3

B = C

1	1
2	2
3	3

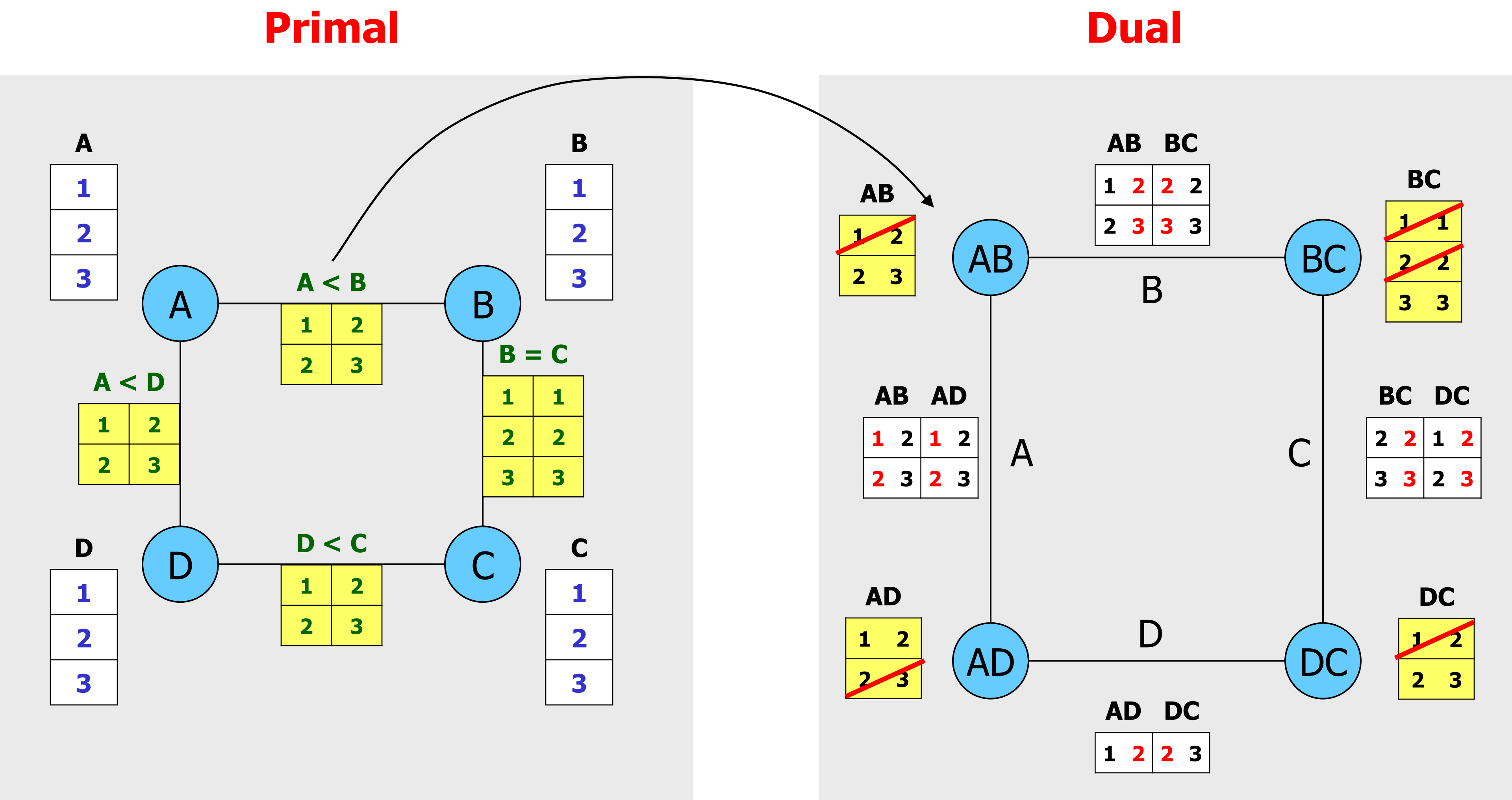
A < D

1	2
2	3

D < C

1	2
2	3

Relational Distributed Arc-Consistency



Primal

A

1
2
3

B

1
2
3

C

1
2
3

D

1
2
3

A < B

1	2
2	3

B = C

1	1
2	2
3	3

A < D

1	2
2	3

D < C

1	2
2	3

Dual

AB

~~1~~	~~2~~
2	3

BC

~~1~~	~~1~~
~~2~~	~~2~~
3	3

AD

1	2
~~2~~	~~3~~

DC

~~1~~	~~2~~
2	3

AB		**BC**	
1	2	2	2
2	3	3	3

AB		**AD**	
1	2	1	2
2	3	2	3

BC		**DC**	
2	2	1	2
3	3	2	3

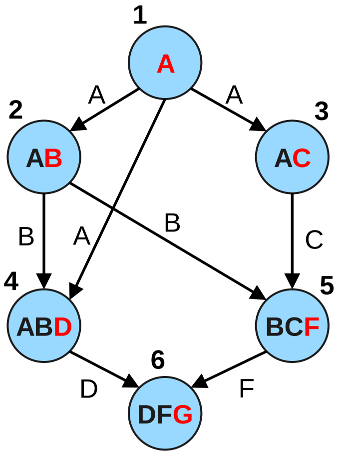
AD		**DC**	
1	2	2	3

Flattening the Bayesian Network

A	P(A)
1	.2
2	.5
3	.3
…	0

A	B	P(B|A)
1	2	.3
1	3	.7
2	1	.4
2	3	.6
3	1	.1
3	2	.9
…	…	0

A	C	P(C|A)
1	2	1
3	2	1
…	…	0



A	B	D	P(D|A,B)
1	2	3	1
1	3	2	1
2	1	3	1
2	3	1	1
3	1	2	1
3	2	1	1
…	…	…	0

B	C	F	P(F|B,C)
1	2	3	1
3	2	1	1
…	…	…	0

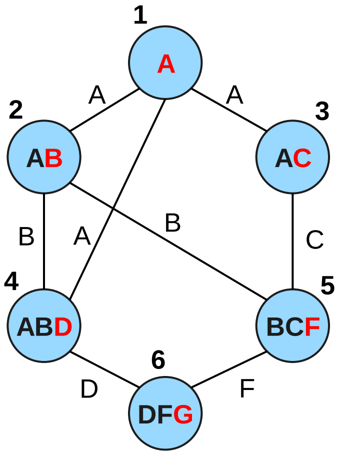
D	F	G	P(G|D,F)
1	2	3	1
2	1	3	1
…	…	…	0

Belief network

A
1
2
3

A	B
1	2
1	3
2	1
2	3
3	1
3	2

A	C
1	2
3	2



A	B	D
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

B	C	F
1	2	3
3	2	1

D	F	G
1	2	3
2	1	3

Flat constraint network

LBP – inference power for zero beliefs

- **Theorem:**

  Trace of zero beliefs of Loopy Belief Propagation =

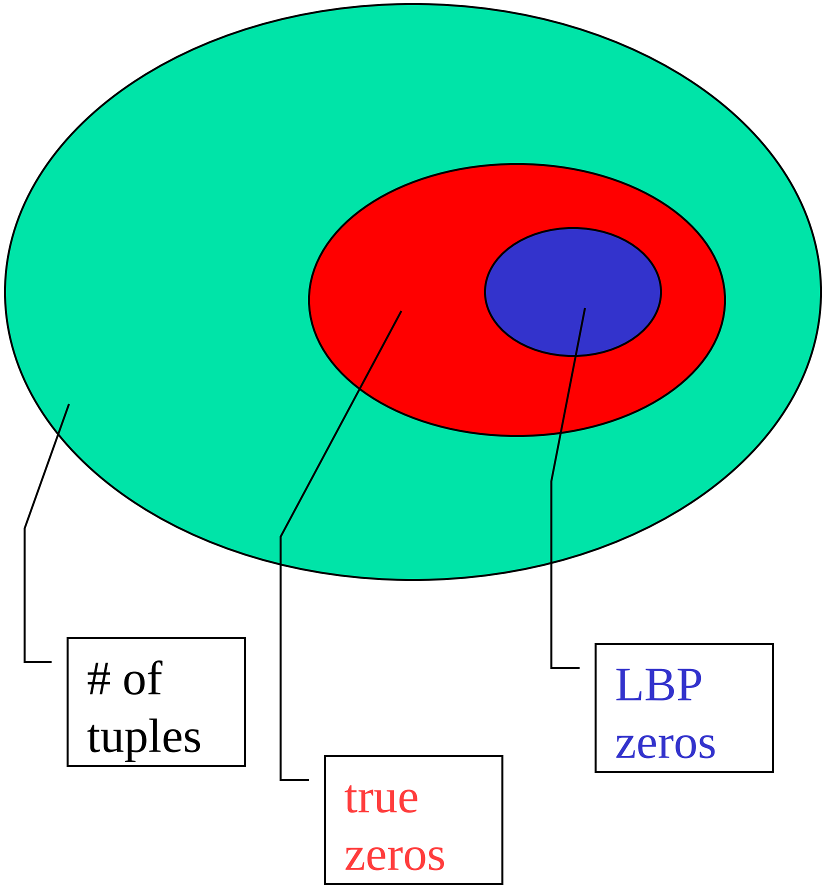
  Trace of invalid tuples of arc-consistency on flat network

- **Soundness:**
  - The inference of zero beliefs by Loopy BP converges in a finite number of iterations
  - all the inferred zero beliefs are correct

- **Incompleteness:**
  - Loopy BP may not infer all the true zero beliefs

Zero and ε-Small Beliefs

Zero beliefs

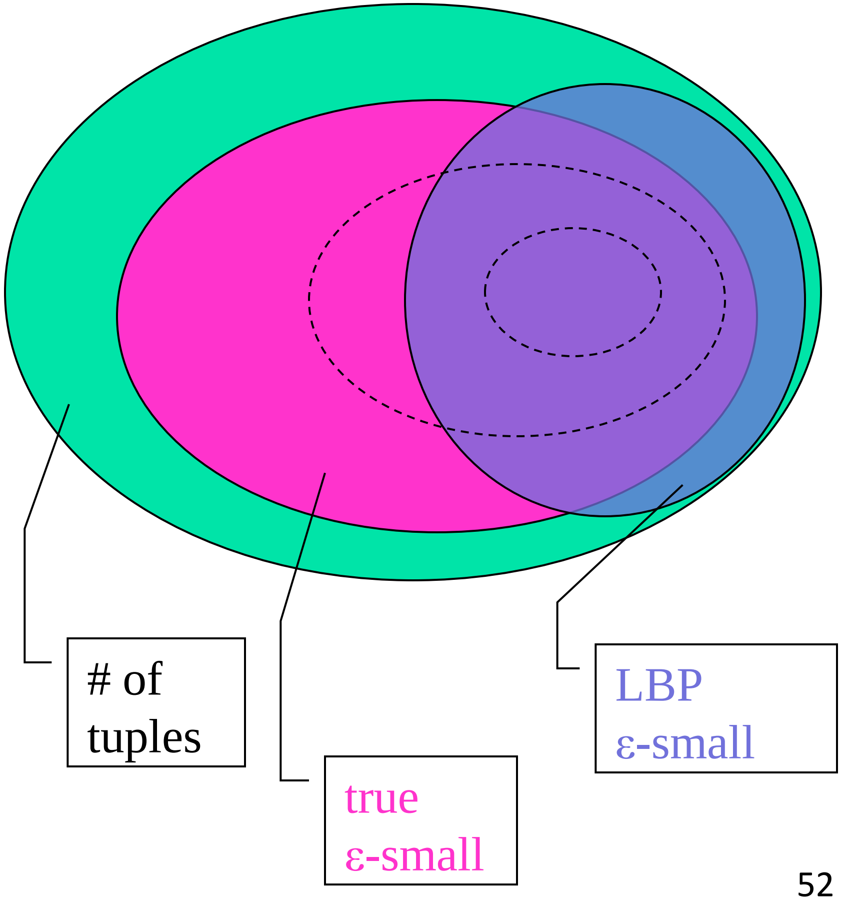


# of tuples

true zeros

LBP zeros

ε-small beliefs

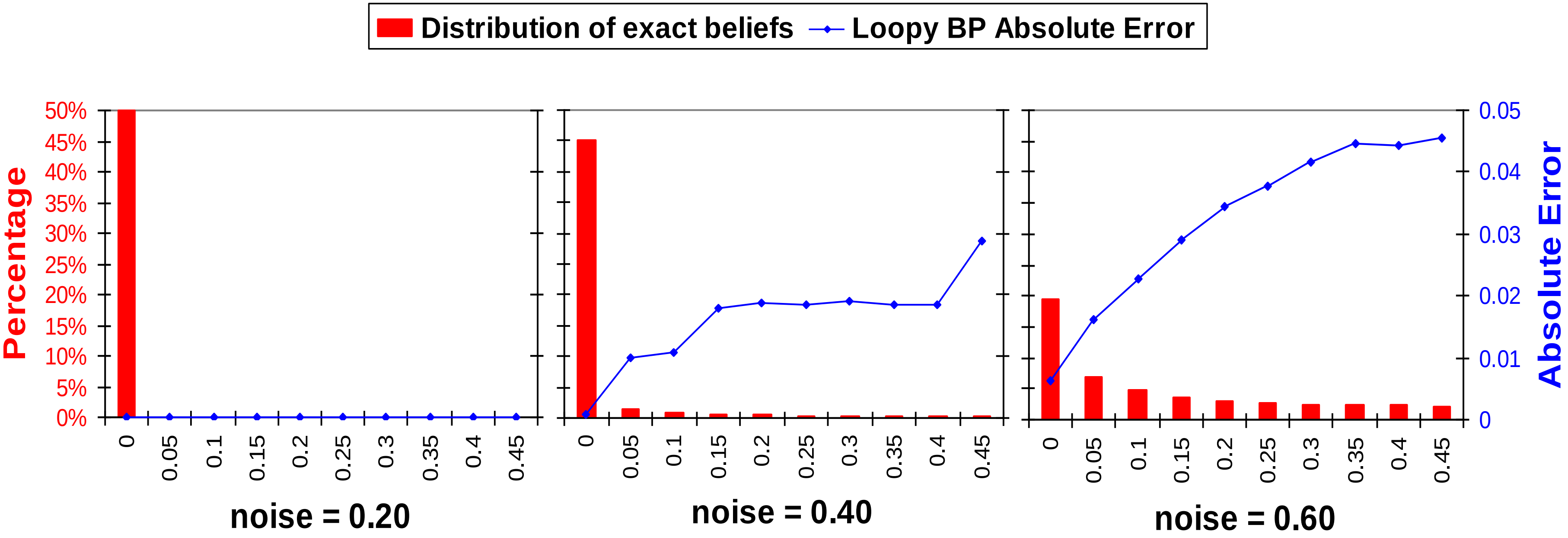


# of tuples

true ε-small

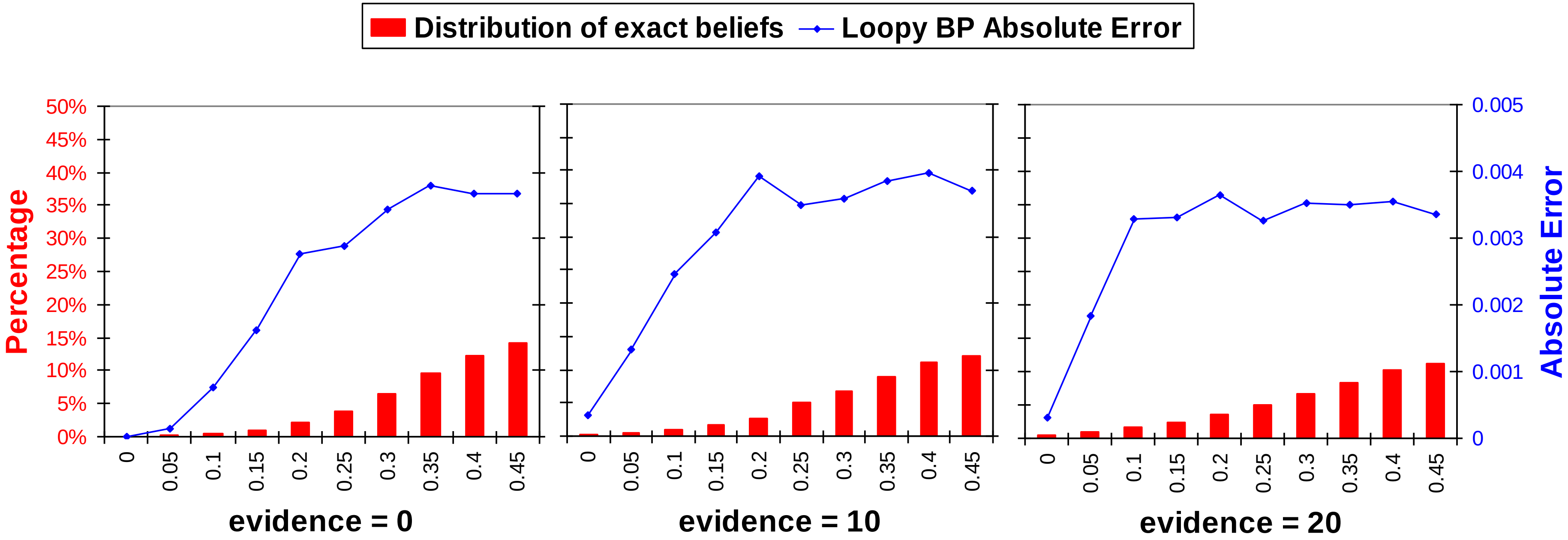
LBP ε-small

Coding Networks



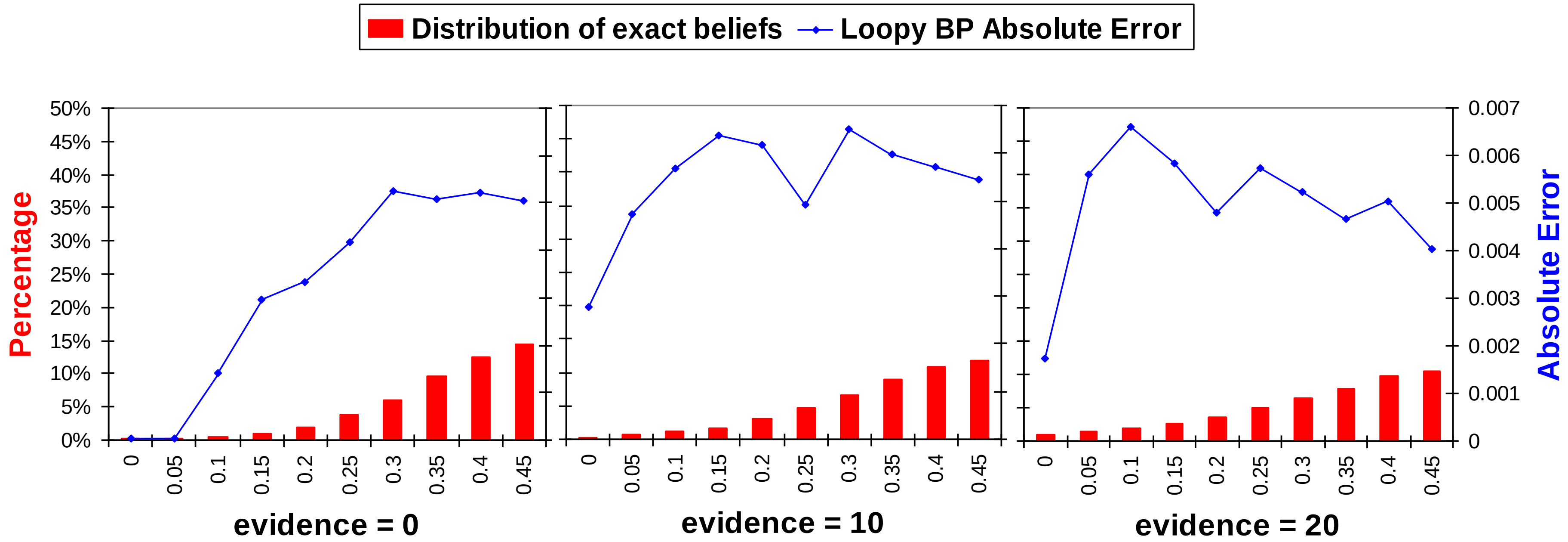
N=200, 1000 instances, treewidth=15

10x10 Grids



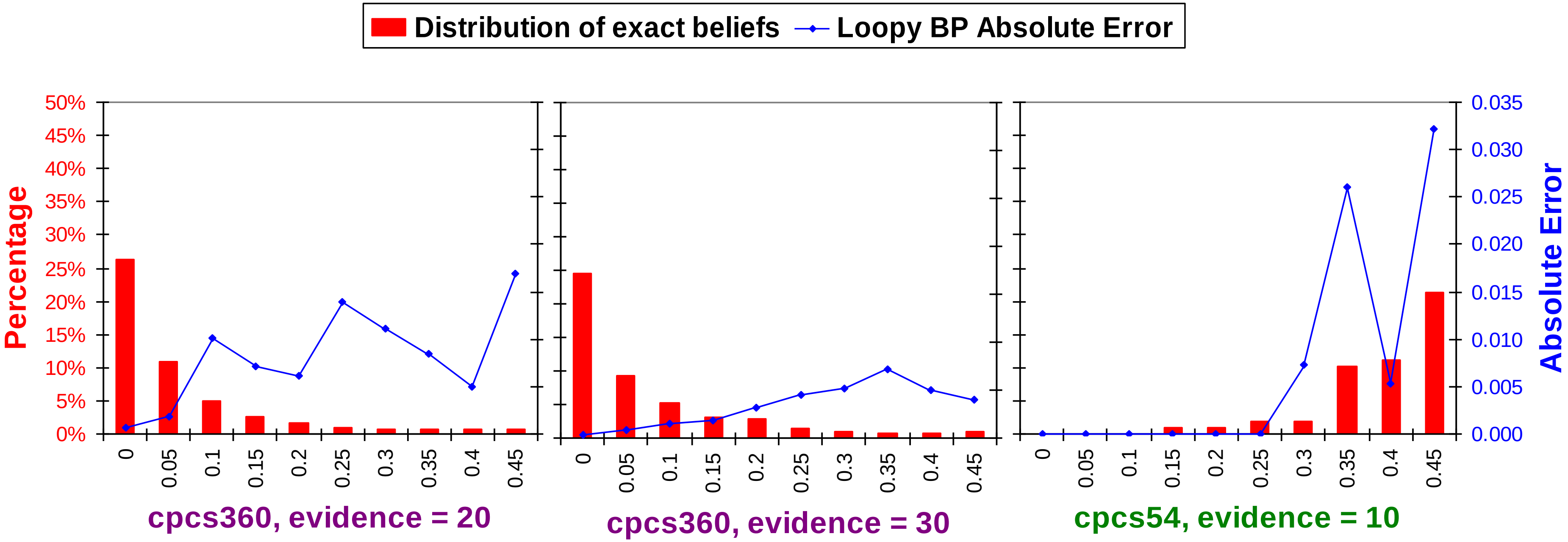
N=100, 100 instances, w*=15

Random Networks



N=80, 100 instances, w*=15

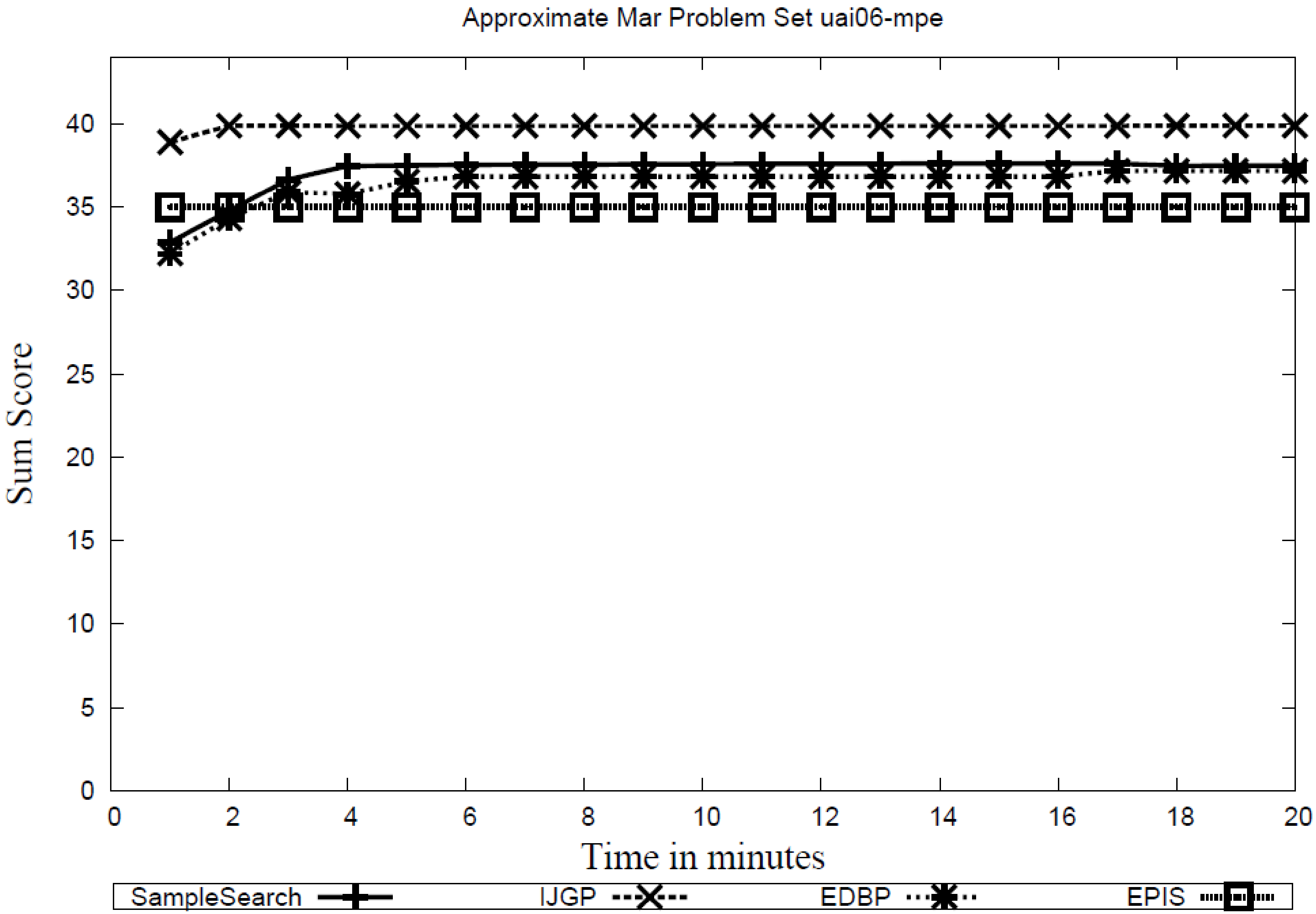
CPCS 54, CPCS360



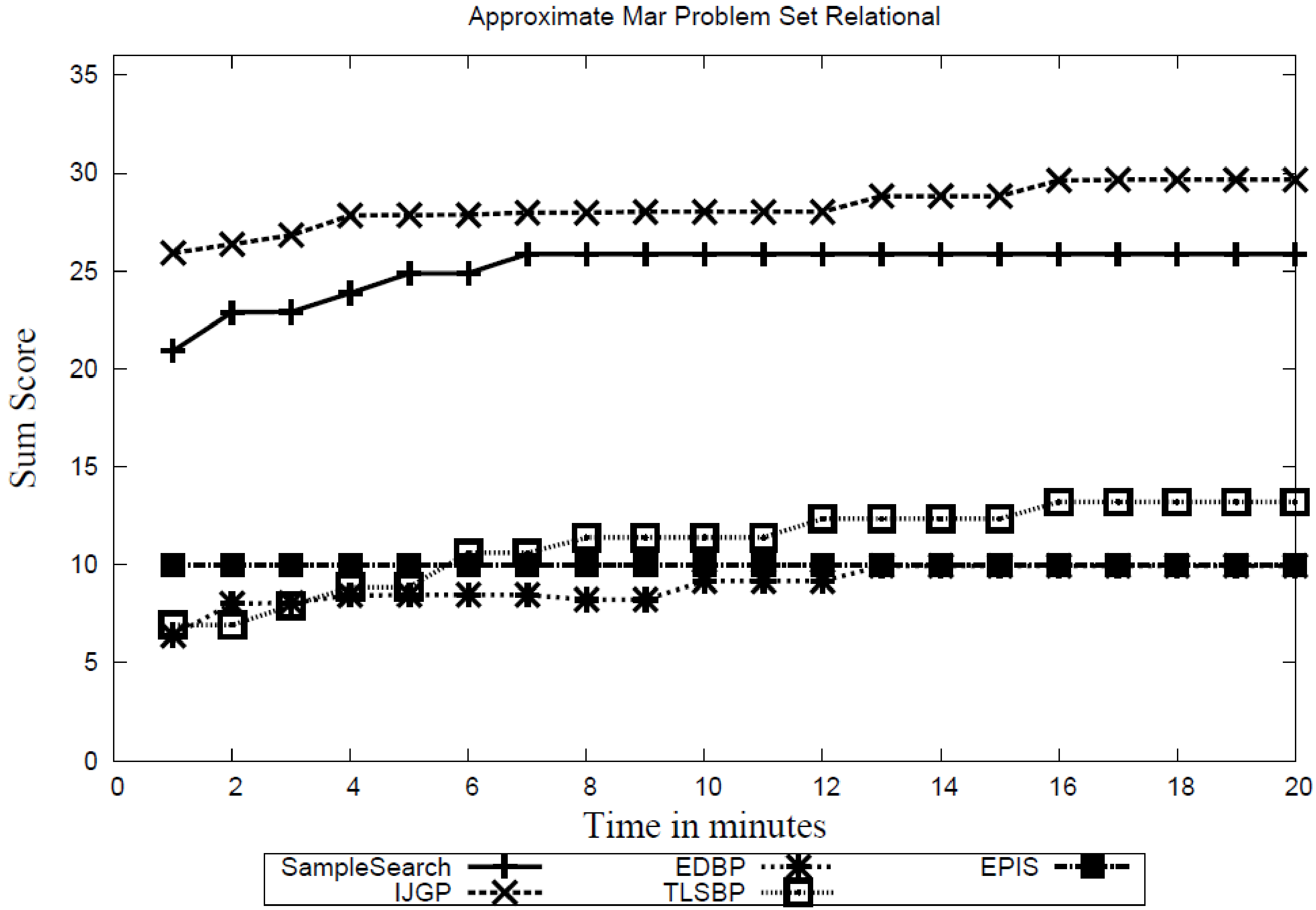
CPCS360: 5 instances, w*=20

CPCS54: 100 instances, w*=15

IJGP on UAI06 problems



IJGP on Set Relational

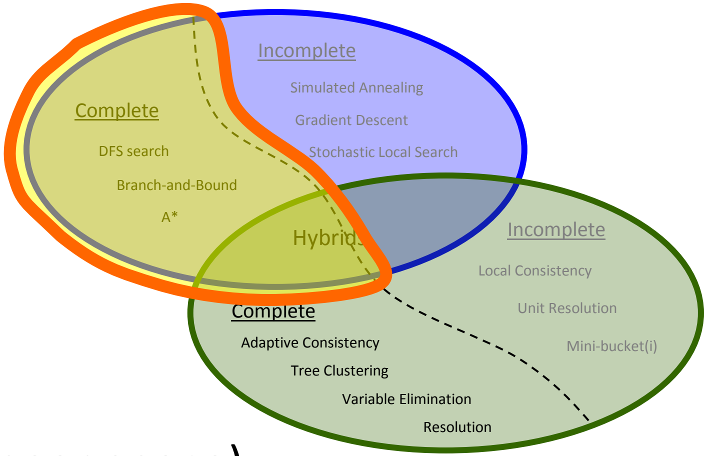


Outline

- Introduction
- Inference
- Search
  - Exact
  - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
- Software

Outline

- Introduction
- Inference
- Search
  - Exact
    - AND/OR search trees (linear space)
    - AND/OR Branch-and-Bound search
    - AND/OR search graphs (caching)
    - AND/OR search for 0-1 integer programming
    - AND/OR search for multi-objective optimization
  - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
- Software



Incomplete

Simulated Annealing

Gradient Descent

Stochastic Local Search

Complete

DFS search

Branch-and-Bound

A*

Hybrids

Incomplete

Local Consistency

Unit Resolution

Mini-bucket(i)

Complete

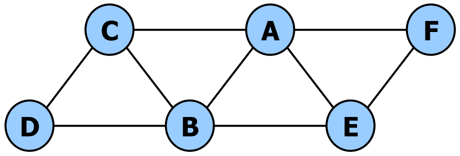
Adaptive Consistency

Tree Clustering

Variable Elimination

Resolution

Classic OR Search Space



A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

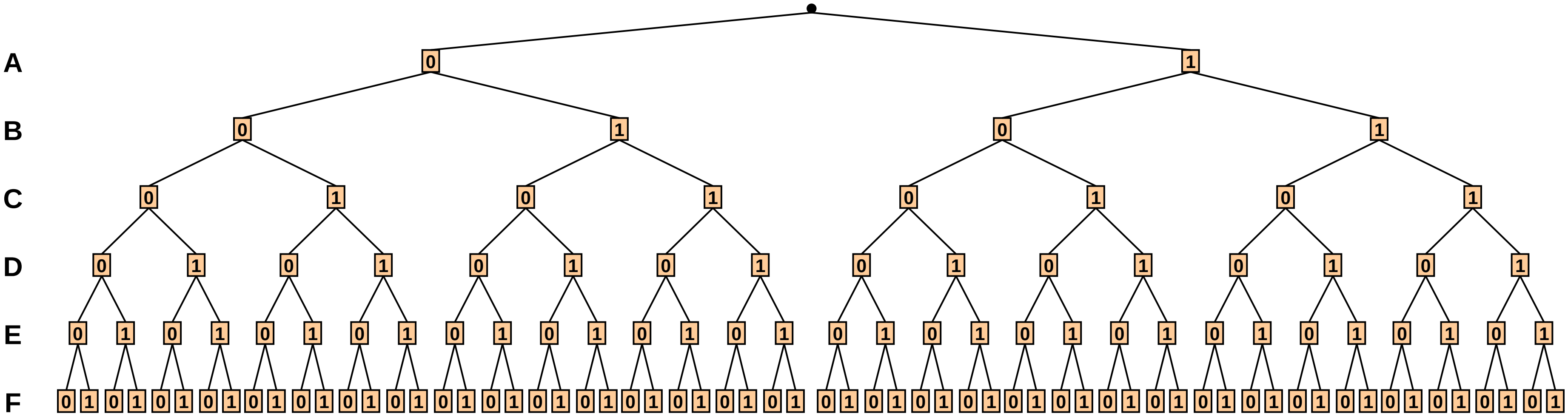
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

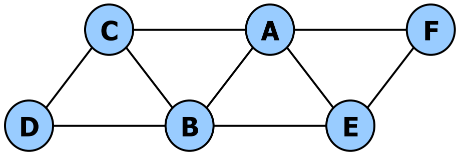
C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

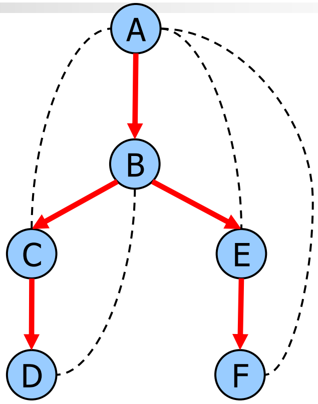
E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$

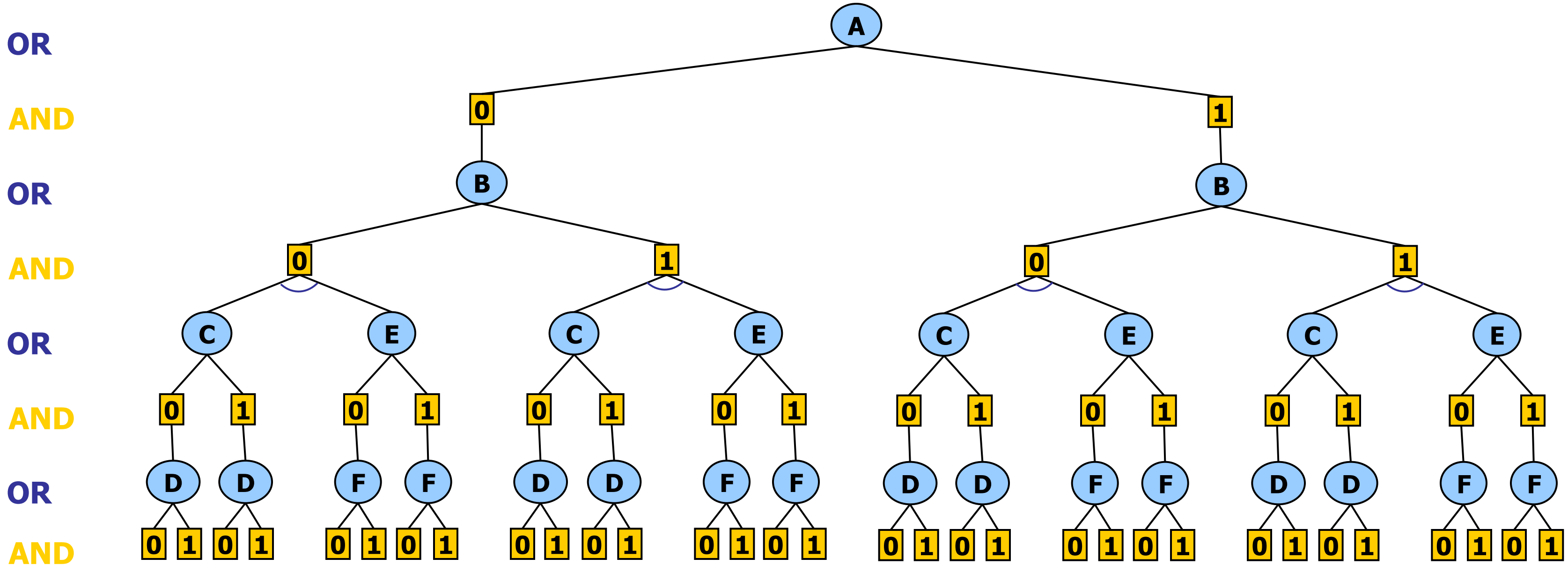


The AND/OR Search Tree

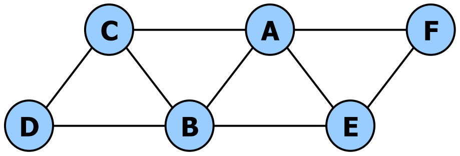


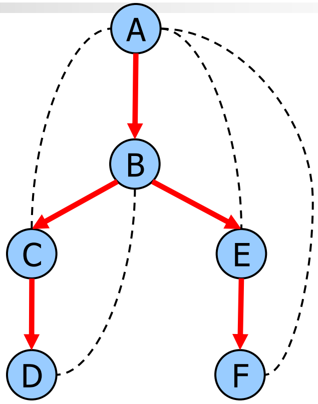


Pseudo tree (Freuder and Quinn,IJCAI85)

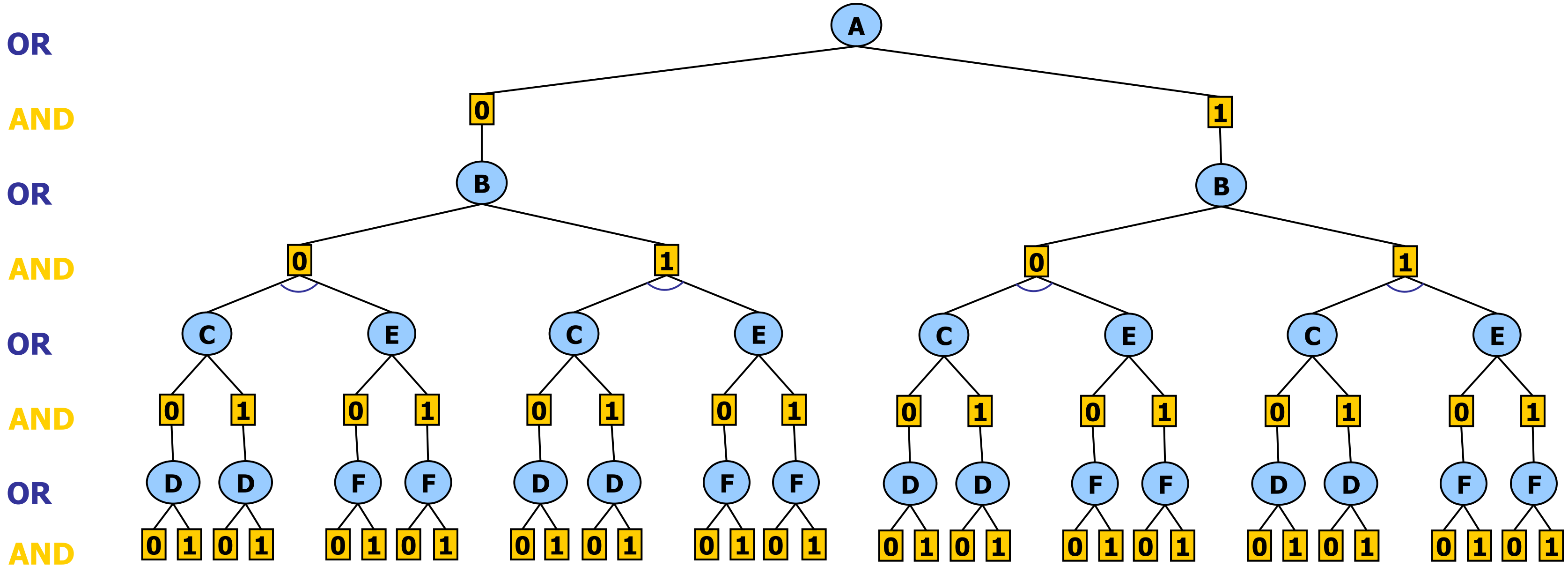


The AND/OR Search Tree



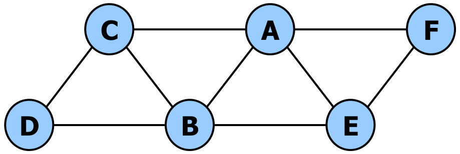


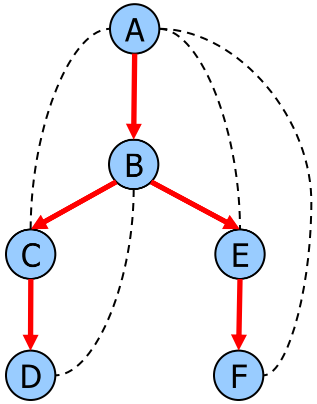
Pseudo tree



A solution subtree is (A=0, B=1, C=0, D=0, E=1, F=1)

Weighted AND/OR Search Tree





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

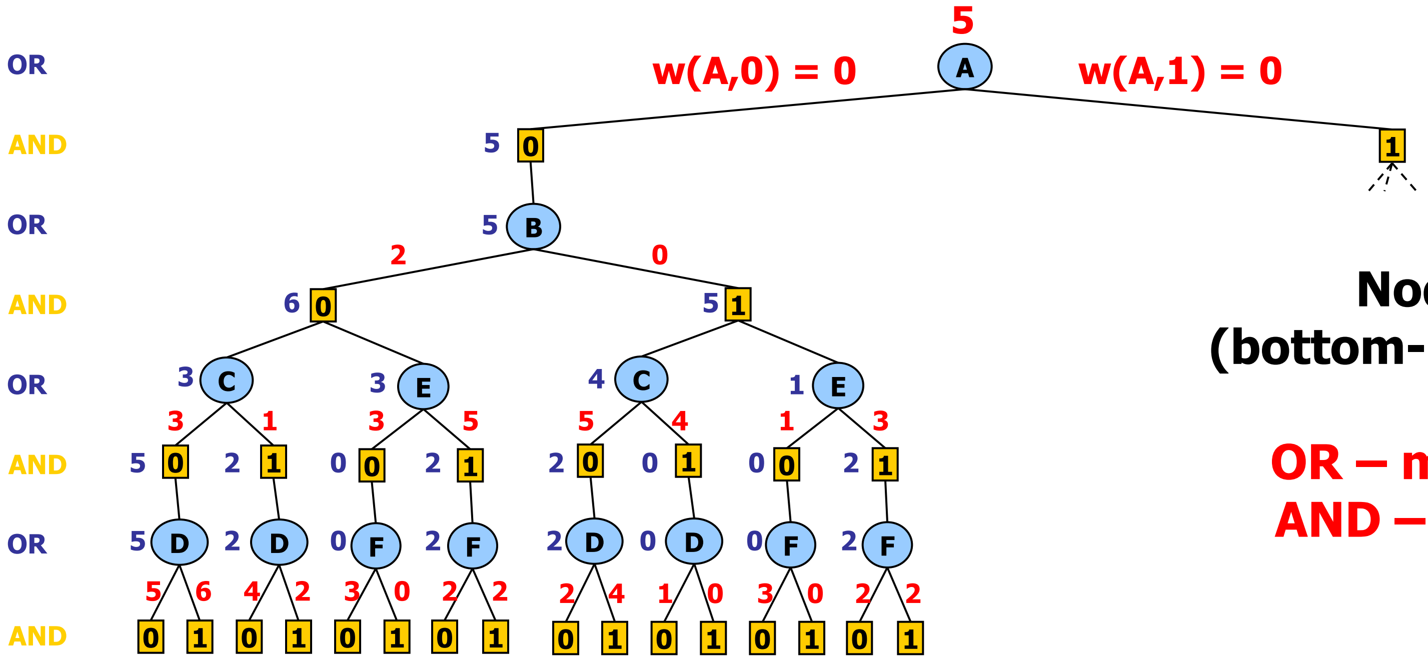
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

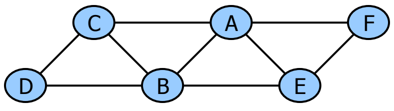
$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$

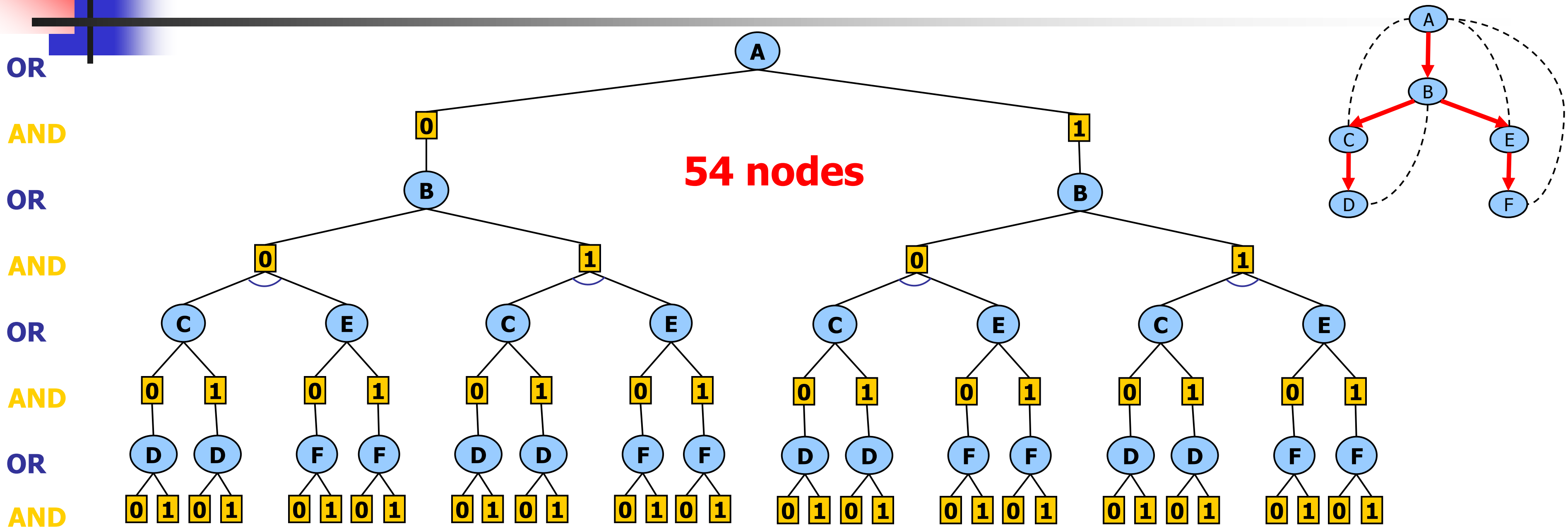


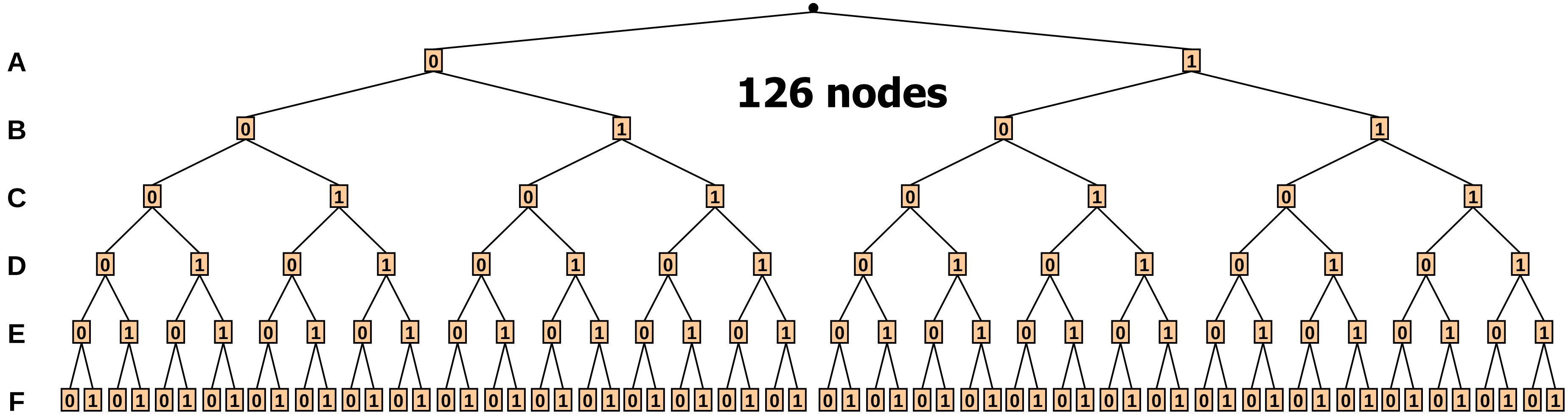
Node Value
(bottom-up evaluation)

OR – minimization
AND – summation

AND/OR vs. OR Spaces







AND/OR vs. OR Spaces

width	**depth**	**OR space**		**AND/OR space**		
		Time (sec.)	Nodes	Time (sec.)	AND nodes	OR nodes
5	10	3.15	2,097,150	0.03	**10,494**	5,247
4	9	3.13	2,097,150	0.01	**5,102**	2,551
5	10	3.12	2,097,150	0.03	**8,926**	4,463
4	10	3.12	2,097,150	0.02	**7,806**	3,903
5	13	3.11	2,097,150	0.10	**36,510**	18,255

Random graphs with 20 nodes, 20 edges and 2 values per node

Complexity of AND/OR Tree Search

	AND/OR tree	**OR tree**
Space	$O(n)$	$O(n)$
Time	$O(n\ d^t)$ $O(n\ d^{w^* \log n})$ (Freuder & Quinn85), (Collin, Dechter & Katz91), (Bayardo & Miranker95), (Darwiche01)	$O(d^n)$

d = domain size
t = depth of pseudo-tree
n = number of variables
w* = treewidth

Constructing Pseudo Trees

- AND/OR search algorithms are influenced by the quality of the pseudo tree
- Finding the minimal induced width / depth pseudo tree is NP-hard
- Heuristics
  - Min-Fill (min induced width)
  - Hypergraph partitioning (min depth)

Constructing Pseudo Trees

- **Min-Fill** (Kjaerulff, 1990)
  - Depth-first traversal of the induced graph obtained along the min-fill elimination order
  - Variables ordered according to the smallest "fill-set"

- **Hypergraph Partitioning** (Karypis and Kumar, 2000)
  - Functions are vertices in the hypergraph and variables are hyperedges
  - Recursive decomposition of the hypergraph while minimizing the separator size at each step
  - Using state-of-the-art software package hMeTiS

Quality of the Pseudo Trees

Network	**hypergraph**		**min-fill**	
	width	depth	width	depth
barley	7	**13**	7	23
diabetes	7	**16**	4	77
link	21	**40**	15	53
mildew	5	**9**	4	13
munin1	12	**17**	12	29
munin2	9	**16**	9	32
munin3	9	**15**	9	30
munin4	9	**18**	9	30
water	11	**16**	10	15
pigs	11	**20**	11	26

Bayesian Networks Repository

Network	**hypergraph**		**min-fill**	
	width	depth	width	depth
spot5	47	152	**39**	204
spot28	108	138	**79**	199
spot29	16	23	**14**	42
spot42	36	48	**33**	87
spot54	12	16	**11**	33
spot404	19	26	**19**	42
spot408	47	52	**35**	97
spot503	11	20	**9**	39
spot505	29	42	**23**	74
spot507	70	122	**59**	160

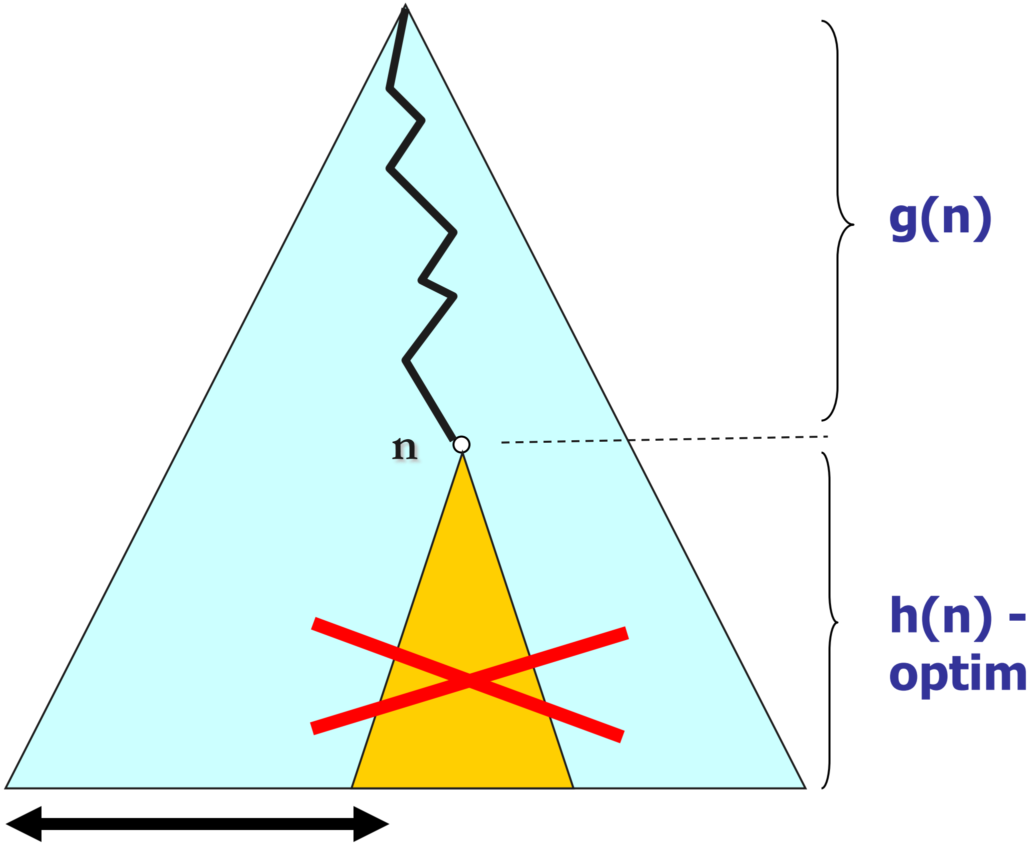
SPOT5 Benchmarks

Outline

- Introduction
- Inference
- Search
  - Exact
    - AND/OR search trees
    - AND/OR Branch-and-Bound search
      - Lower bounding heuristics
      - Dynamic variable orderings
    - AND/OR search graphs (caching)
    - AND/OR search for 0-1 integer programming
    - AND/OR search for multi-objective optimization
  - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
- Software

Classic Branch-and-Bound Search

Each node is a COP subproblem
(defined by current conditioning)

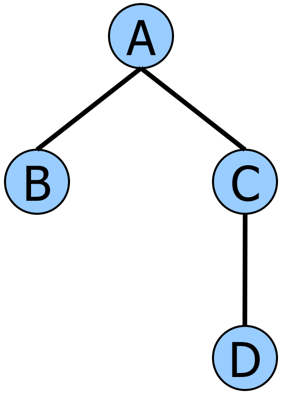


f(n) = g(n) + h(n)
f(n) = lower bound

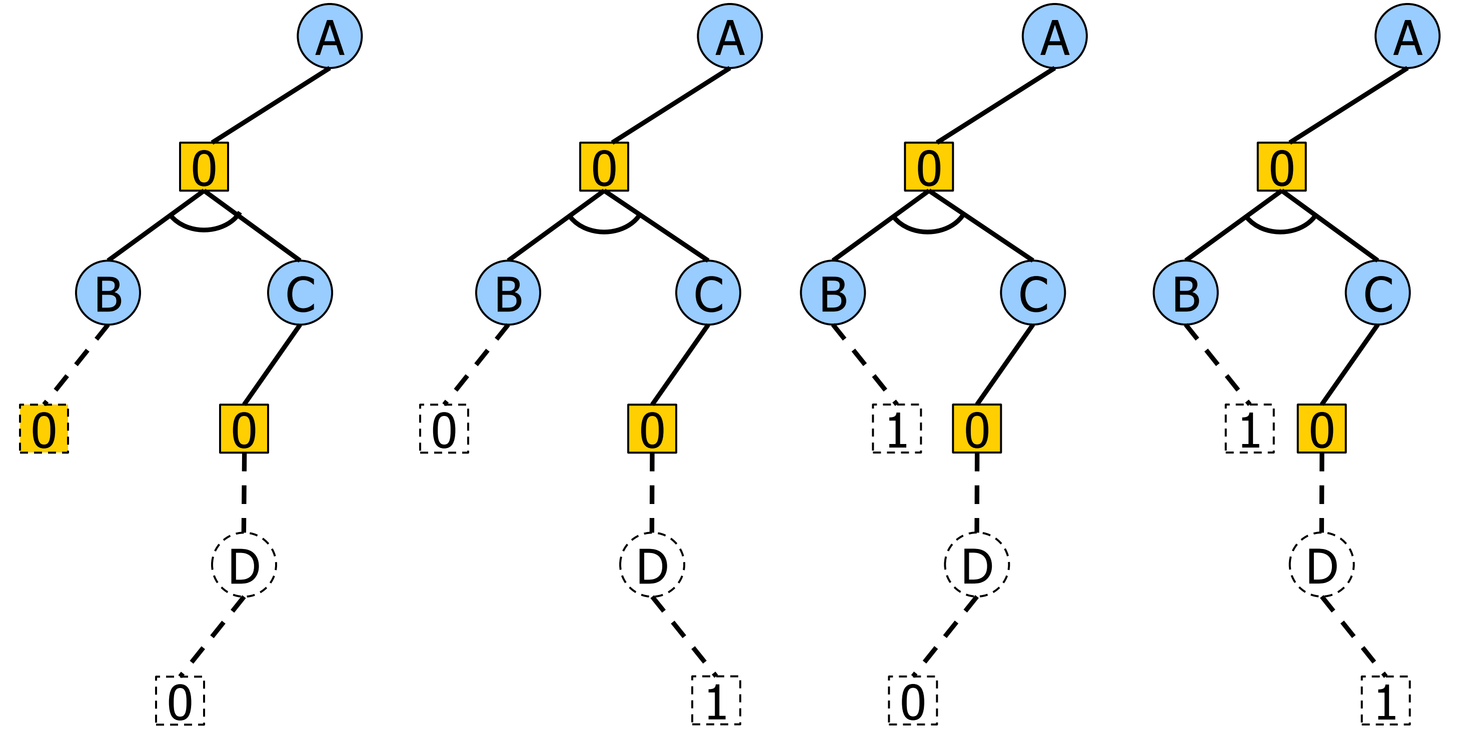
Prune if f(n) ≥ UB

(UB) Upper Bound = best solution so far

Partial Solution Tree



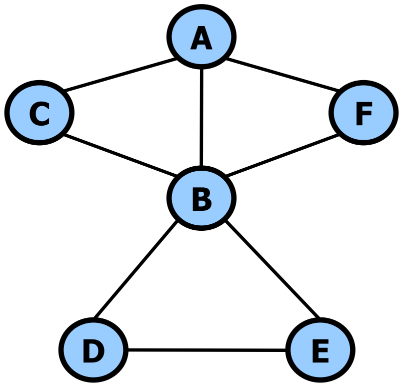
Pseudo tree



(A=0, B=0, C=0, D=0) (A=0, B=0, C=0, D=1) (A=0, B=1, C=0, D=0) (A=0, B=1, C=0, D=1)

Extension(T’) – solution trees that extend T’

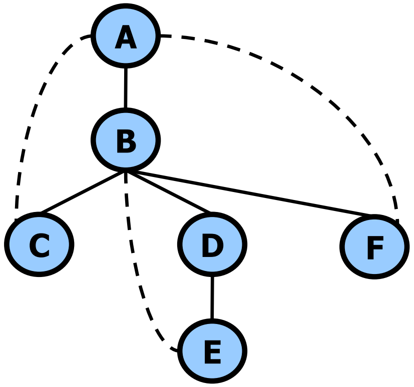
Exact Evaluation Function

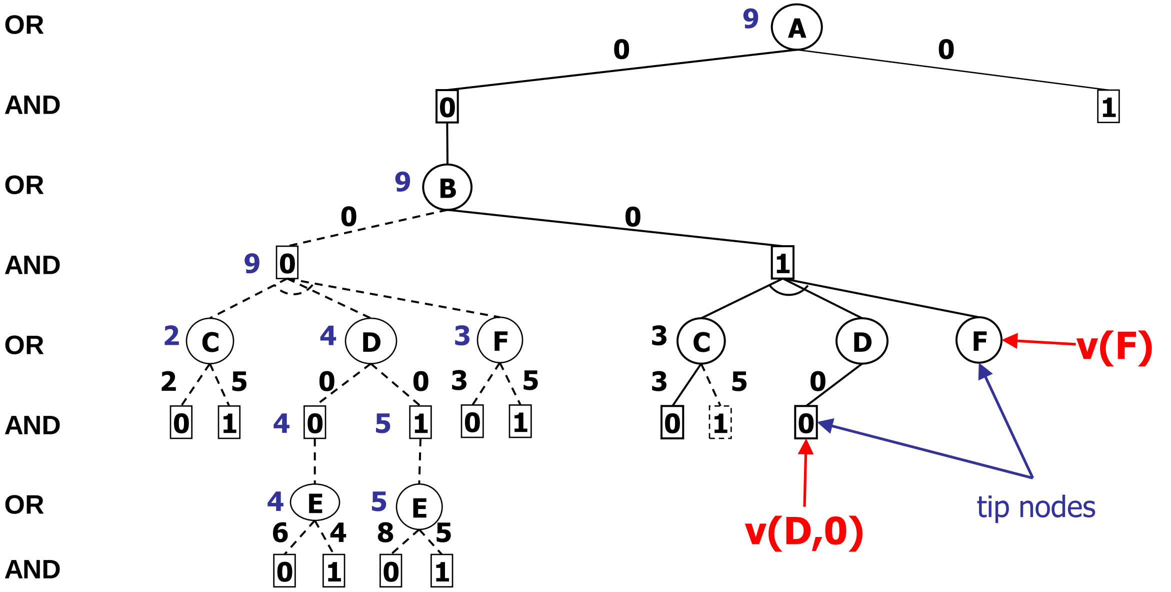


A	B	C	$f_1(ABC)$
0	0	0	**2**
0	0	1	**5**
0	1	0	**3**
0	1	1	**5**
1	0	0	**9**
1	0	1	**3**
1	1	0	**7**
1	1	1	**2**

A	B	F	$f_2(ABF)$
0	0	0	**3**
0	0	1	**5**
0	1	0	**1**
0	1	1	**4**
1	0	0	**6**
1	0	1	**5**
1	1	0	**6**
1	1	1	**5**

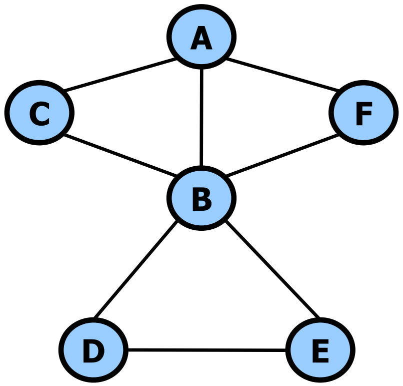
B	D	E	$f_3(BDE)$
0	0	0	**6**
0	0	1	**4**
0	1	0	**8**
0	1	1	**5**
1	0	0	**9**
1	0	1	**3**
1	1	0	**7**
1	1	1	**4**





$f^*(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + v(D,0) + v(F)$

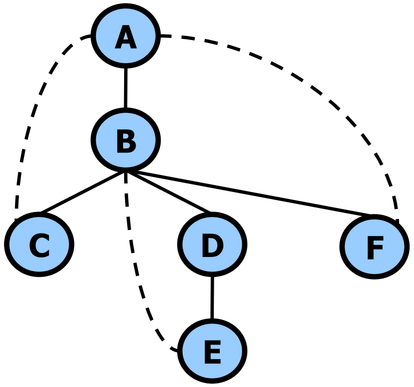
Heuristic Evaluation Function

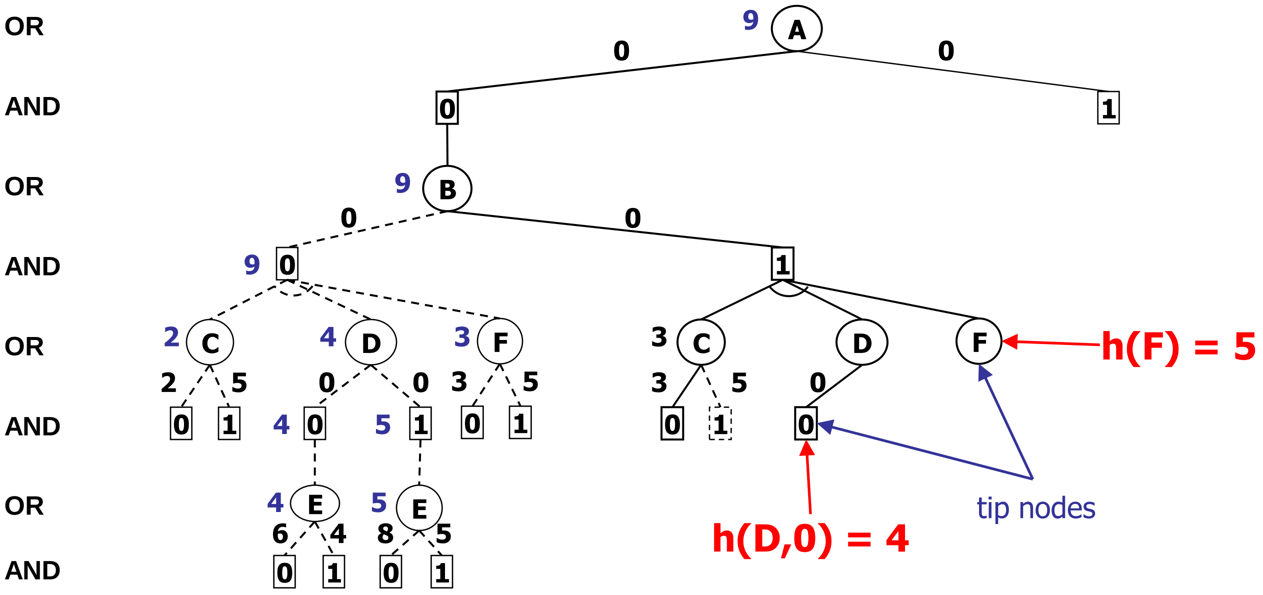


A	B	C	f_1(ABC)
0	0	0	2
0	0	1	5
0	1	0	3
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	2

A	B	F	f_2(ABF)
0	0	0	3
0	0	1	5
0	1	0	1
0	1	1	4
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

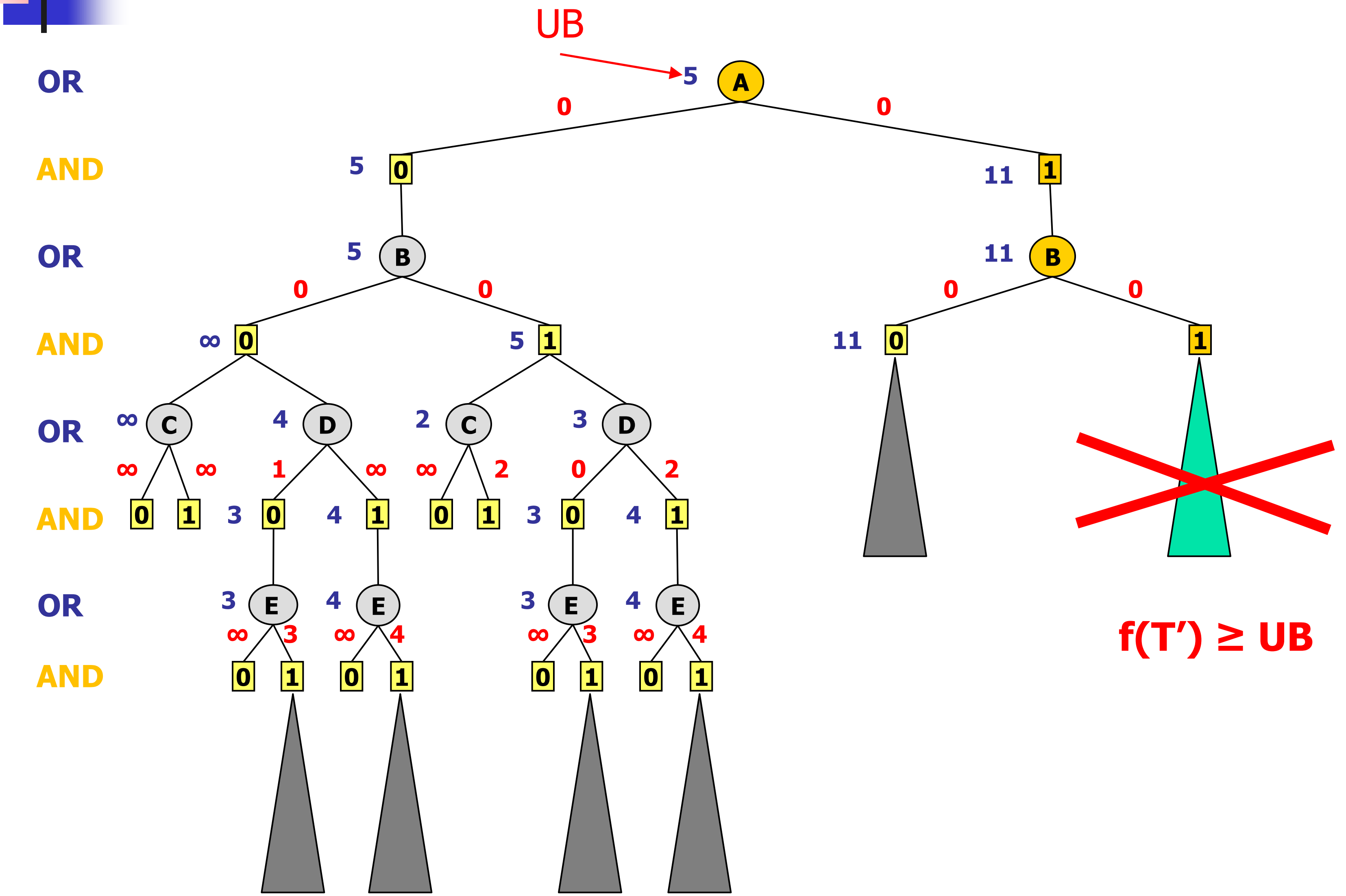
B	D	E	f_3(BDE)
0	0	0	6
0	0	1	4
0	1	0	8
0	1	1	5
1	0	0	9
1	0	1	3
1	1	0	7
1	1	1	4





$$f(T') = w(A,0) + w(B,1) + w(C,0) + w(D,0) + h(D,0) + h(F) = 12 \leq f^*(T')$$

AND/OR Branch-and-Bound Search



f(T′) ≥ UB

AND/OR Branch-and-Bound Search (AOBB)

(Marinescu and Dechter, IJCAI2005, AIJ2009)

- Associate each node n with a heuristic lower bound h(n) on v(n)
- EXPAND (top-down)
  - Evaluate f(T') and prune search if f(T') ≥ UB
  - Expand the tip node n
- PROPAGATE (bottom-up)
  - Update value of the parent p of n
    - OR nodes: minimization
    - AND nodes: summation

Heuristics for AND/OR Branch-and-Bound

- In the AND/OR search space **h(n)** can be computed using any heuristic. We used:
  - Static Mini-Bucket heuristics

    (Kask and Dechter, AIJ2001), (Marinescu and Dechter, IJCAI2005)
  - Dynamic Mini-Bucket heuristics

    (Marinescu and Dechter, IJCAI2005)
  - Maintaining local consistency

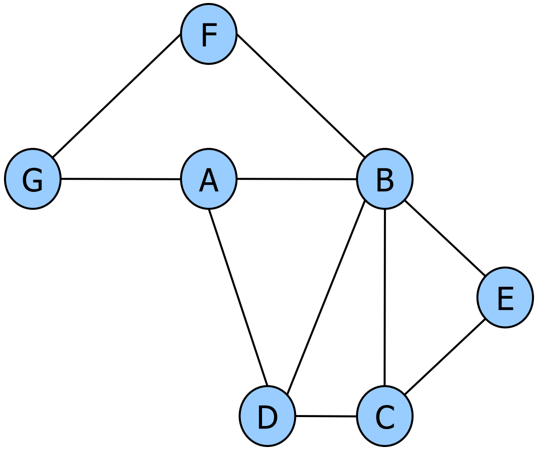
    (Larrosa and Schiex, AAAI2003), (de Givry et al., IJCAI2005)
  - LP relaxations

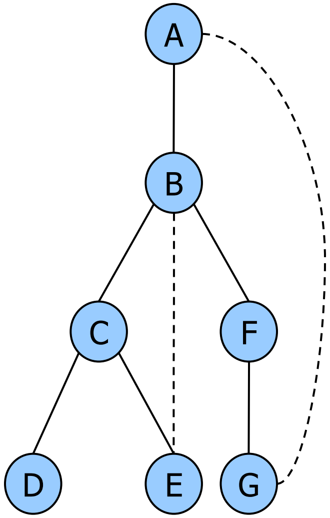
    (Nemhauser and Wosley, 1998)

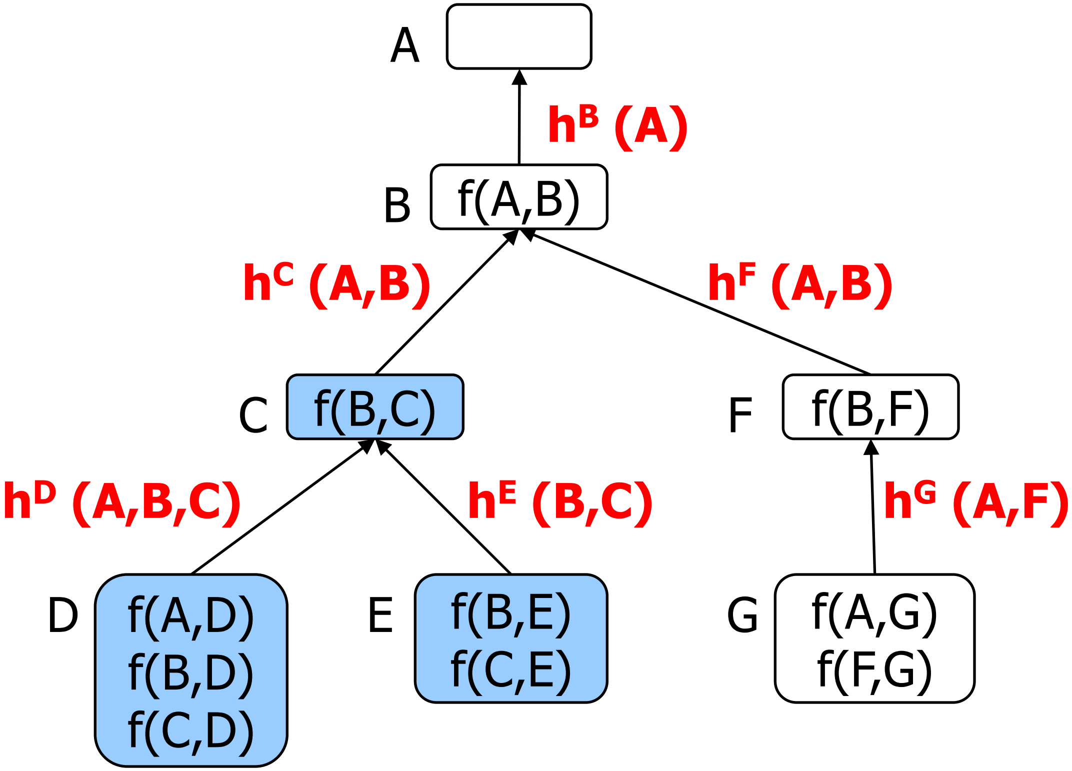
Mini-Bucket Heuristics

- Static Mini-Buckets
  - Pre-compiled
  - Reduced overhead
  - Less accurate
  - Static variable ordering
- Dynamic Mini-Buckets
  - Computed dynamically
  - Higher overhead
  - High accuracy
  - Dynamic variable ordering

Bucket Elimination







A

h^B (A)

B f(A,B)

h^C (A,B)

h^F (A,B)

C f(B,C)

F f(B,F)

h^D (A,B,C)

h^E (B,C)

h^G (A,F)

D f(A,D)
f(B,D)
f(C,D)

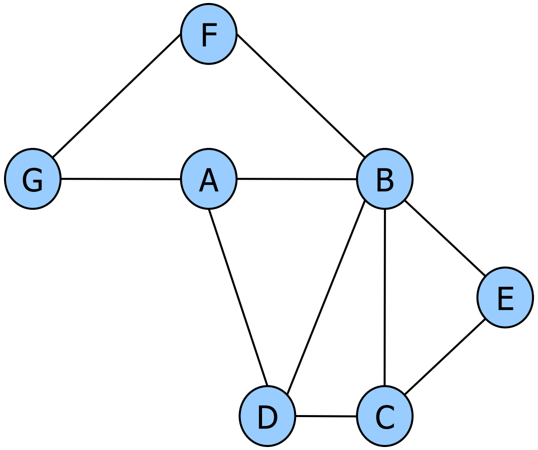
E f(B,E)
f(C,E)

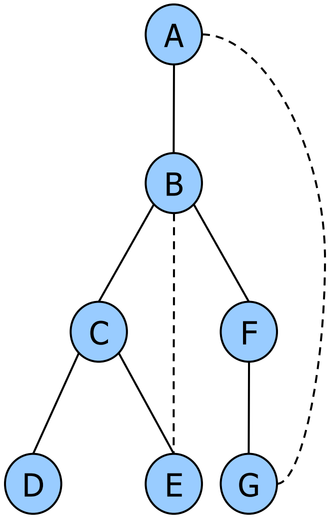
G f(A,G)
f(F,G)

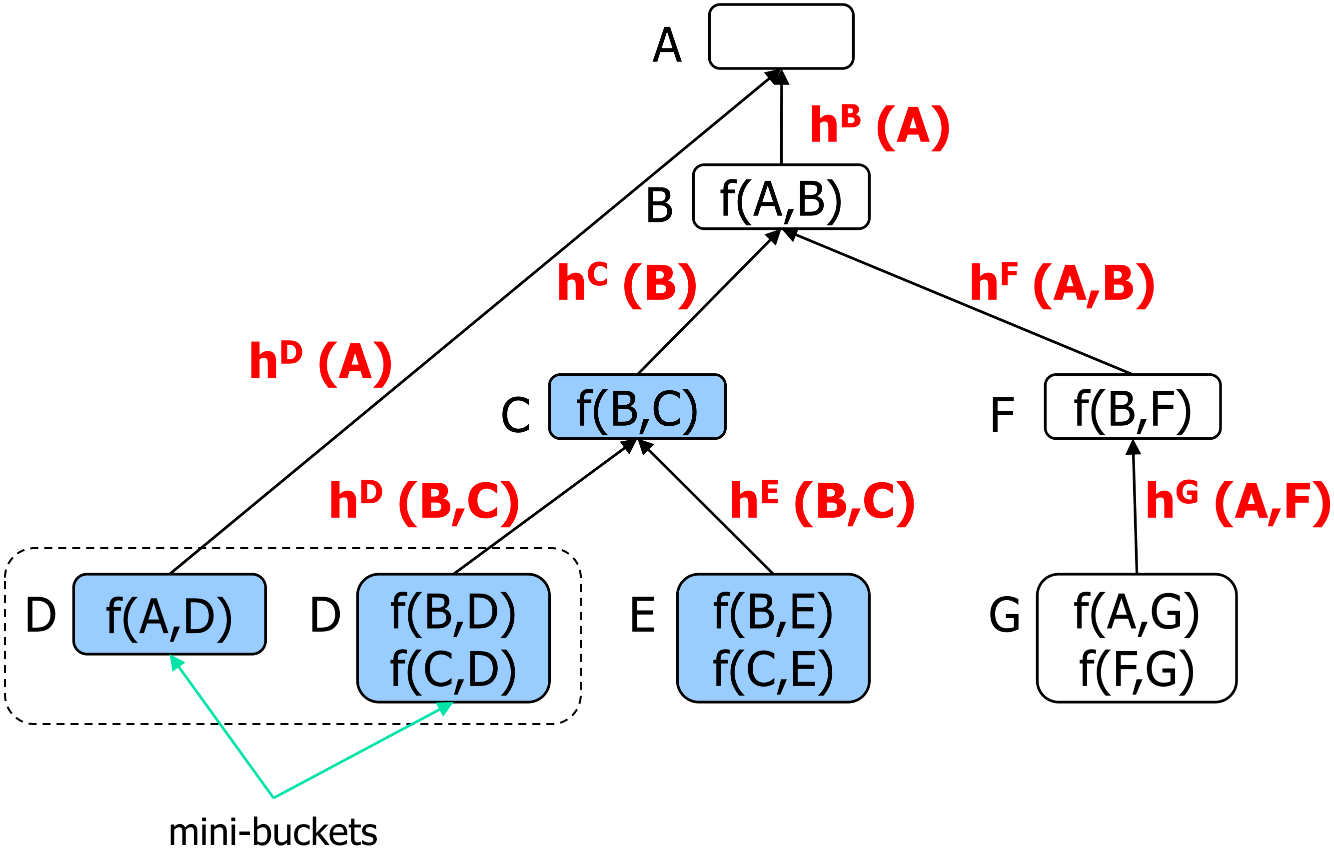
$$h^*(a, b, c) = h^D(a, b, c) + h^E(b, c)$$

Ordering: (A, B, C, D, E, F, G)

Static Mini-Bucket Heuristics



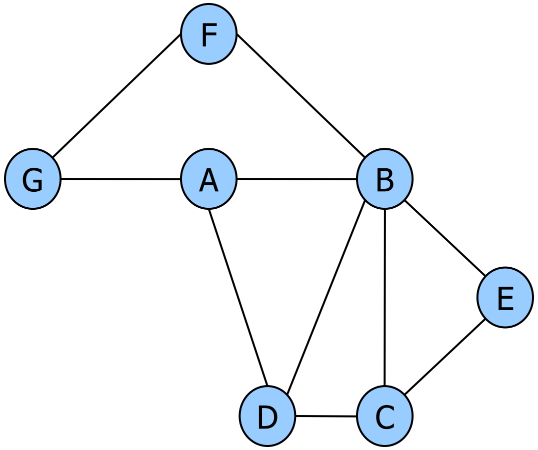


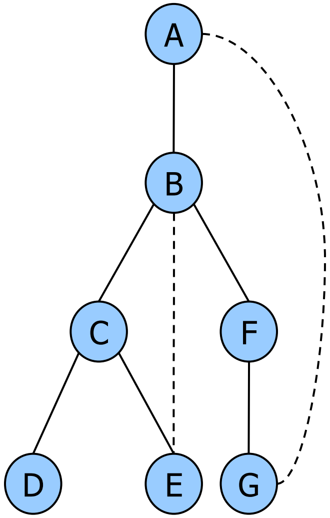


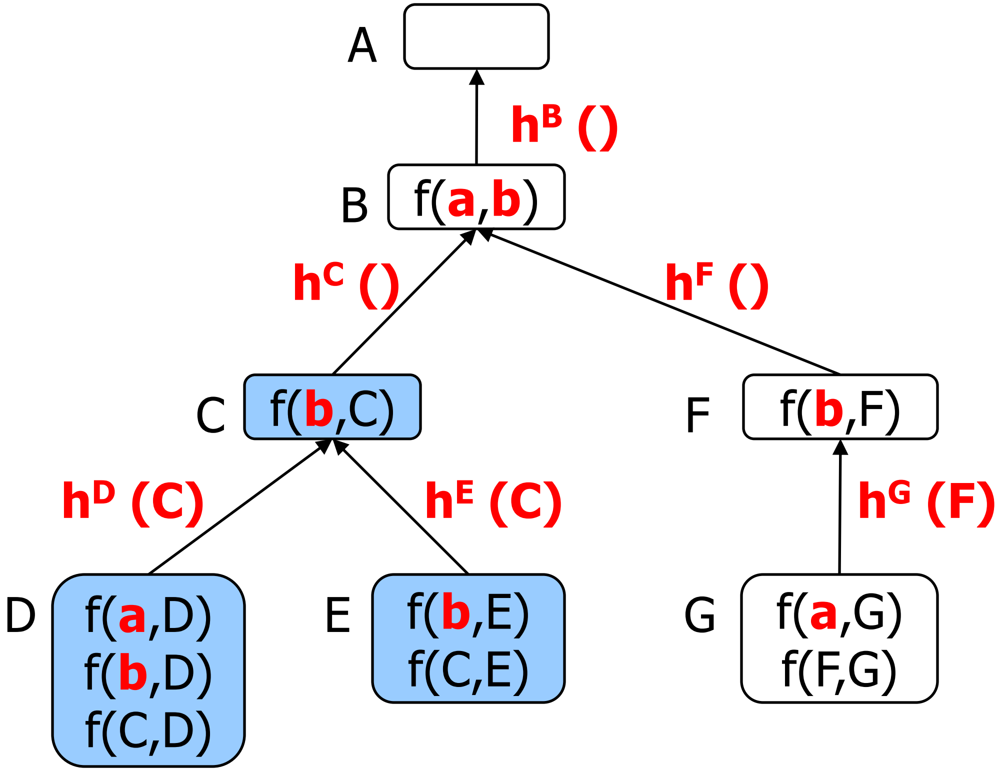
$$h(a, b, c) = h^D(a) + h^D(b, c) + h^E(b, c)$$
$$\leq h^*(a, b, c)$$

Ordering: (A, B, C, D, E, F, G)

Dynamic Mini-Bucket Heuristics







$$\begin{aligned} h(a, b, c) &= h^D(c) + h^E(c) \\ &= h^*(a, b, c) \end{aligned}$$

Ordering: (A, B, C, D, E, F, G)

Outline

- Introduction
- Inference
- Search
  - Exact
    - AND/OR search trees
    - AND/OR Branch-and-Bound search
      - Lower bounding heuristics
      - Dynamic variable orderings
    - AND/OR search graphs (caching)
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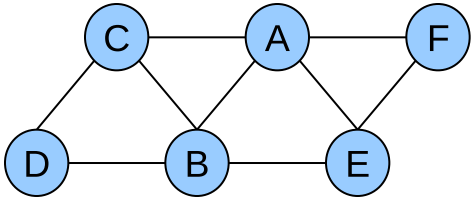
Dynamic Variable Orderings

(Marinescu and Dechter, ECAI2006)

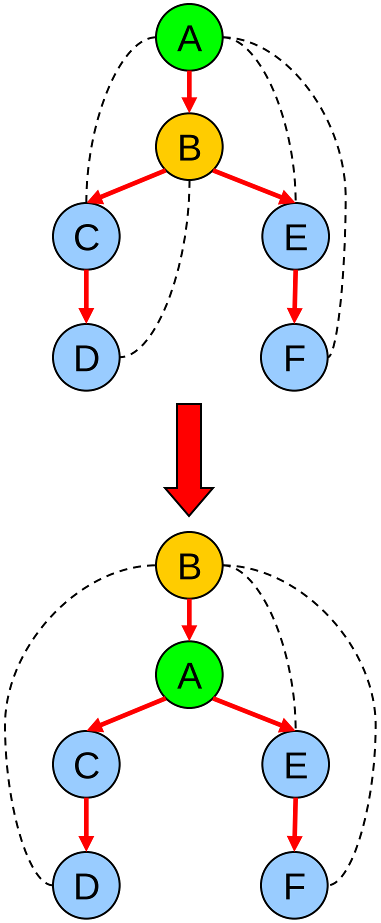
- Variable ordering heuristics:
  - **Semantic-based**
    - Aim at shrinking the size of the search space based on context and current value assignments
      - e.g. min-domain, min-dom/deg, min reduced cost
  - **Graph-based**
    - Aim at maximizing the problem decomposition
      - e.g. pseudo-tree arrangement

Orthogonal forces, use one as primary and break ties based on the other

Partial Variable Ordering



Primal graph



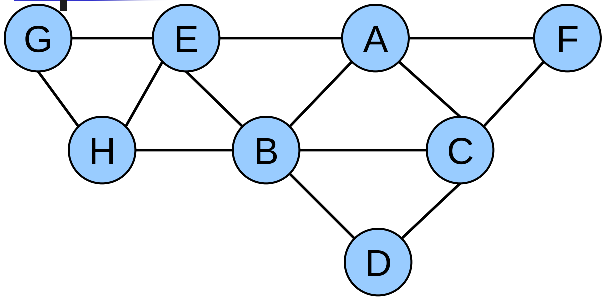
Variable Groups/Chains:
- {A,B}
- {C,D}
- {E,F}

Instantiate {A,B}
before {C,D} and {E,F}

*{A,B} is a separator/chain

Variables on **chains**
in the pseudo tree
can be instantiated
dynamically, based
on some semantic
ordering heuristic

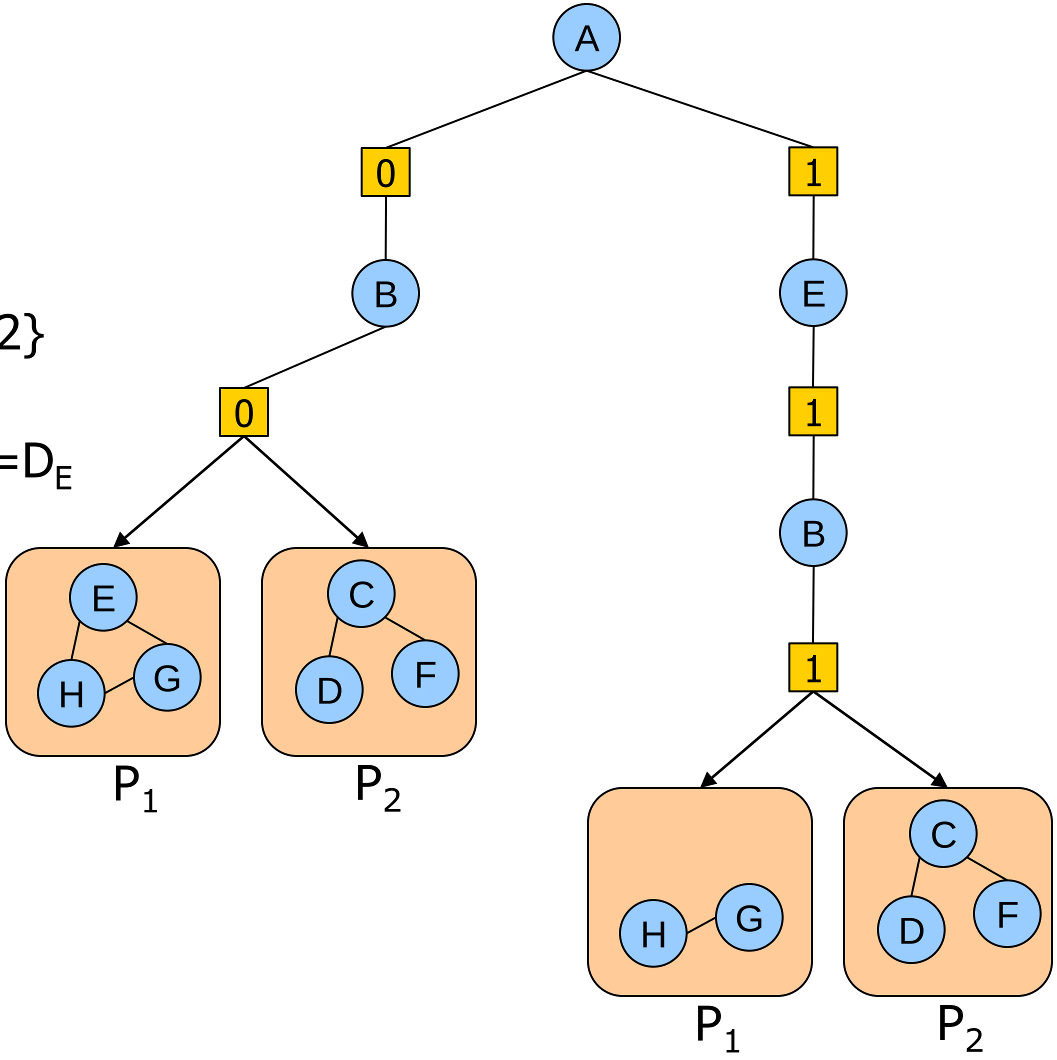
Full Dynamic Variable Ordering



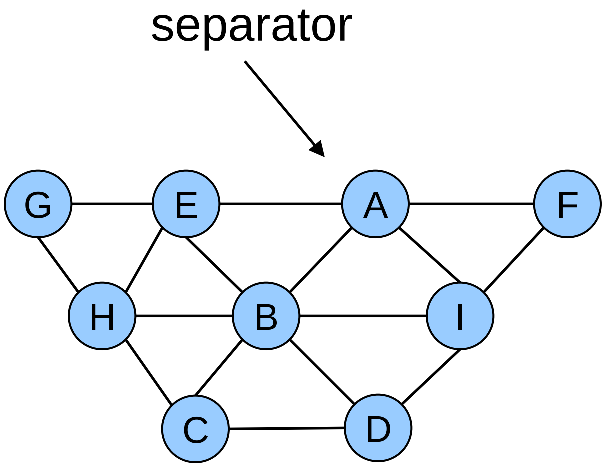
domains $D_A=\{0,1\}$ $D_B=\{0,1,2\}$
$D_E=\{0,1,2,3\}$
$D_C=D_D=D_F=D_G=D_H=D_E$

A	B	f(AB)
0	0	**3**
0	1	**∞**
0	2	**∞**
1	0	**4**
1	1	**0**
1	2	**6**

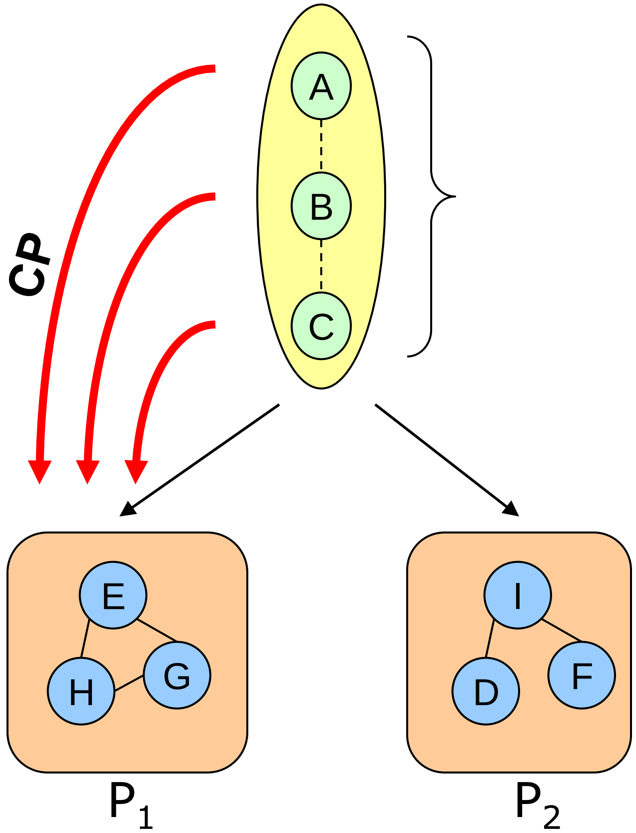
A	E	f(AE)
0	0	**0**
0	1	**5**
0	2	**1**
0	3	**4**
1	0	**∞**
1	1	**∞**
1	2	**0**
1	3	**5**



Dynamic Separator Ordering


separator

Primal graph



Constraint Propagation may create **singleton** variables in **P1** and **P2** (changing the problem’s structure), which in turn may yield smaller separators

Experiments

- **Benchmarks**
  - Belief Networks (BN)
  - Weighted CSPs (WCSP)
- **Algorithms**
  - AOBB
  - Samlam (BN)
  - Superlink (Genetic linkage analysis)
  - Toolbar (ie, DFBB+EDAC)
- **Heuristics**
  - Mini-Bucket heuristics (BN, WCSP)
  - EDAC heuristics (WCSP)

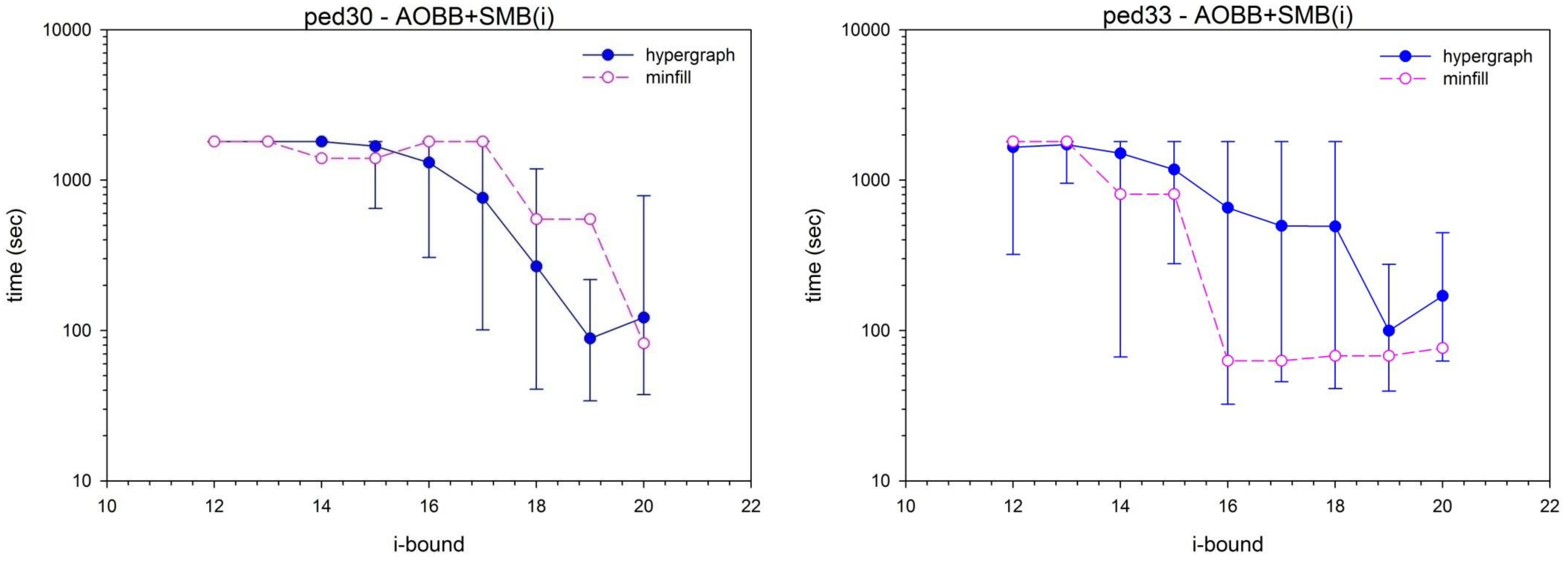
Genetic Linkage Analysis

(Fishelson and Geiger, UAI2002)

pedigree (n, d) (w*, h)	**Superlink** v. 1.6	**SamIam** v. 2.3.2	**BB+SMB(i)** **AOBB+SMB(i)** i=12		**BB+SMB(i)** **AOBB+SMB(i)** i=16		**BB+SMB(i)** **AOBB+SMB(i)** i=20	
	time	time	time	nodes	time	nodes	time	nodes
ped18 (1184, 5) (21, 119)	139.06	157.05	- -	- -	- 270.96	- 2,555,078	- **20.27**	- 7,689
ped25 (994, 5) (29, 53)	-	out	- -	- -	- -	- -	- **1894.17**	- 11,709,153
ped30 (1016, 5) (25, 51)	13095.83	out	- 5563.22	- 63,068,960	- 1811.34	- 20,275,620	- **82.25**	- 588,558
ped33 (581, 5) (26, 48)	-	out	- 2335.28	- 32,444,818	- **62.91**	- 807,071	- 76.47	- 320,279
ped39 (1272, 5) (23, 94)	322.14	out	- -	- -	- 4041.56	- 52,804,044	- 141.23	- 407,280

Min-fill pseudo tree. Time limit 3 hours.

Impact of the Pseudo Tree



Runtime distribution for hypergraph pseudo trees over 20 independent runs.
ped30 and **ped33** linkage networks.

Dynamic Variable Orderings

(Bensana et al., Constraints 1999)

spot5	**n**	**w***		**toolbar**	**BBEDAC**	**AOEDAC**	**AOEDAC+PVO**	**DVO+AOEDAC**	**AOEDAC+DSO**
	c	h							
29	16	7	time	4.56	109.66	613.79	545.43	**0.83**	11.36
	57	8	nodes	218,846	710,122	8,997,894	7,837,447	8,698	92,970
42b	14	9	time	-	-	-	-	-	**6825.4**
	75	9	nodes	-	-	-	-	-	27,698,614
54	14	9	time	0.31	0.97	31.34	9.11	**0.06**	0.75
	75	9	nodes	21,939	8,270	823,326	90,495	688	6,614
404	16	10	time	151.11	2232.89	255.83	152.81	12.09	**1.74**
	89	12	nodes	6,215,135	7,598,995	3,260,610	1,984,747	88,079	14,844
408b	18	10	time	-	-	-	-	-	**747.71**
	106	13	nodes	-	-	-	-	-	2,134,472
503	22	11	time	-	-	-	-	-	**53.72**
	131	15	nodes	-	-	-	-	-	231,480

SPOT5 benchmark. Time limit 2 hours.

Summary

- New generation of depth-first AND/OR Branch-and-Bound search
- Heuristics based on
  - Mini-Bucket approximation (static, dynamic)
  - Local consistency (FDAC, EDAC, VAC, …)
- Dynamic variable orderings
- Superior to state-of-the-art solvers traversing the classic OR search space

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            - AND/OR Branch-and-Bound with caching
            - Best-First AND/OR search
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        - AND/OR search for multi-objective optimization
    - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
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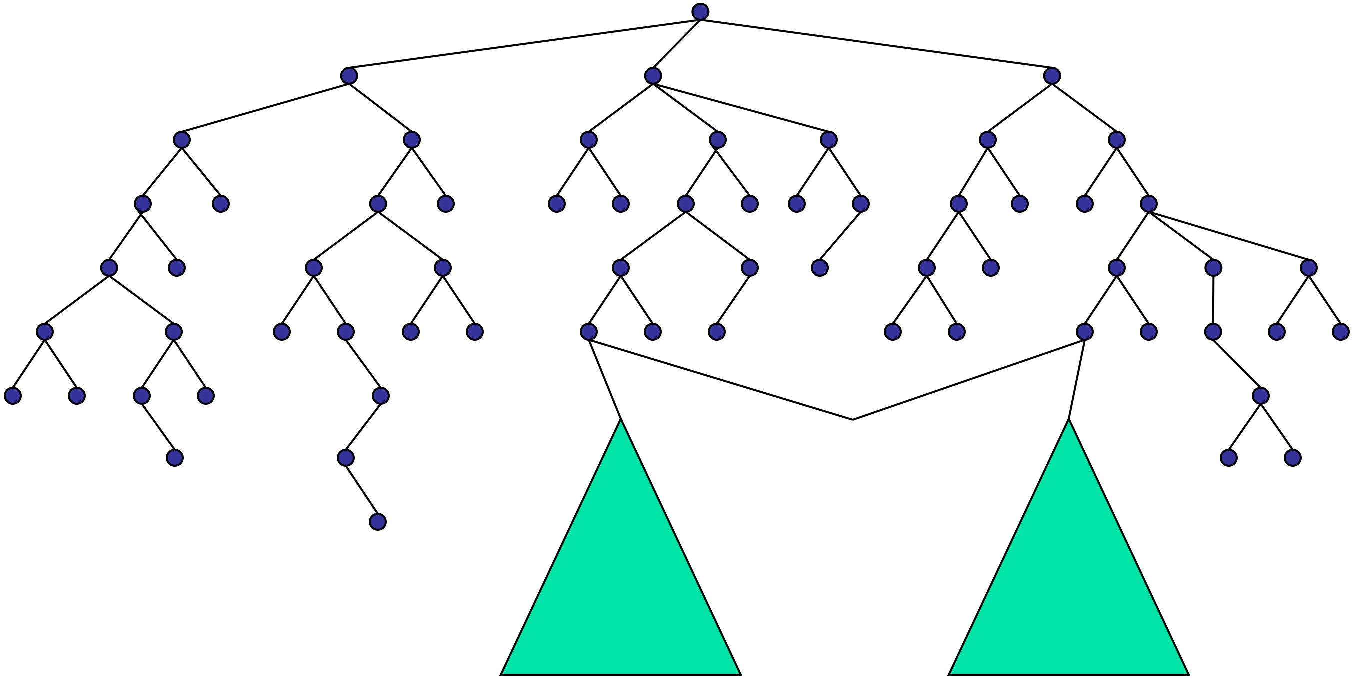
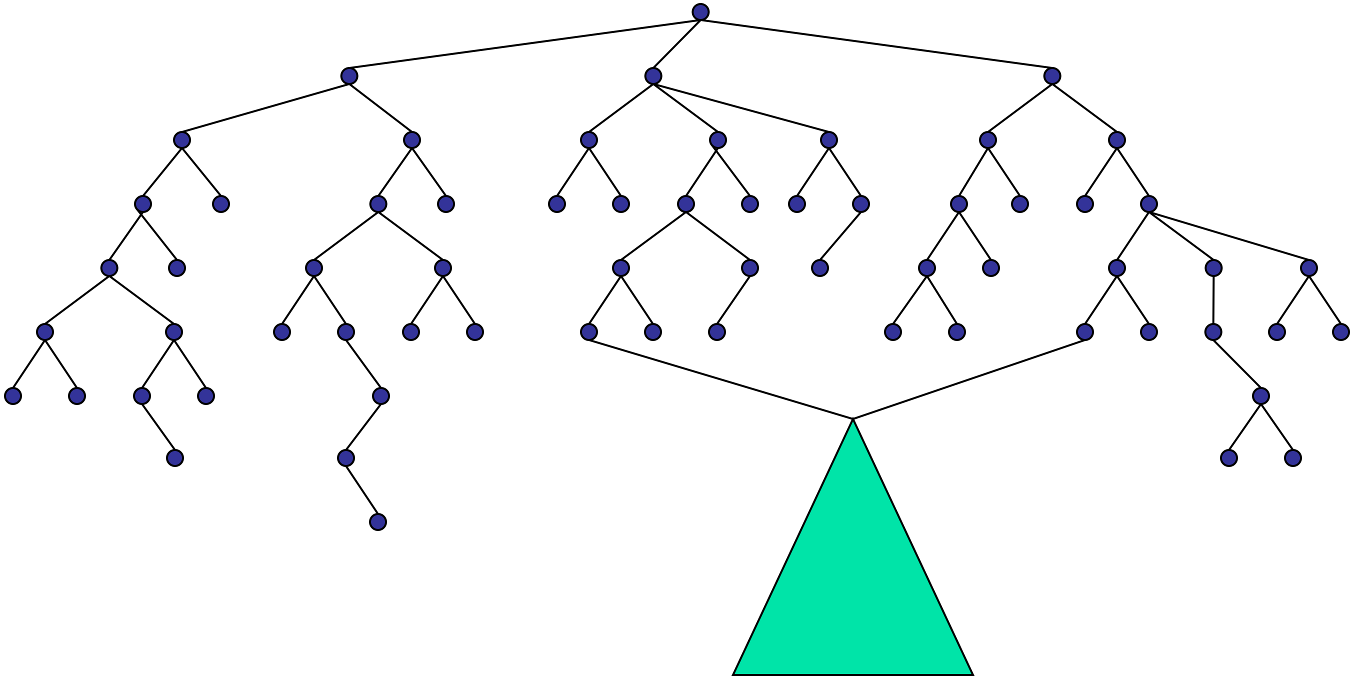
From search trees to search graphs

- Any two nodes that root identical subtrees (subgraphs) can be merged



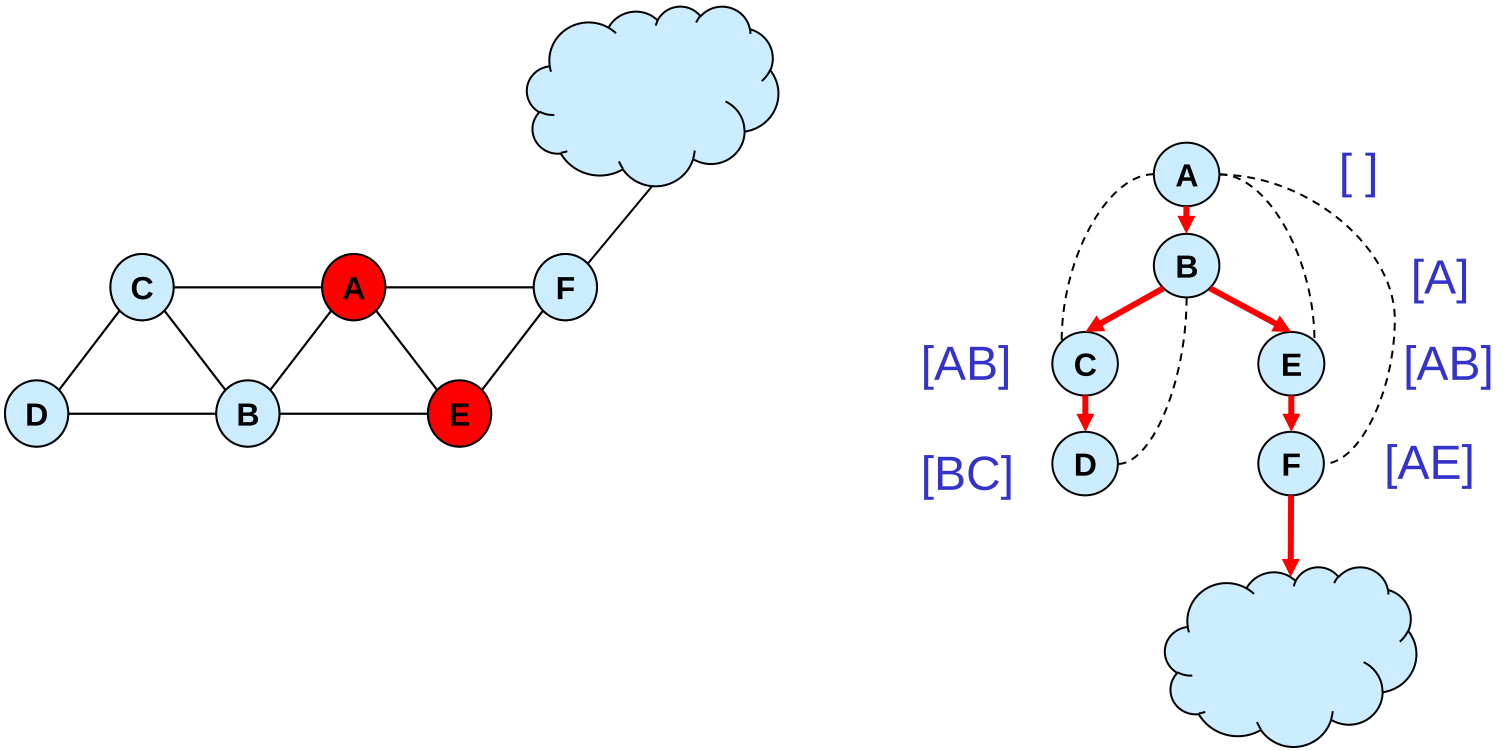
From search trees to search graphs

- Any two nodes that root identical subtrees (subgraphs) can be merged

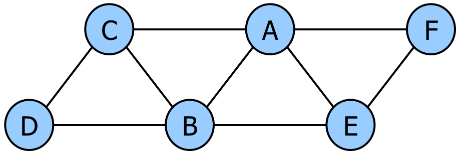


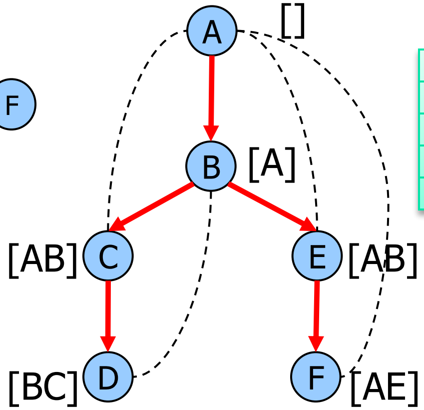
Merging based on context

context (X) = ancestors of X connected to
- X
- descendants of X



AND/OR Search Graph





A	B	f_1
0	0	2
0	1	0
1	0	1
1	1	4

A	C	f_2
0	0	3
0	1	0
1	0	0
1	1	1

A	E	f_3
0	0	0
0	1	3
1	0	2
1	1	0

A	F	f_4
0	0	2
0	1	0
1	0	0
1	1	2

B	C	f_5
0	0	0
0	1	1
1	0	2
1	1	4

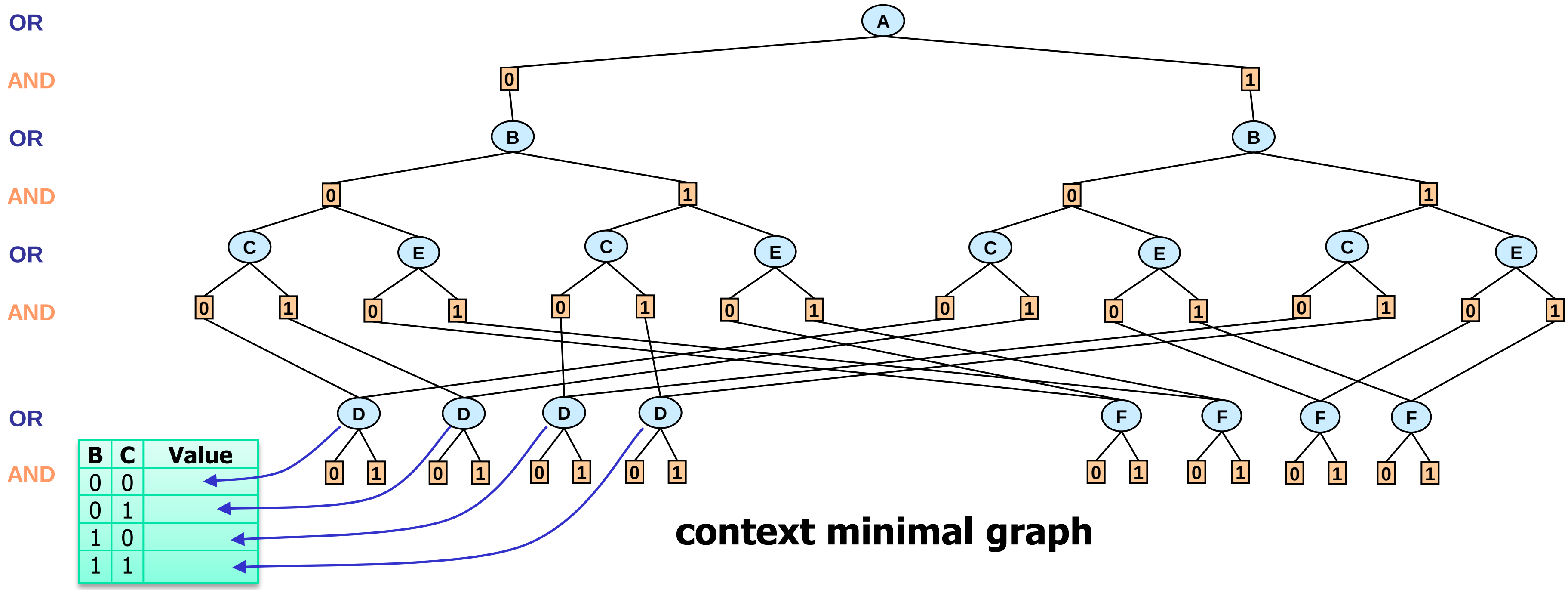
B	D	f_6
0	0	4
0	1	2
1	0	1
1	1	0

B	E	f_7
0	0	3
0	1	2
1	0	1
1	1	0

C	D	f_8
0	0	1
0	1	4
1	0	0
1	1	0

E	F	f_9
0	0	1
0	1	0
1	0	0
1	1	2

$$f(X) = \min_X \sum_{i=1}^{9} f_i(X)$$



B	C	Value
0	0	
0	1	
1	0	
1	1	

Cache table for D

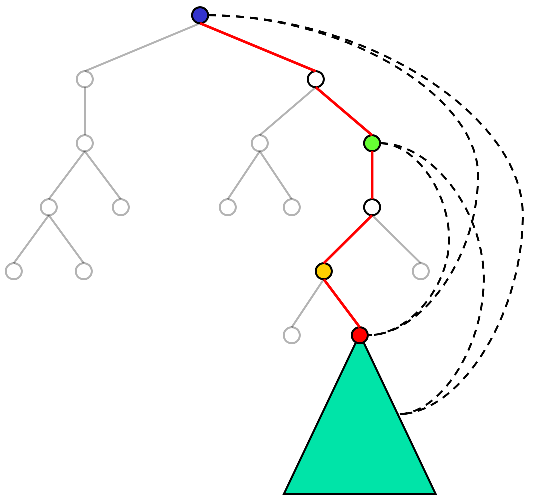
context minimal graph

How big is the context?

context (X) = ancestors of X in pseudo tree that are
connected to X, or to descendants of X

context (X) = parents in the induced graph

max |context| = induced width = treewidth



pseudo tree

context(●) = [● ● ●]

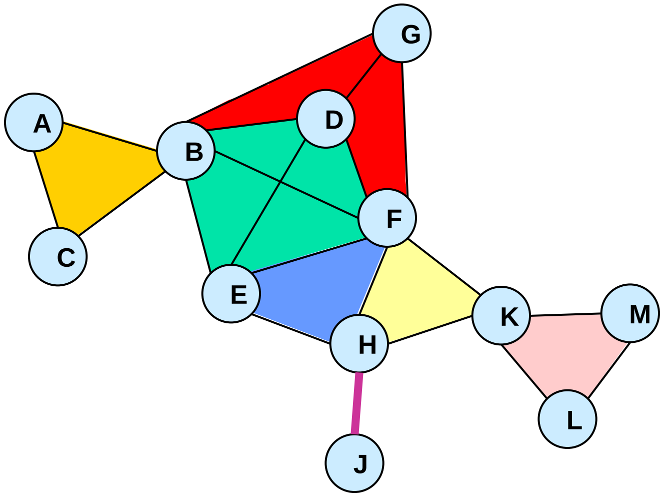
Complexity of AND/OR Graph Search

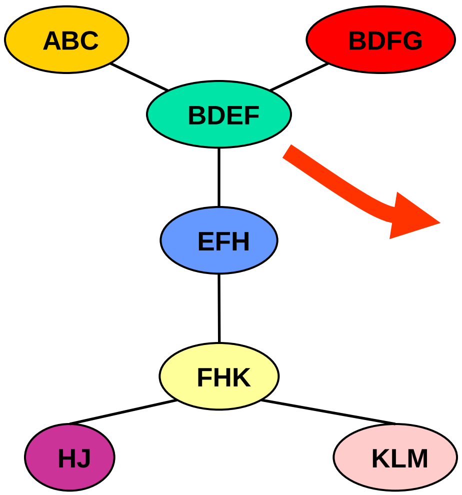
	AND/OR graph	**OR graph**
Space	$O(n\ d^{w^*})$	$O(n\ d^{pw^*})$
Time	$O(n\ d^{w^*})$	$O(n\ d^{pw^*})$

d = domain size
n = number of variables
w* = treewidth
pw* = pathwidth

$w^* \leq pw^* \leq w^* \log n$

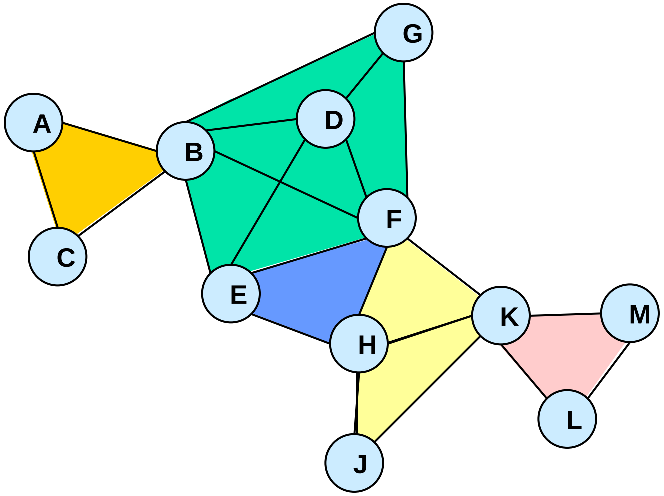
Treewidth vs. pathwidth

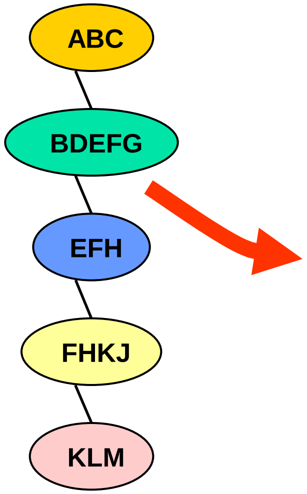




***treewidth* = 3**

= (max cluster size) - 1

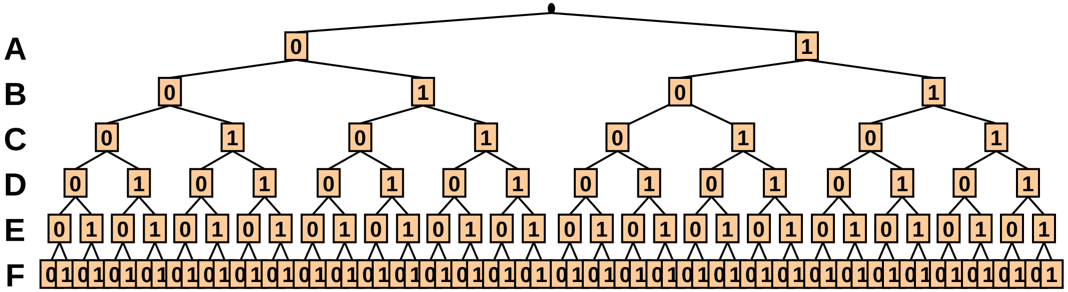




***pathwidth* = 4**

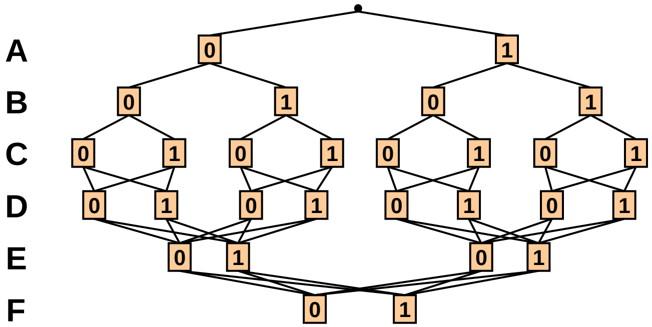
= (max cluster size) - 1

All Four Search Spaces



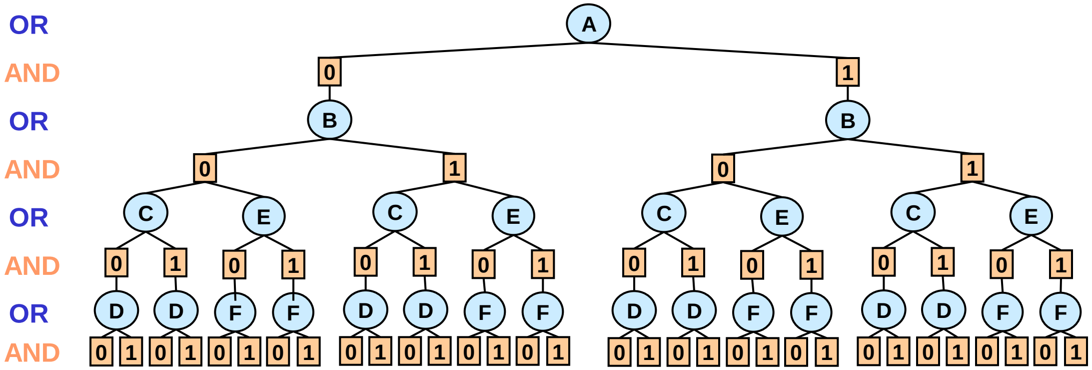
Full OR search tree

126 nodes



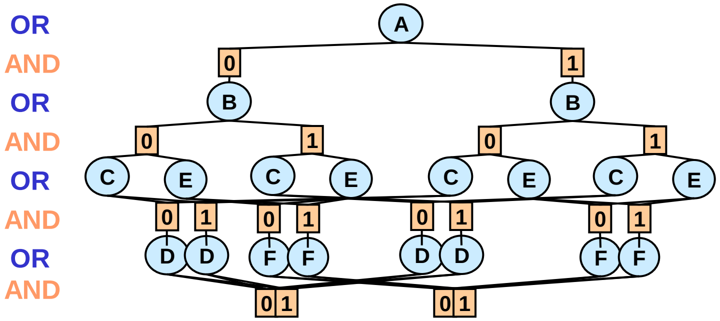
Context minimal OR search graph

28 nodes



Full AND/OR search tree

54 AND nodes



Context minimal AND/OR search graph

18 AND nodes

AND/OR Branch-and-Bound with Caching

(Marinescu and Dechter, AAAI2006, AIJ2009)

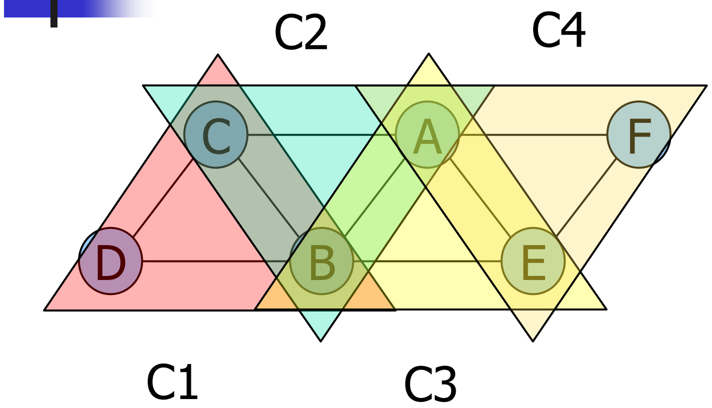
- Associate each node n with a heuristic lower bound h(n) on v(n)

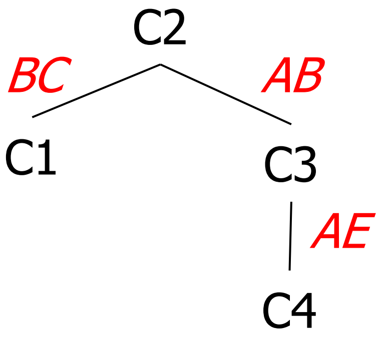
- EXPAND (top-down)
  - Evaluate f(T') and prune search if f(T') ≥ UB
  - If not in cache, expand the tip node n

- PROPAGATE (bottom-up)
  - Update value of the parent p of n
    - OR nodes: minimization
    - AND nodes: summation
  - Cache value of n, based on context

Backtrack with Tree Decomposition

(Jegou and Terrioux, ECAI2004)

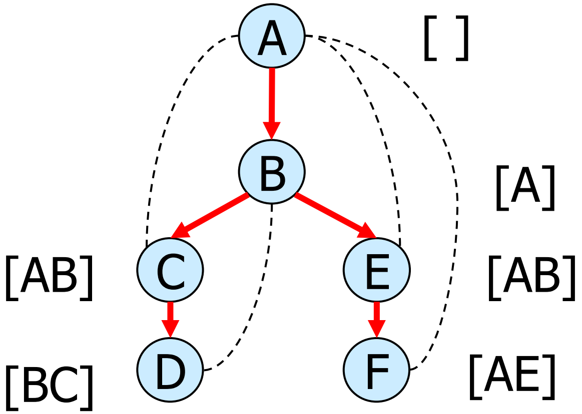




tree decomposition (w=2)

BTD:

- AND/OR graph search (caching on separators)
- Partial variable ordering (dynamic inside clusters)
- Maintaining local consistency



pseudo tree (w=2)

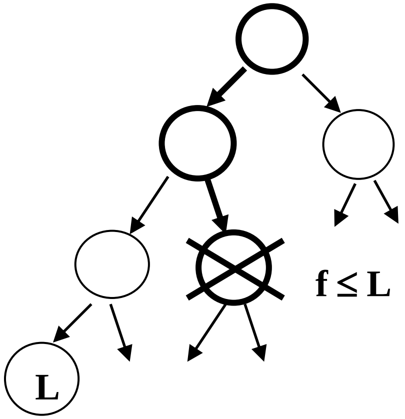
Basic Heuristic Search Schemes

Heuristic function **$f(x^p)$** computes a lower bound on the best extension of **x^p** and can be used to guide a heuristic search algorithm. We focus on:

1. DF Branch-and-Bound

Use heuristic function **$f(x^p)$** to prune the depth-first search tree

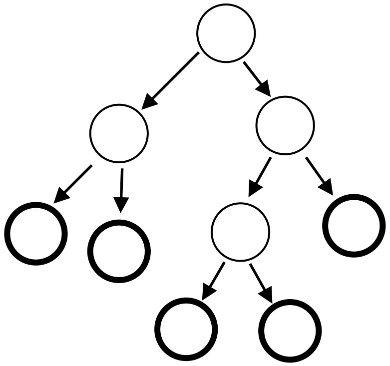
Linear space



2. Best-First Search

Always expand the node with the highest heuristic value **$f(x^p)$**

Needs lots of memory



Experiments

- **Benchmarks**
  - Belief Networks (BN)
  - Weighted CSPs (WCSP)
- **Algorithms**
  - AOBB-C – AND/OR Branch-and-Bound w/ caching
  - AOBF-C – Best-first AND/OR Search
  - Samlam
  - Superlink
  - Toolbar (DFBB+EDAC), Toolbar-BTD (BTD+EDAC)
- **Heuristics**
  - Mini-Bucket heuristics

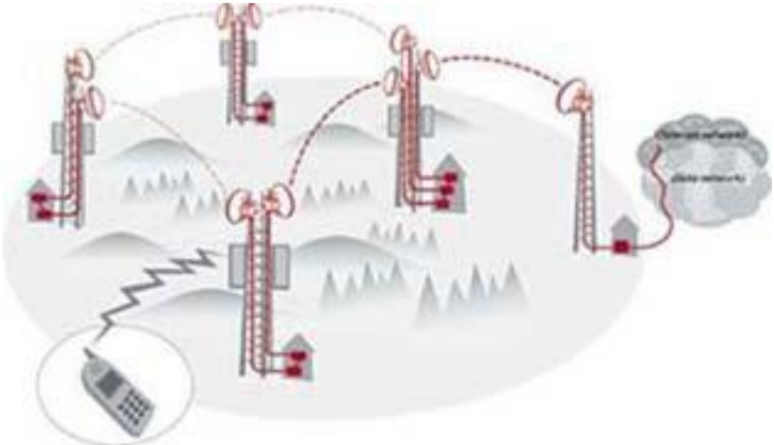
Genetic Linkage Analysis

pedigree (w*, h) (n, d)	**SamIam** **Superlink**	**BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=12		**BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=14		**BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=16		**BB-C+SMB(i)** **AOBB+SMB(i)** **AOBB-C+SMB(i)** **AOBF-C+SMB(i)** i=18	
		time	nodes	time	nodes	time	nodes	time	nodes
ped30 (23, 118) (1016, 5)	out 13095.83	- - 10212.70 out	- - 93,233,570	- - 8858.22 out	- - 82,552,957	- - - out	- - -	- 214.10 34.19 **30.39**	- 1,379,131 193,436 72,798
ped33 (37, 165) (581, 5)	out -	- 2804.61 1426.99 out	- 34,229,495 11,349,475	- 737.96 307.39 140.61	- 9,114,411 2,504,020 407,387	- 3896.98 1823.43 out	- 50,072,988 14,925,943	- 159.50 86.17 **74.86**	- 1,647,488 453,987 134,068
ped42 (25, 76) (448, 5)	out 561.31	- - - out	- - -	- - - out	- - -	- - 2364.67 **133.19**	- - 22,595,247 93,831	out	

Min-fill pseudo tree. Time limit 3 hours.

Radio Link Frequency Assignment Problem

(Cabon et al., *Constraints* 1999), (Koster *et al.*, *4OR* 2003)



- Given a telecommunication network
- …find the best frequency for each communication link, avoiding interferences

- Best can be:
    - Minimize the maximum frequency, no interference (max)
    - **Minimize the global interference (sum)**
- Generalizes graph coloring problems: $|f_1 - f_2| \geq a$

CELAR *problem size: n=100—458 ; d=44 ; m=1,000—5,000*

- CELAR SCEN-06

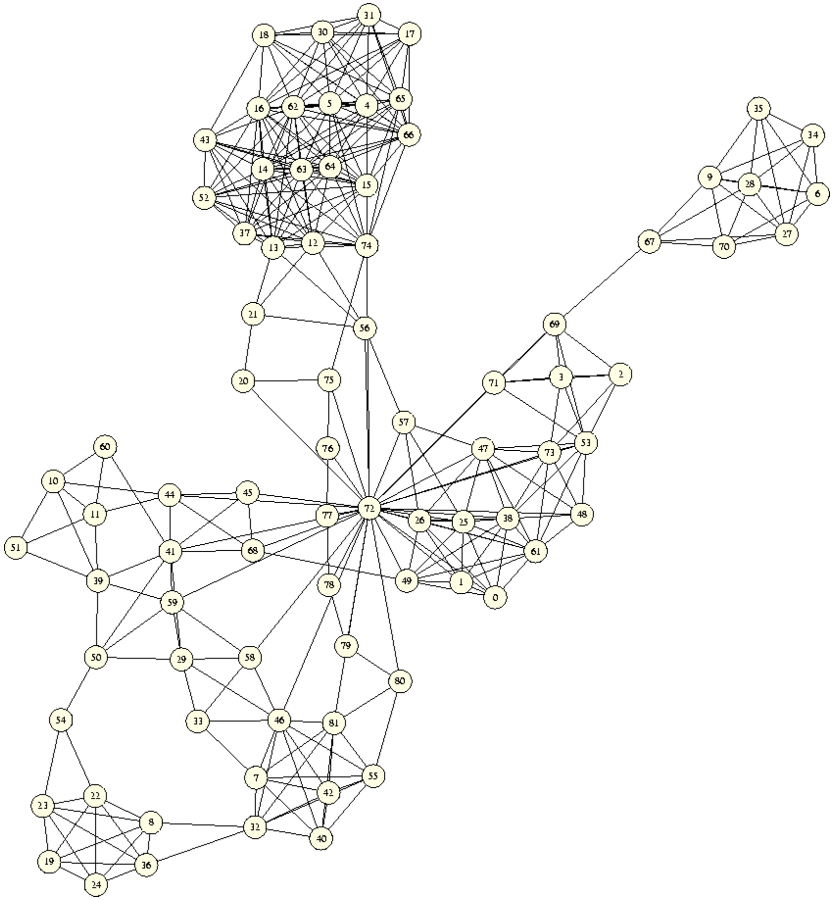
  n=100, d=44,

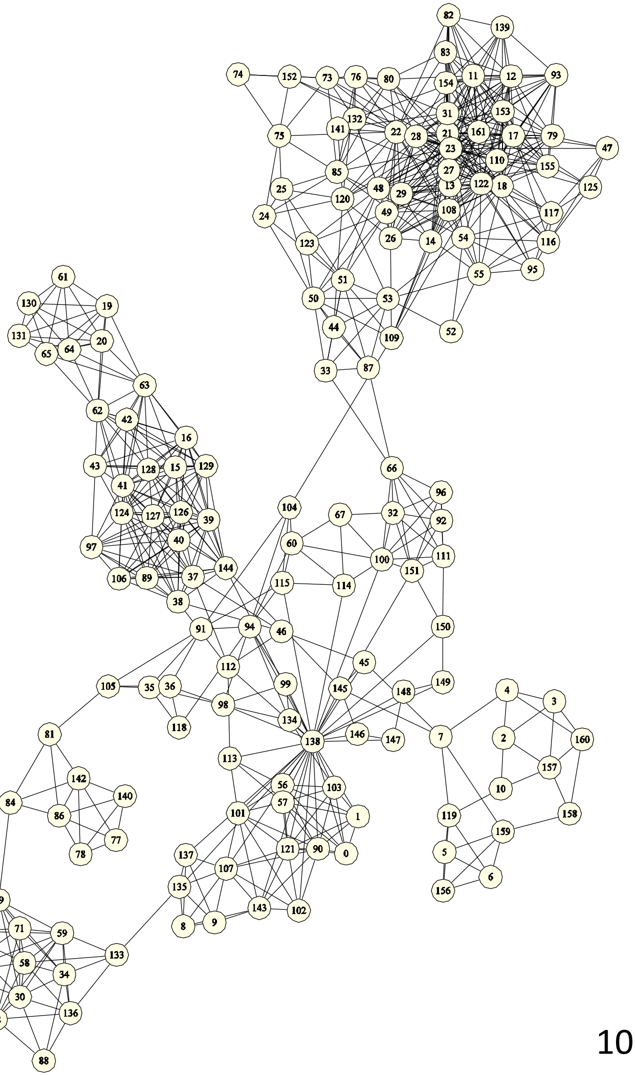
  m=350, optimum=3389

- CELAR SCEN-07r **(OPEN)**

  n=162, d=44,

  m=764, optimum=343592





CELAR

toulbar2 v0.8 running on a 2.6 GHz computer with 32 GB

(Sanchez et al., IJCAI2009)

- Maximum Cardinality Search tree decomposition heuristic
- Root selection: largest (SCEN-06) / most costly (SCEN-07) cluster
- Last-conflict variable ordering and dichotomic branching

- Closed 1 open problem by exploiting tree decompostion and EDAC

CELAR	**n**	**d**	**m**	**k**	**p**	**w**	**DFBB**	**BTD**	**RDS-BTD**
SCEN-06	**100**	**44**	**350**	**∞**	**∞**	**11**	**2588 sec.**	**221 sec.**	**316 sec.**
SCEN-07r	**162**	**44**	**764**	**354008**	**3**	**53**	**- > 50days**	**6 days**	**4.5 days**

Summary

- New memory intensive AND/OR search algorithms for optimization in graphical models

- Depth-first and best-first control strategies

- Superior to state-of-the-art OR and AND/OR Branch-and-Bound tree search algorithms

Outline

- Introduction
- Inference
- Search
  - Exact
    - AND/OR search trees
    - AND/OR Branch-and-Bound search
    - AND/OR search graphs (caching)
    - AND/OR search for 0-1 integer programming
    - AND/OR search for multi-objective optimization
  - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
- Software

0-1 Integer Linear Programming

minimize: $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subjectto:

$$
\begin{aligned}
&a_1^1 x_1 + a_2^1 x_2 + \ldots + a_n^1 x_n \le b^1 \\
&a_1^2 x_1 + a_2^2 x_2 + \ldots + a_n^2 x_n \le b^2 \\
&\qquad\qquad \ldots \\
&a_1^m x_1 + a_2^m x_2 + \ldots + a_n^m x_n \le b^m \\
&x_1, x_2, \ldots, x_n \in \{0,1\}
\end{aligned}
$$

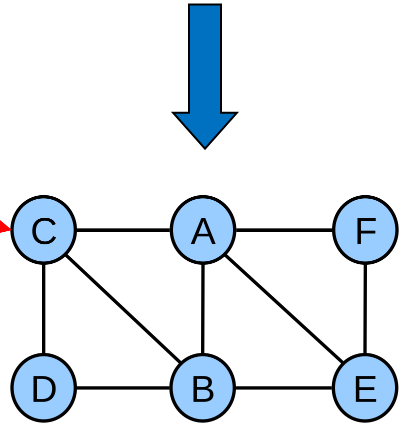
:

- VLSI circuit design
- Scheduling
- Routing
- Combinatorial auctions
- Facility location
- …

$\text{minimize}: z = 7A + 3B - 2C + 5D - 6E + 8F$

subject to:

$$
\begin{aligned}
&\boxed{3A - 12B + C \le 3} \\
&-2B + 5C - 3D \le -2 \\
&2A + B - 4E \le 2 \\
&A - 3E + F \le 1 \\
&A, B, C, D, E, F \in \{0,1\}
\end{aligned}
$$



primal graph

AND/OR Search Tree

$$\text{minimize}: z = 7A + 3B - 2C + 5D - 6E + 8F$$

subject to:

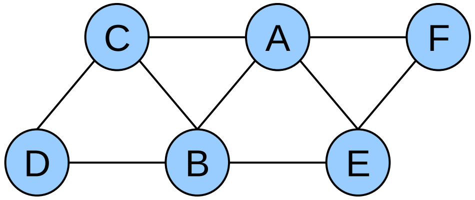
$$3A - 12B + C \leq 3$$

$$-2B + 5C - 3D \leq -2$$

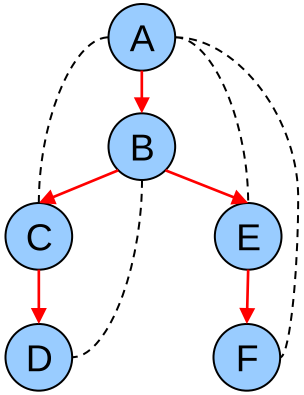
$$2A + B - 4E \leq 2$$

$$A - 3E + F \leq 1$$

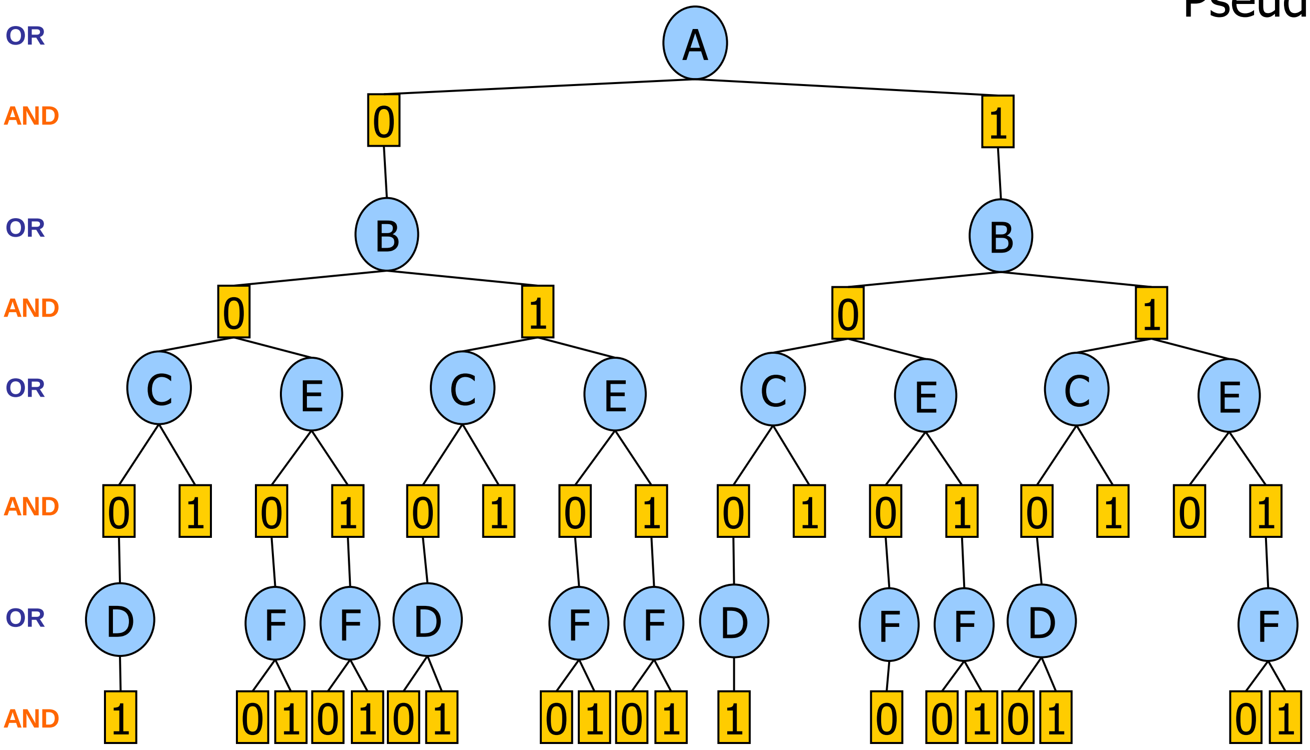
$$A, B, C, D, E, F \in \{0,1\}$$



Primal graph



Pseudo tree



Weighted AND/OR Search Tree

$$\text{minimize}: z = 7A + 3B - 2C + 5D - 6E + 8F$$

subject to:

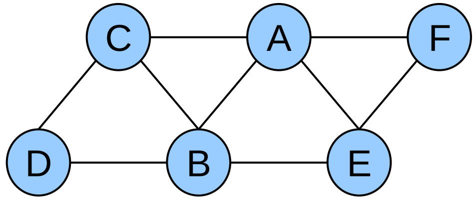
$$3A - 12B + C \leq 3$$

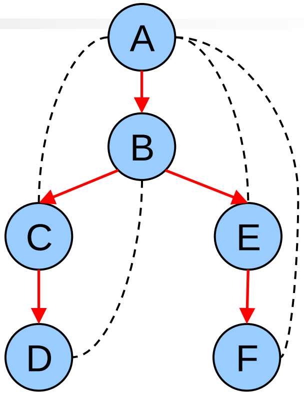
$$-2B + 5C - 3D \leq -2$$

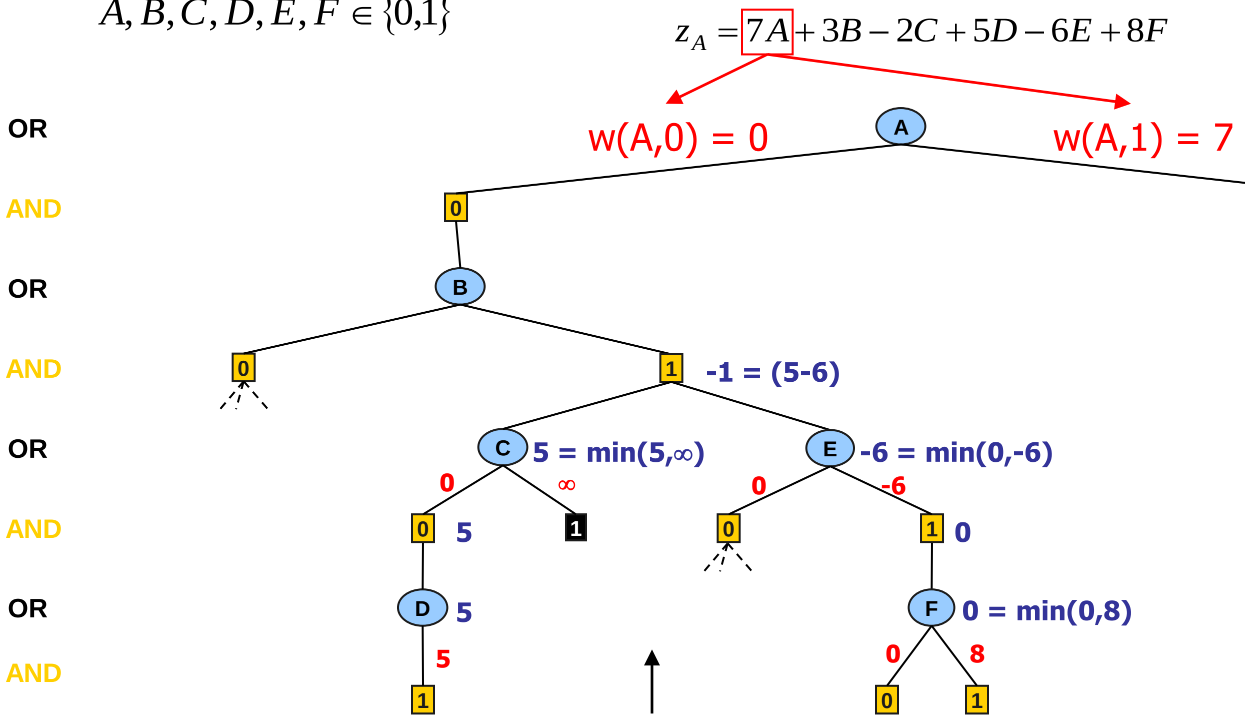
$$2A + B - 4E \leq 2$$

$$A - 3E + F \leq 1$$

$$A, B, C, D, E, F \in \{0,1\}$$







$$z_A = \boxed{7A} + 3B - 2C + 5D - 6E + 8F$$

Node Value (bottom up)

OR – minimization
AND – summation

AND/OR Search Graph

$$\text{minimize}: z = 7A + 3B - 2C + 5D - 6E + 8F$$

subject to:

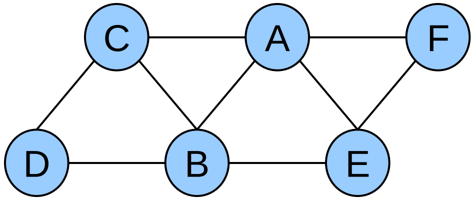
$$3A - 12B + C \le 3$$

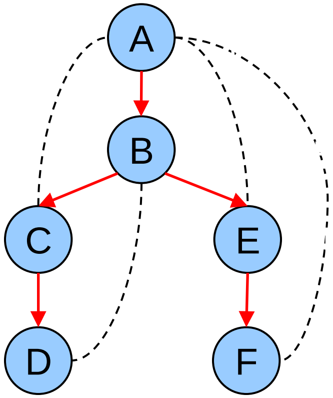
$$-2B + 5C - 3D \le -2$$

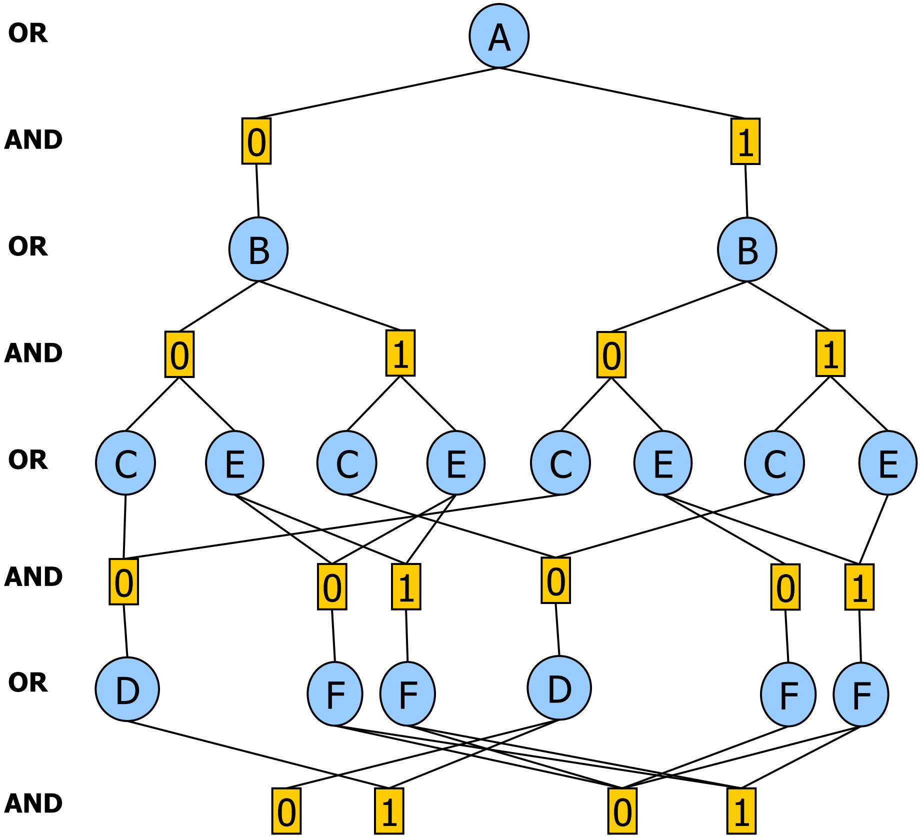
$$2A + B - 4E \le 2$$

$$A - 3E + F \le 1$$

$$A, B, C, D, E, F \in \{0,1\}$$







16 nodes (graph) vs. 54 nodes (tree)

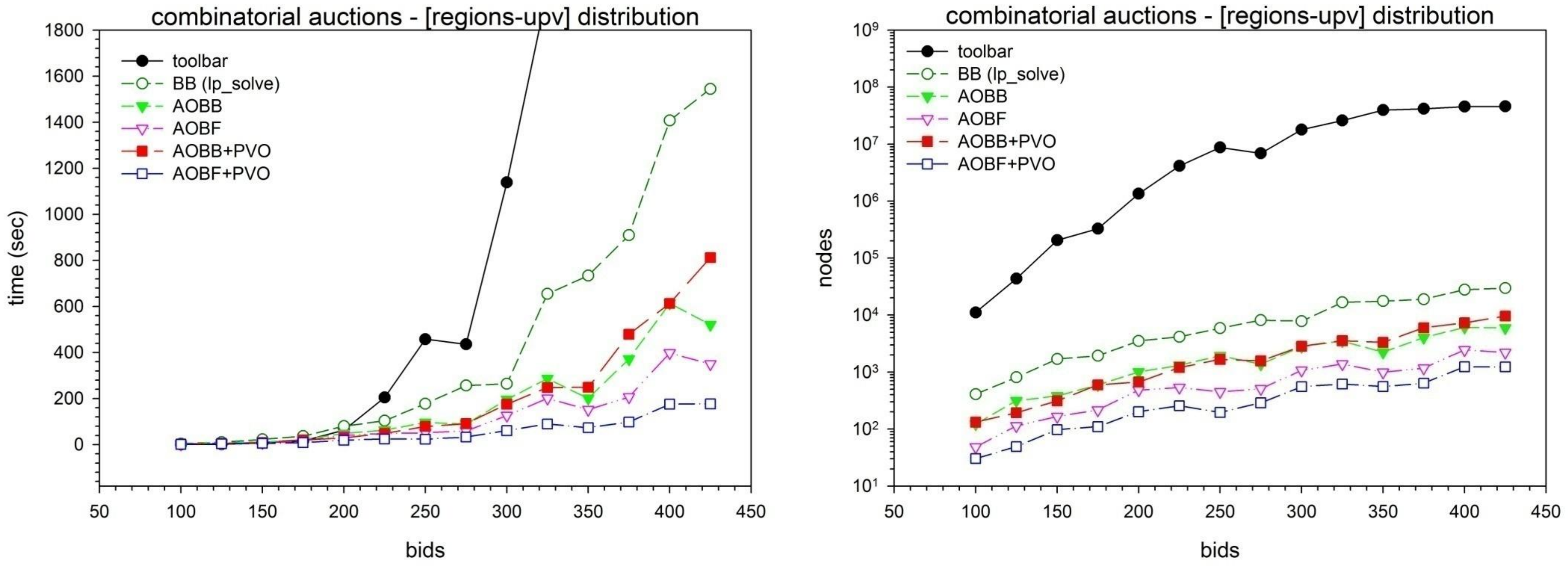
Experiments

- **Algorithms**
  - AOBB, AOBF – tree search
  - AOBB+PVO, AOBF+PVO – tree search
  - AOBB-C, AOBF-C – graph search
  - lp_solve 5.5, CPLEX 11.0, toolbar (DFBB+EDAC)

- **Benchmarks**
  - Combinatorial auctions
  - MAX-SAT instances

- **Implementation**
  - LP relaxation solved by lp_solve 5.5 library
  - BB (lp_solve) baseline solver

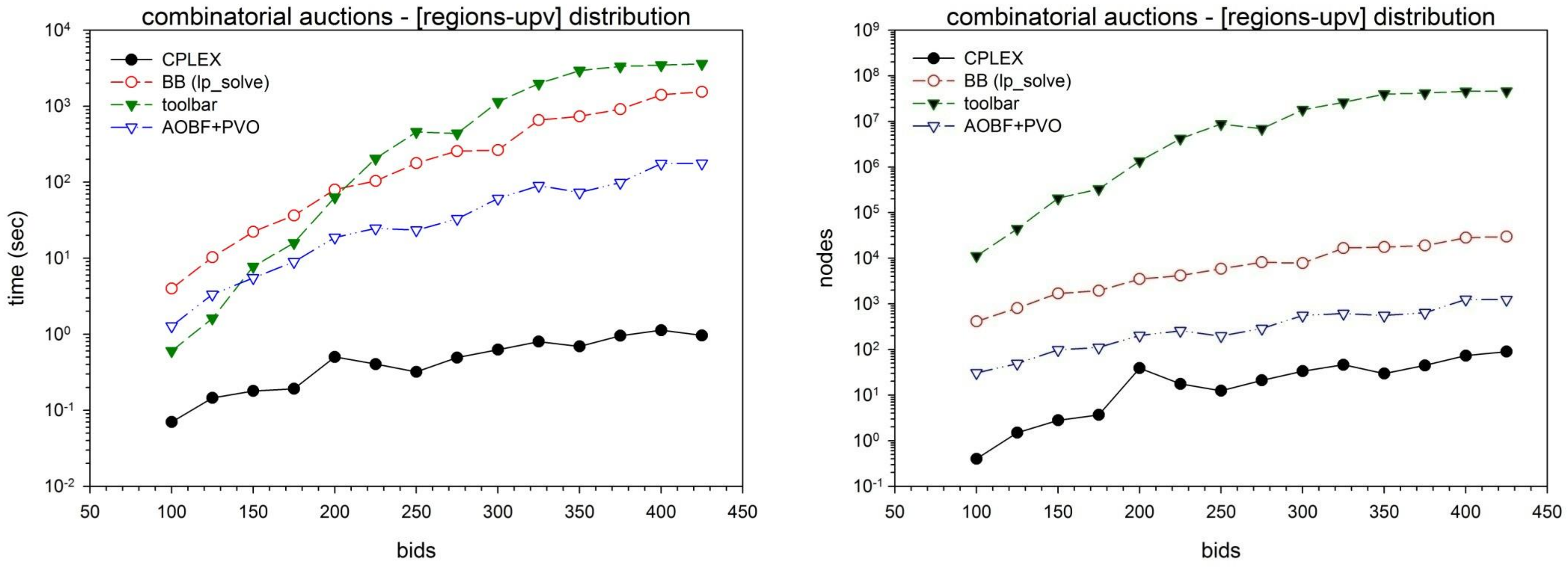
Combinatorial Auctions



Combinatorial auctions from **`regions-upv`** distribution with 100 goods and increasing number of bids. Time limit 1 hour.

Very large treewidth $\in$ [68, 184]

Combinatorial Auctions



Combinatorial auctions from `regions-upv` distribution with 100 goods and increasing number of bids. Time limit 1 hour.

Very large treewidth $\in$ [68, 184]

MAX-SAT Instances (pret)

			Tree search		Tree search		Graph search	
pret (w*, h)	**BB** **CPLEX**		**AOBB** **AOBF**		**AOBB+PVO** **AOBF+PVO**		**AOBB-C** **AOBF-C**	
	time	nodes	time	nodes	time	nodes	time	nodes
pret60-40 (6, 13)	- 676.94	- 3,926,422	7.88 7.56	1,255 1,202	8.41 8.70	1,216 1,326	7.38 **3.58**	1,216 568
pret60-60 (6, 13)	- 535.05	- 2,963,435	8.56 8.08	1,259 1,184	8.70 8.31	1,247 1,206	7.30 **3.56**	1,140 538
pret60-75 (6, 13)	- 402.53	- 2,005,738	6.97 7.38	1,124 1,145	6.80 8.42	1,089 1,149	6.34 **3.08**	1,067 506
pret150-40 (6, 15)	- out	-	95.11 101.78	6,625 6,535	108.84 101.97	7,152 6,246	75.19 **19.70**	5,625 1,379
pret150-60 (6, 15)	- out	-	98.88 106.36	6,851 6,723	112.64 102.28	7,347 6,375	78.25 **19.75**	5,813 1,393
pret150-75 (6, 15)	- out	-	108.14 98.95	7,311 6,282	115.16 103.03	7,452 6,394	84.97 **20.95**	6,114 1,430

pret MAX-SAT instances. Time limit 10 hours.

BB solver could not solve any instance.

Summary

- New AND/OR search algorithms for 0-1 Integer Programming
- Dynamic variable orderings
- Superior to baseline OR Branch-and-Bound from the lp_solve library
- Outperform CPLEX on selected MAX-SAT instances

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- Introduction
- Inference
- Search
  - Exact
    - AND/OR search trees
    - AND/OR Branch-and-Bound search
    - AND/OR search graphs (caching)
    - AND/OR search for 0-1 integer programming
    - AND/OR search for multi-objective optimization
  - Approximate: Sampling etc.
- Compilation: AND/OR Decision Diagrams
- Software

Algorithms for AND/OR Space

- **Back-jumping** for CSPs
  (Gaschnig 1977), (Dechter 1990), (Prosser, Bayardo and Mirankar, 1995)
- **Pseudo-search re-arrangement**, for any CSP task
  (Freuder and Quinn 1985)
- **Pseudo-tree search for soft constraints**
  (Larrosa, Meseguer and Sanchez, 2002)
- **Recursive Conditioning**
  (Darwiche, 2001), explores the AND/OR tree or graph for any query
- **BTD: Searching tree-decompositions** for optimization
  (Jeagou and Terrioux, 2004)
- **Value Elimination**
  (Bacchus, Dalmao and Pittasi, 2003)

Outline

- Introduction
- Inference
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  - Exact
  - Approximate: Sampling
- Compilation: AND/OR Decision Diagrams
- Software

Sampling: Approximation of Search

1. Importance Sampling
2. Markov Chain Monte Carlo: Gibbs Sampling
3. Sampling in presence of Determinism
4. Rao-Blackwellisation
5. AND/OR importance sampling

See :Sampling Techniques for Probabilistic and Deterministic Graphical models PDF
Tutorial, AAAI 2010, Atlanta, GA, July 12, 2010:

http://www.ics.uci.edu/~dechter/talks.html

Outline

- Introduction
- Inference
- Search
- Compilation: AND/OR Decision Diagrams
- Software

Outline

- Introduction
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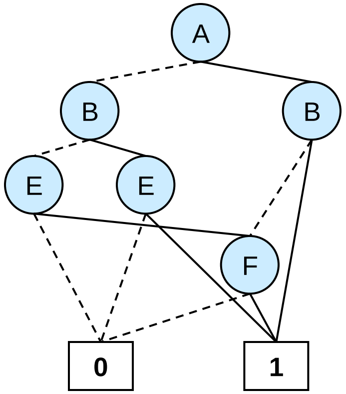
- Compilation
  - AND/OR Decision Diagrams
  - Apply Operator
  - Bottom up (Variable elimination)
  - Top down (AND/OR search)
- Software

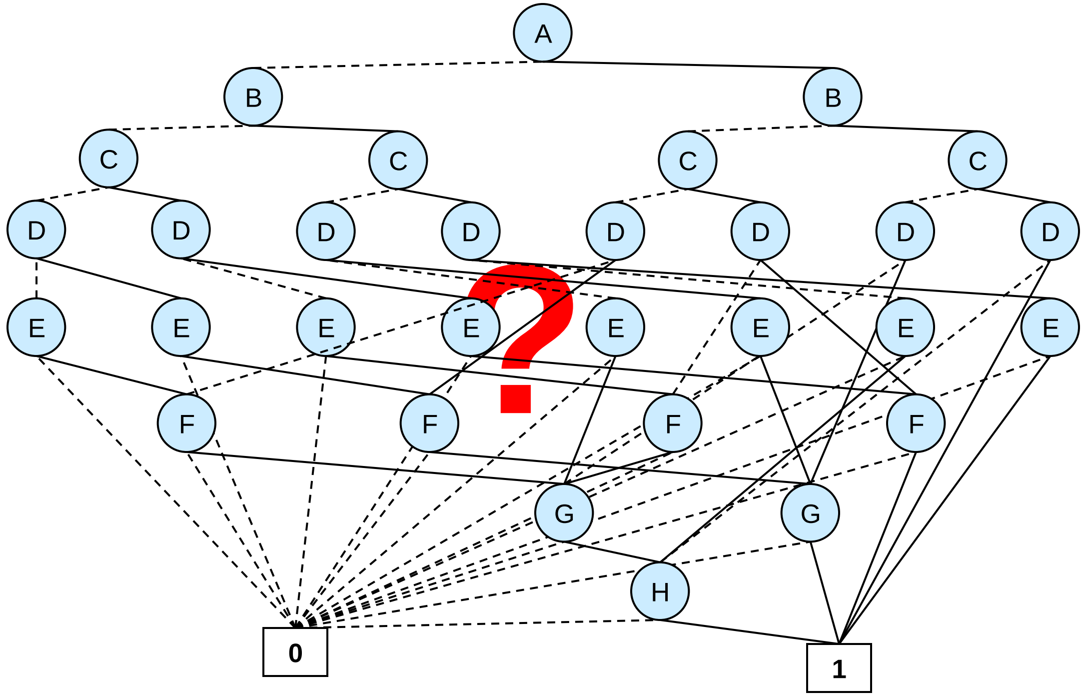
Exploiting Structure

$f = (A \lor E) \land (B \lor F)$

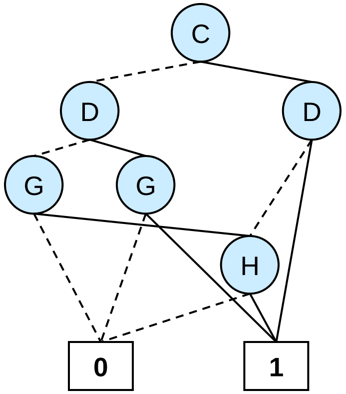


OBDD

$f \land g$ =

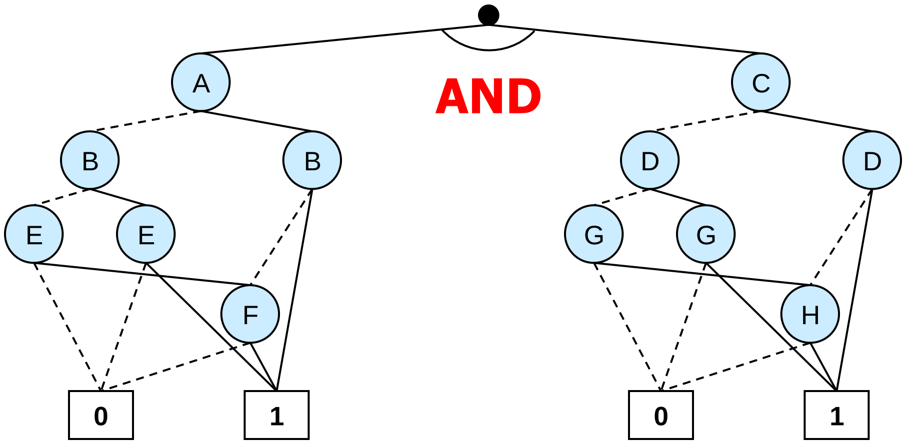


$g = (C \lor G) \land (D \lor H)$

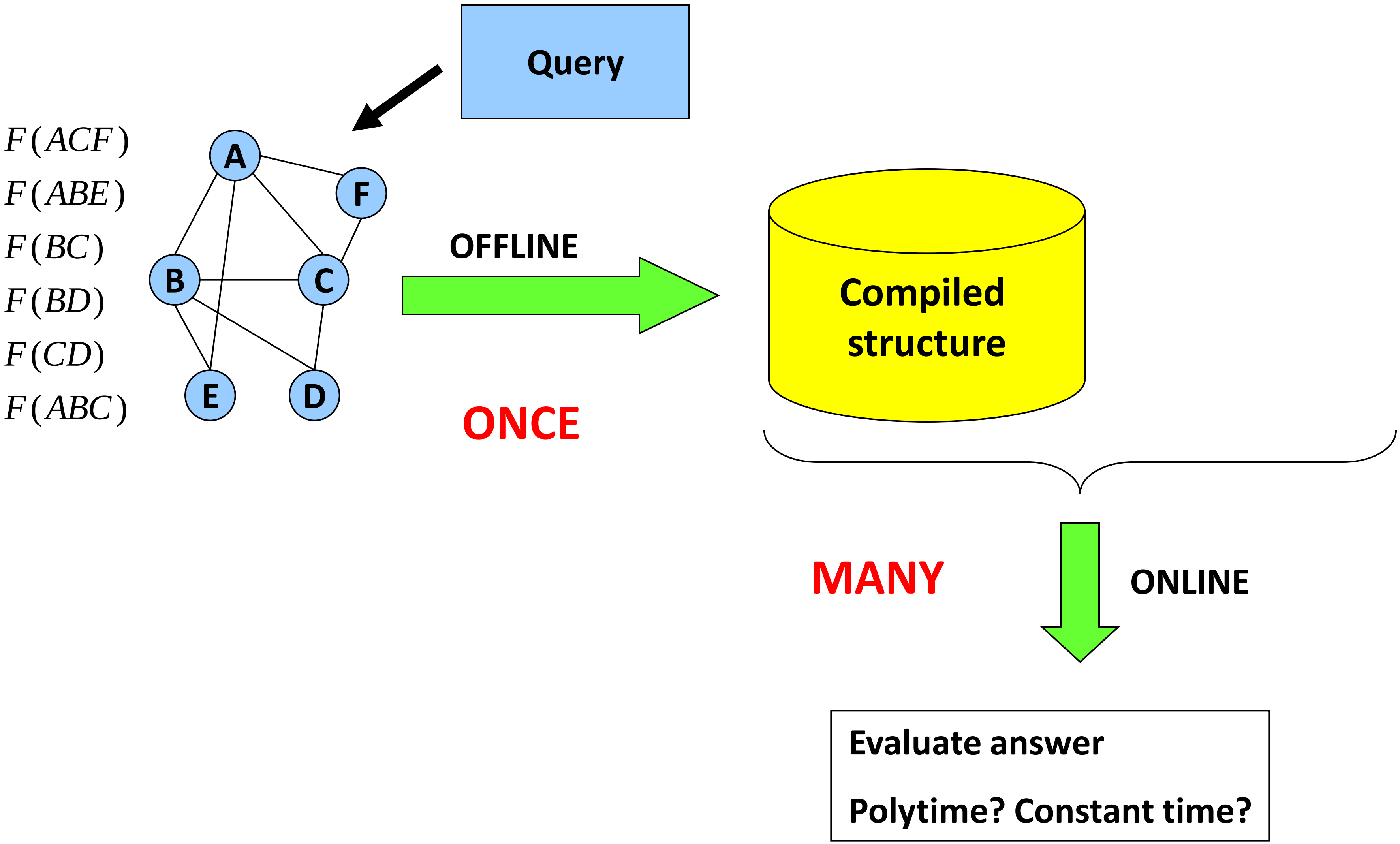


OBDD

$f \land g$ =



Compilation of Graphical Models



$F(ACF)$

$F(ABE)$

$F(BC)$

$F(BD)$

$F(CD)$

$F(ABC)$

Query

OFFLINE

ONCE

Compiled structure

MANY

ONLINE

Evaluate answer

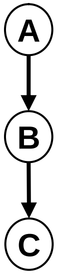
Polytime? Constant time?

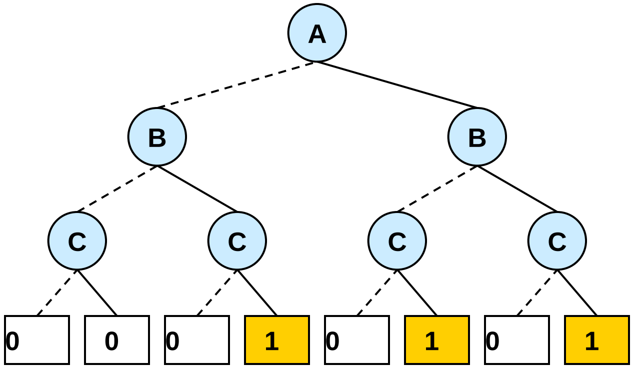
Ordered Binary Decision Diagram

B = {0,1}   f : B^3 → B

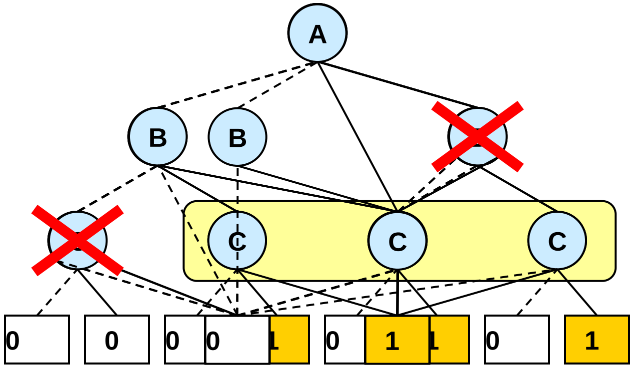
A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Table





Decision tree



1) Merge nodes with identical children

Merge terminal nodes

Simplify

OBDD

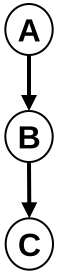
[Bryant86]

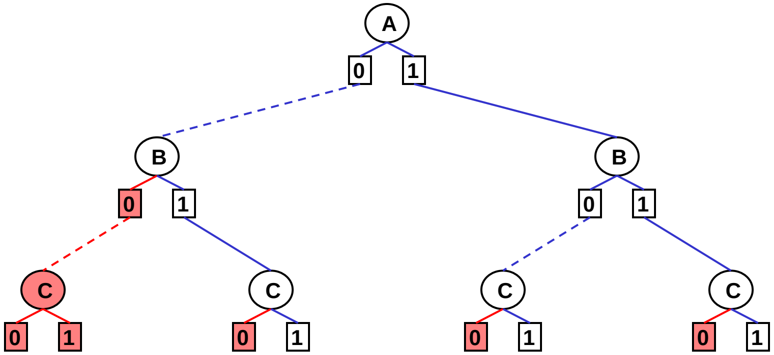
2) Remove redundant nodes

Ordering enables efficient operations

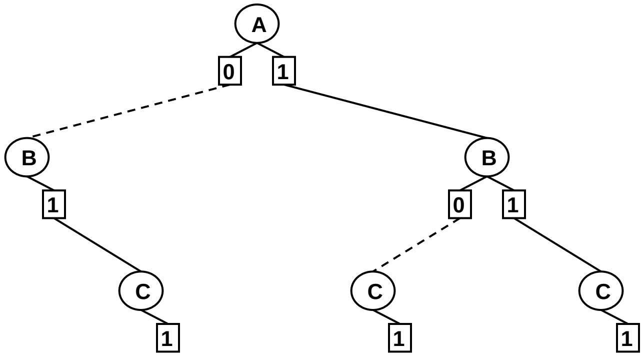
Minimal AND/OR Graphs (MAO)

A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

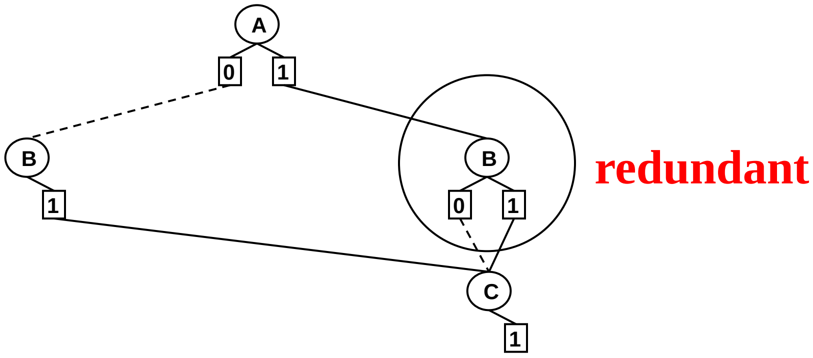




Full AND/OR search tree



Backtrack free AND/OR search tree

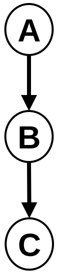


Minimal AND/OR search graph

Minimal AND/OR graph = closure under "merge"
= unique fix point of "merge"

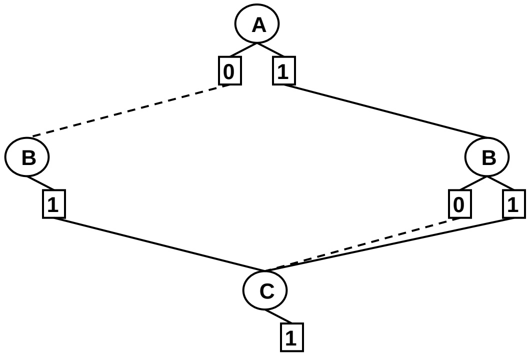
From MAO to Decision Diagram

A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

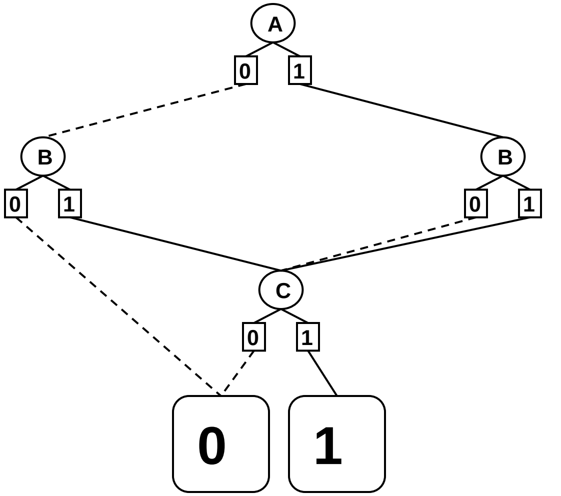


Point dead-ends to terminal node "0"

Point goods to terminal node "1"



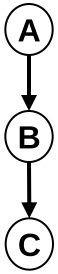
Minimal AND/OR graph



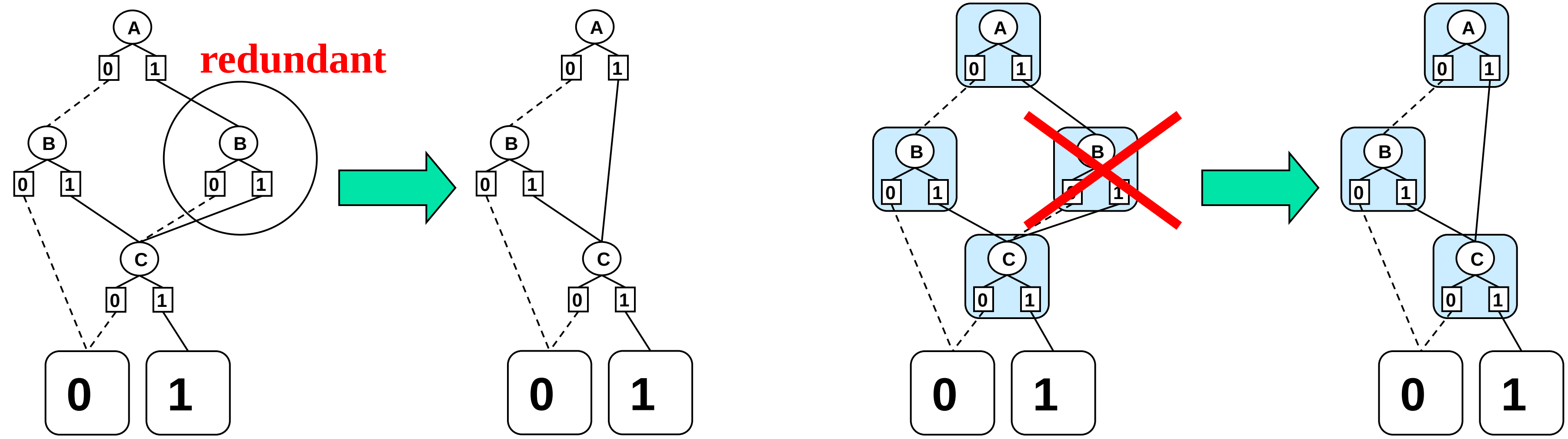
Decision Diagram

Removing Redundancy

A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



Group OR node together with its AND children into a meta-node

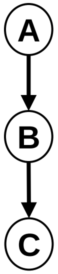


OBDD

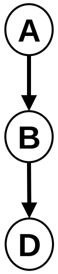
(pseudo tree is a chain)

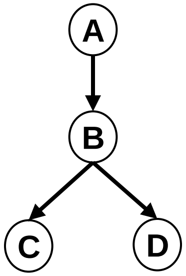
AOBDD

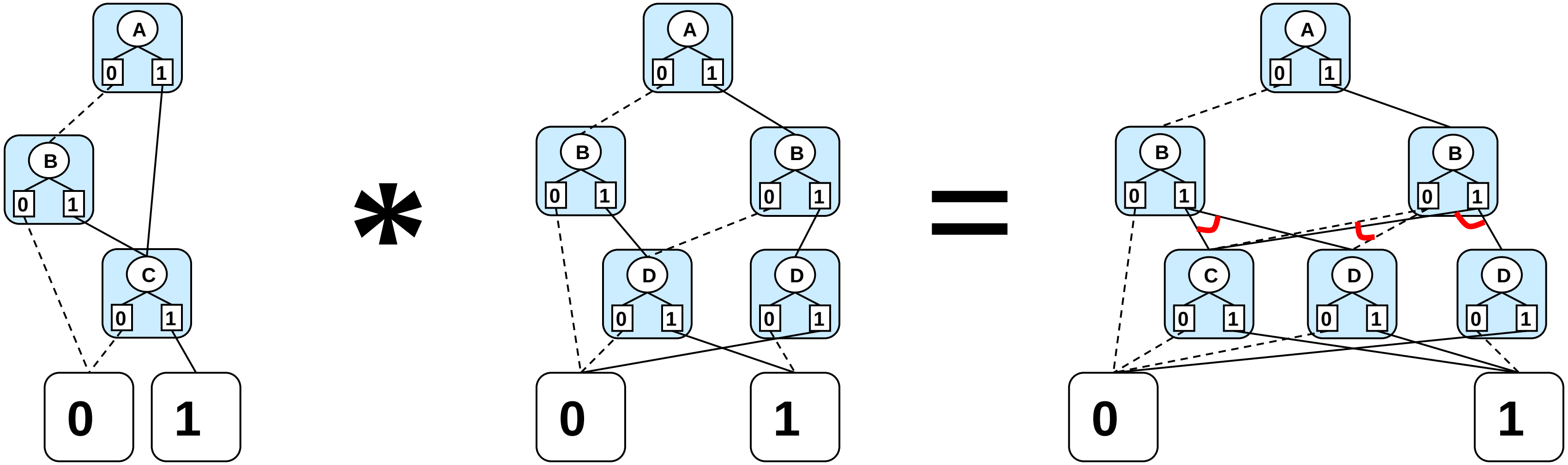
A	B	C	f(ABC)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



A	B	D	g(ABD)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0







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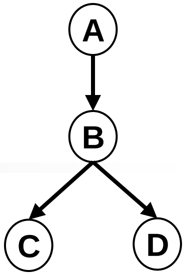
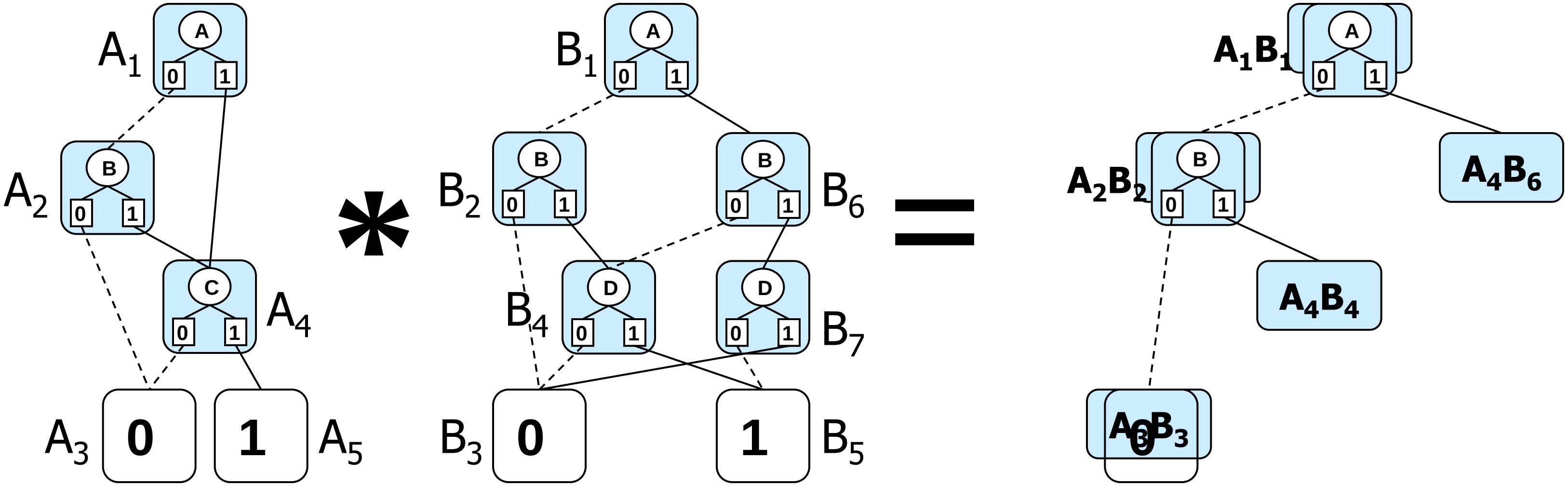
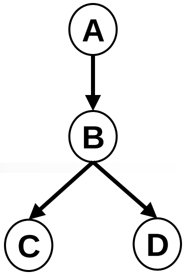
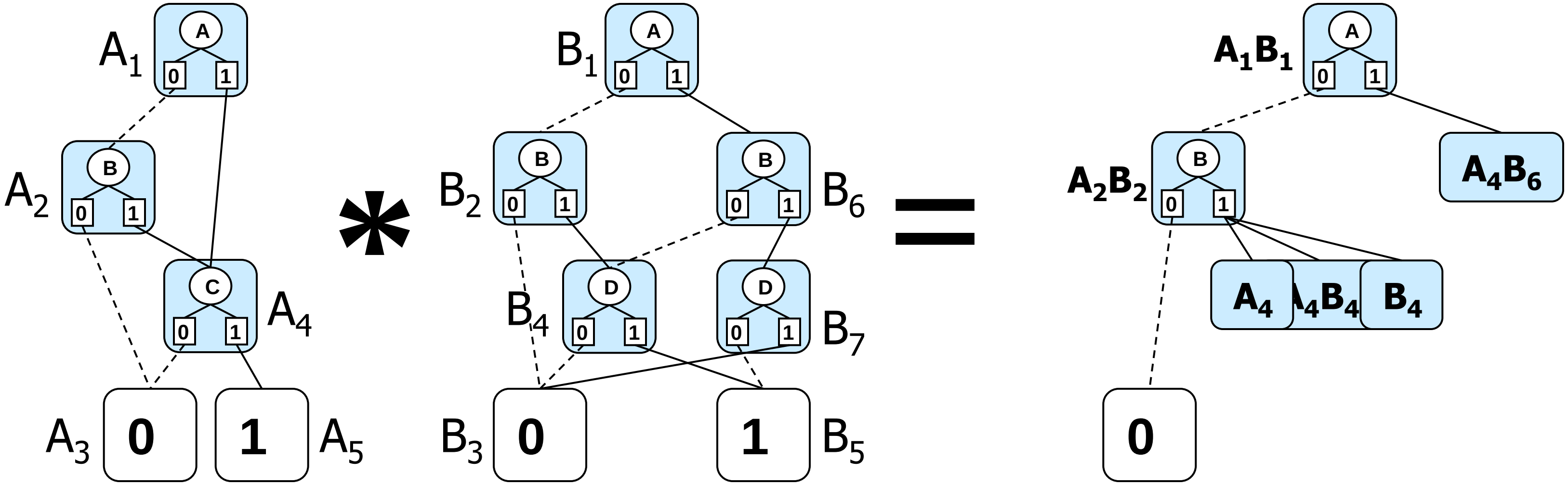
Apply Operator



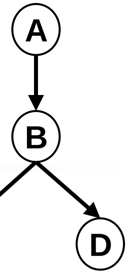


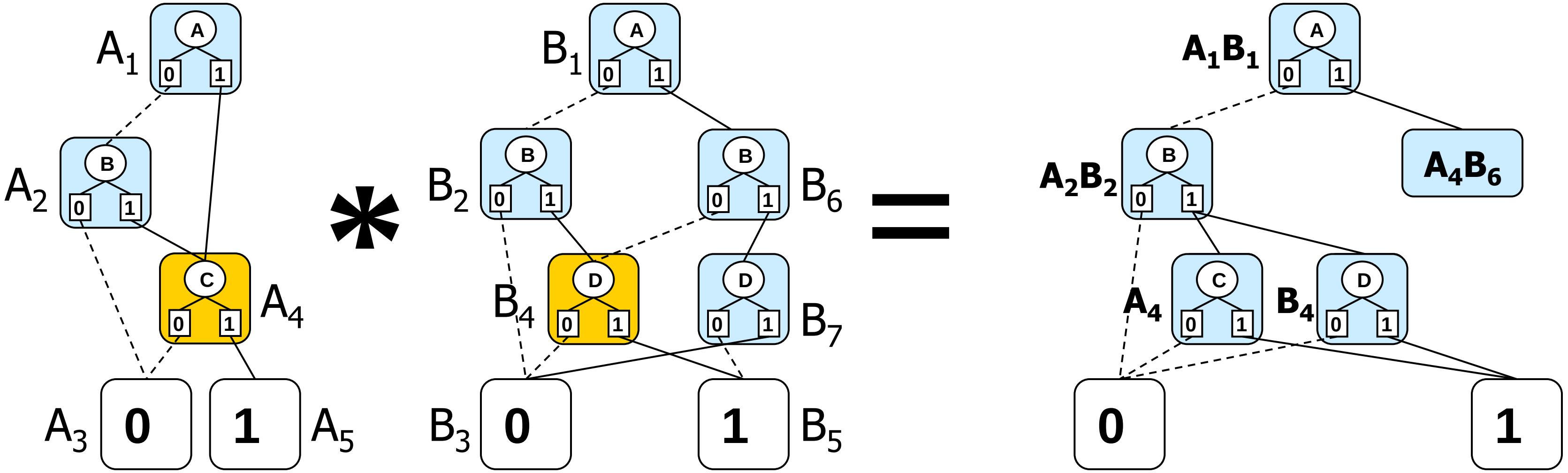
Apply Operator



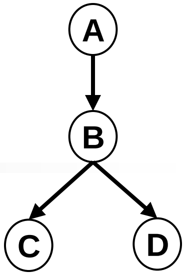


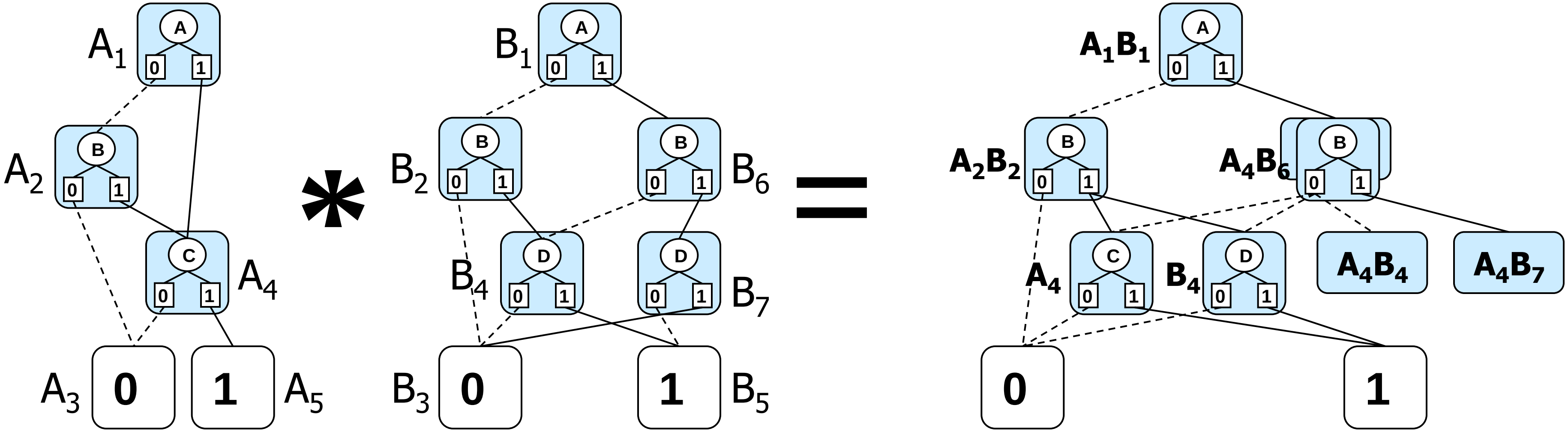
Apply Operator



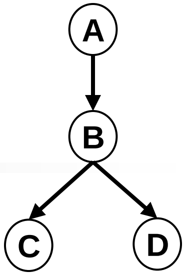


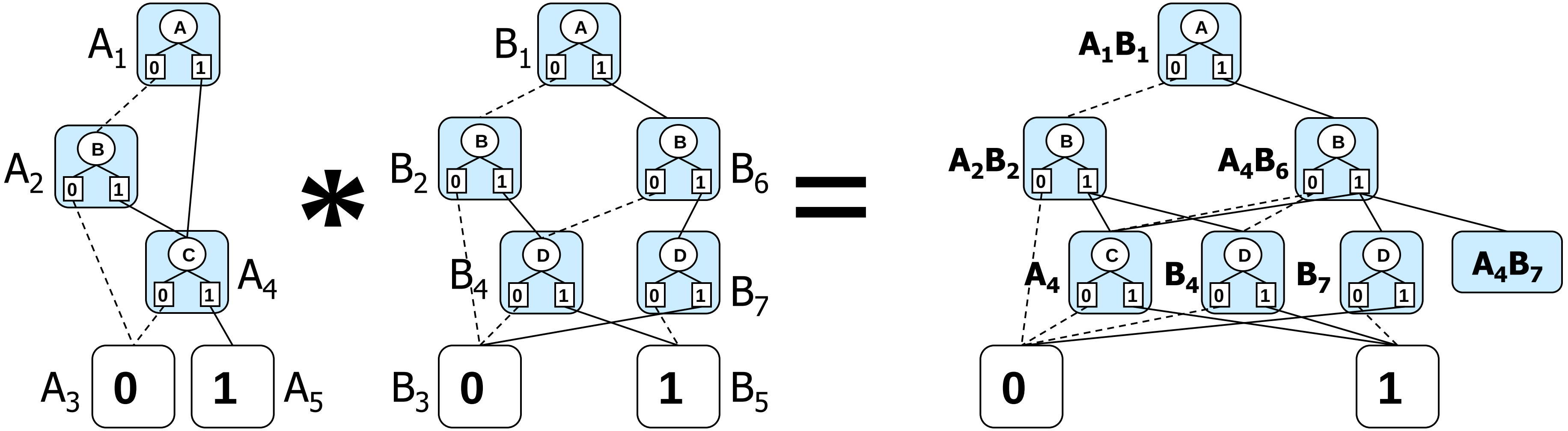
Apply Operator





Apply Operator





And/Or Multi-Valued Decision Diagrams

- AOMDDs are:
  - AND/OR search graphs
  - canonical representations, given a pseudo tree
  - Defined by two rules:
    - All isomorphic subgraphs are merged
    - There are no redundant (meta) nodes

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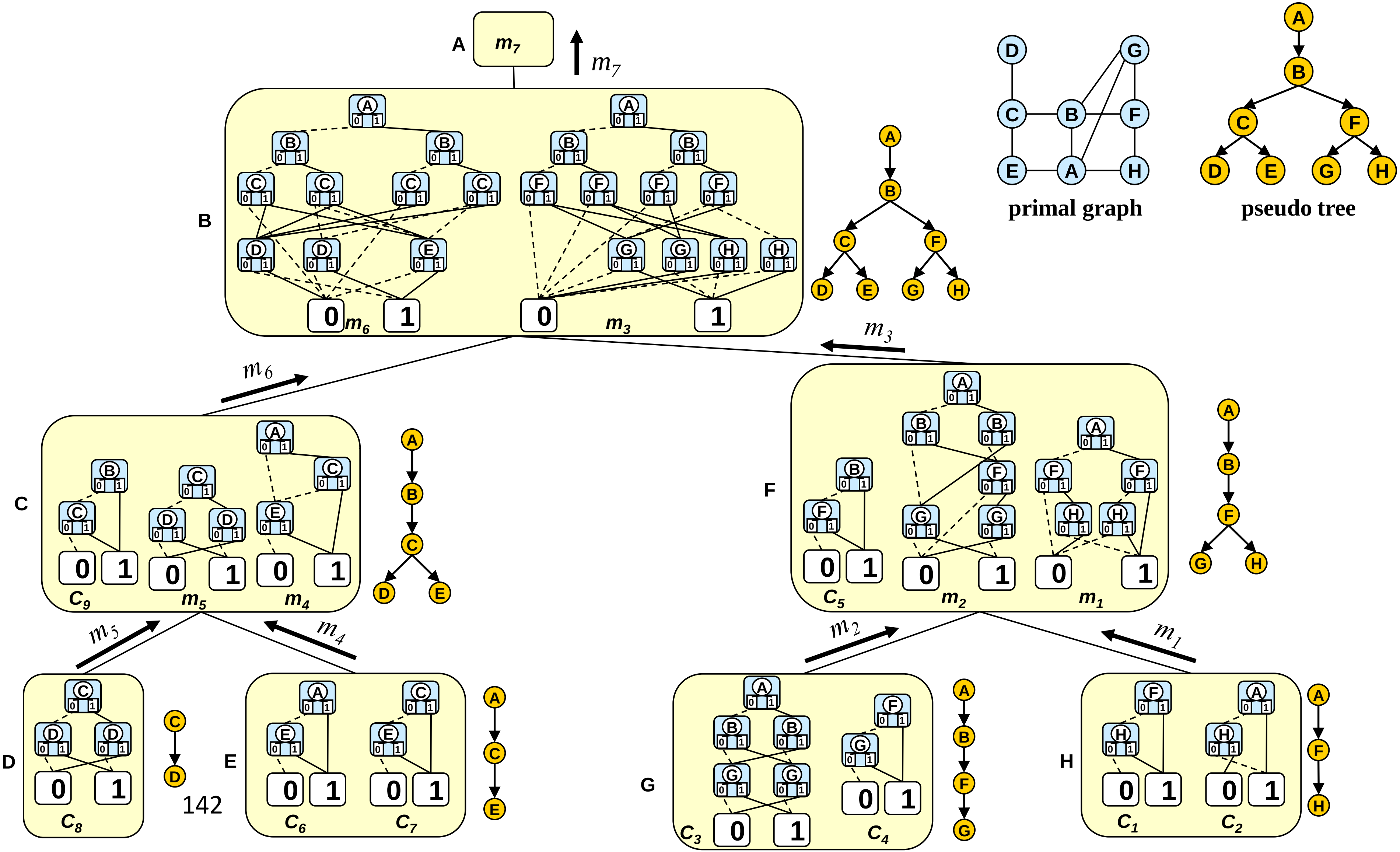
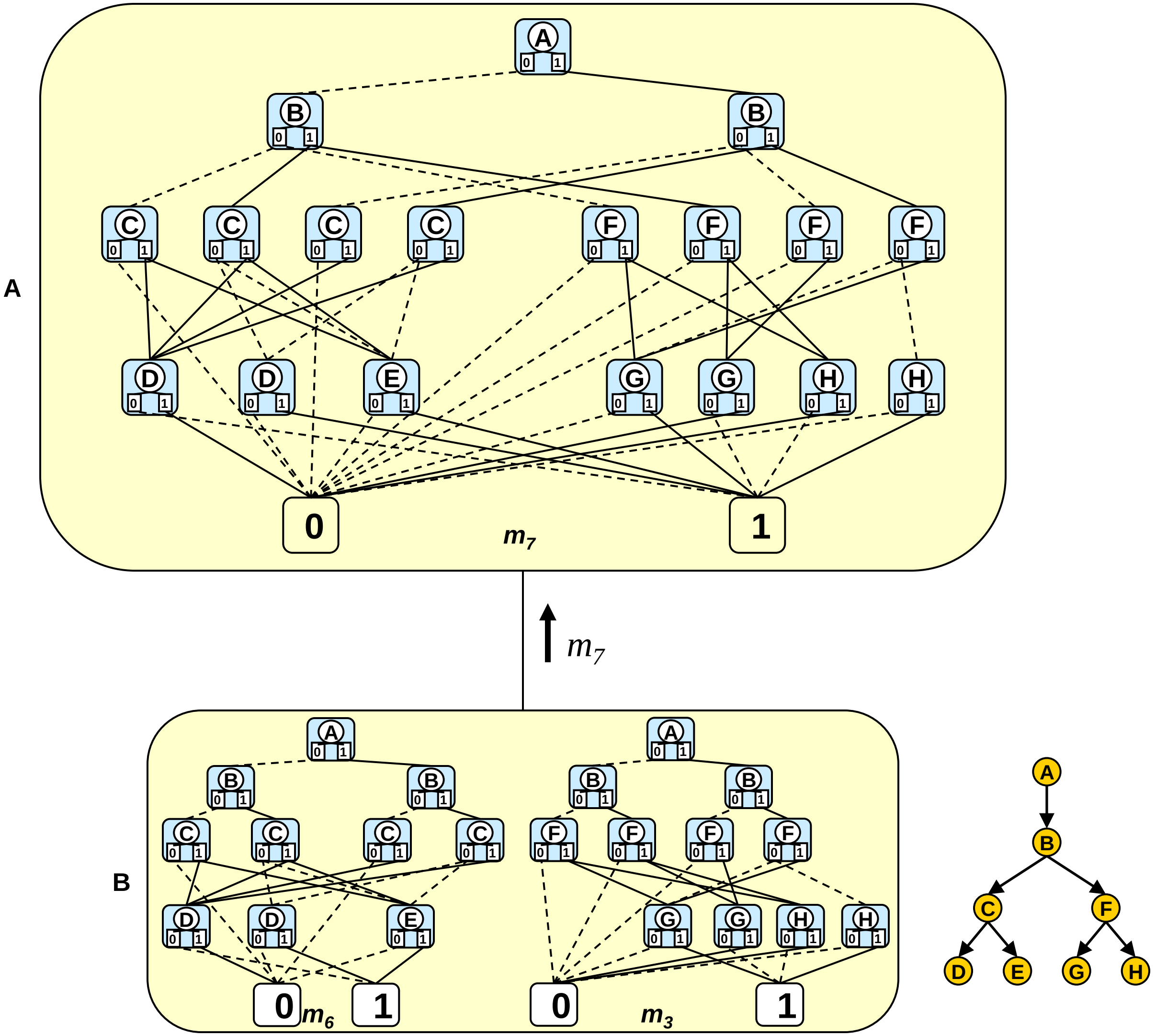
Example:

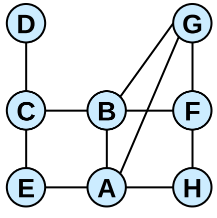
$(f \lor h) \land (a \lor !h) \land (a\#b\#g) \land (f \lor g) \land$
$(b \lor f) \land (a \lor e) \land (c \lor e) \land (c\#d) \land (b \lor c)$



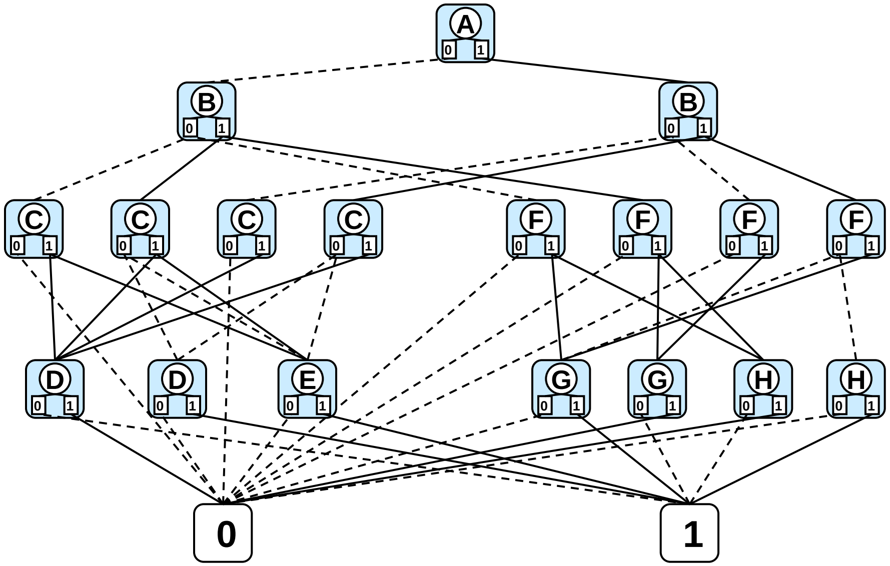
Example (continued)

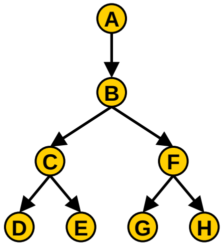


AOBDD vs. OBDD



primal graph

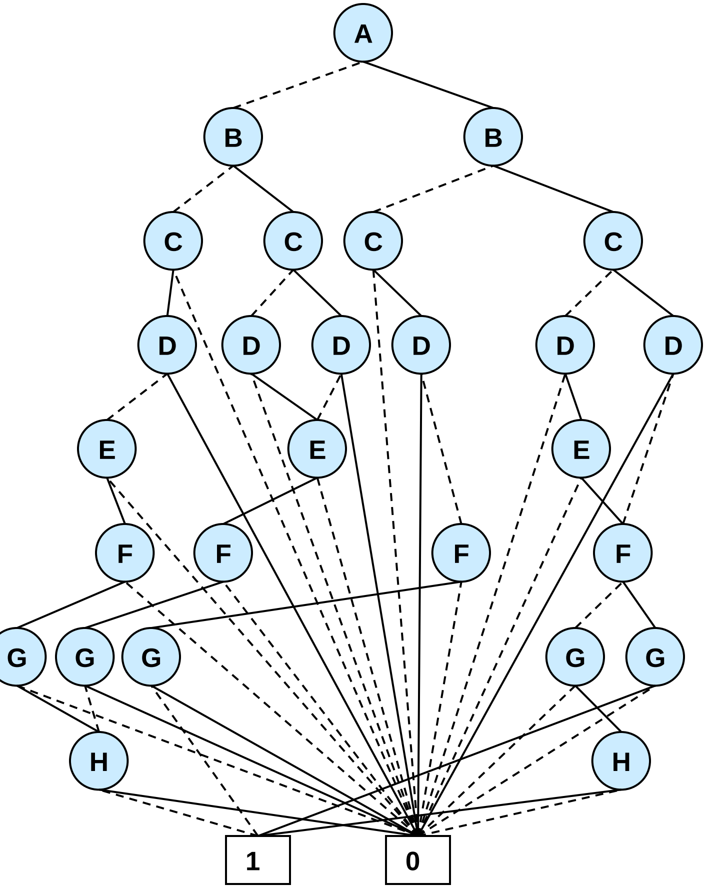


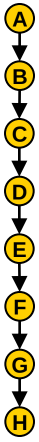


AOBDD

18 nonterminals

47 arcs





OBDD

27 nonterminals

54 arcs

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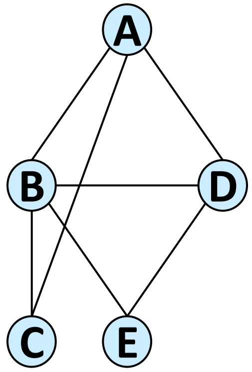
Constraint Optimization - AND/OR Tree

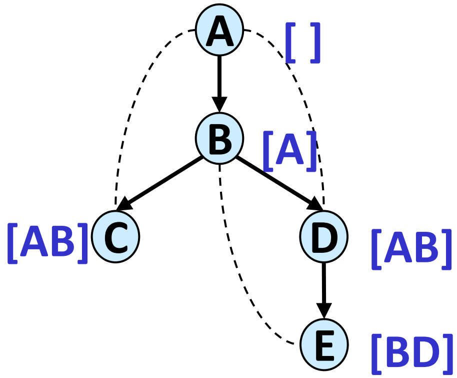
A	B	C	$f_1(ABC)$
0	0	0	∞
0	0	1	∞
0	1	0	∞
0	1	1	2
1	0	0	∞
1	0	1	2
1	1	0	∞
1	1	1	2

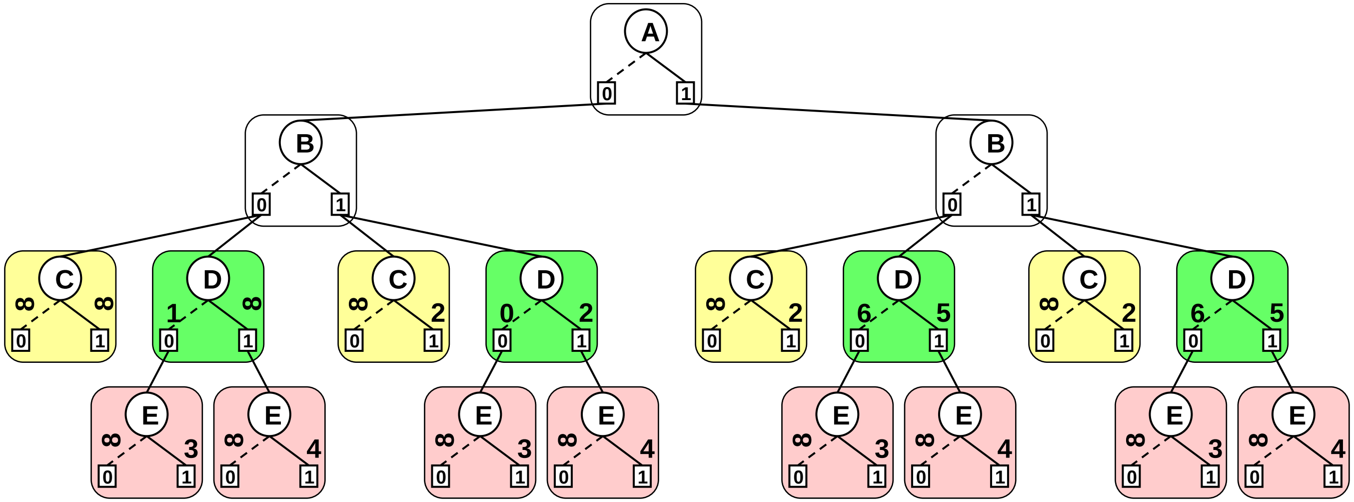
A	B	D	$f_2(ABD)$
0	0	0	1
0	0	1	∞
0	1	0	0
0	1	1	2
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

B	D	E	$f_3(BDE)$
0	0	0	∞
0	0	1	3
0	1	0	∞
0	1	1	4
1	0	0	∞
1	0	1	3
1	1	0	∞
1	1	1	4

$f_1(ABC)$
$f_2(ABD)$
$f_3(BDE)$







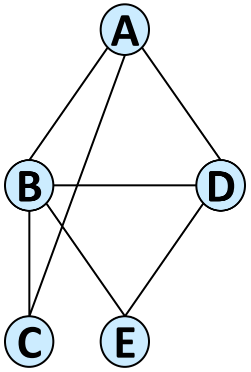
AND/OR Context Minimal Graph

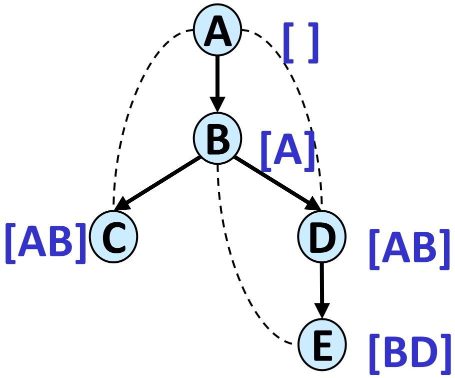
A	B	C	f_1(ABC)
0	0	0	∞
0	0	1	∞
0	1	0	∞
0	1	1	2
1	0	0	∞
1	0	1	2
1	1	0	∞
1	1	1	2

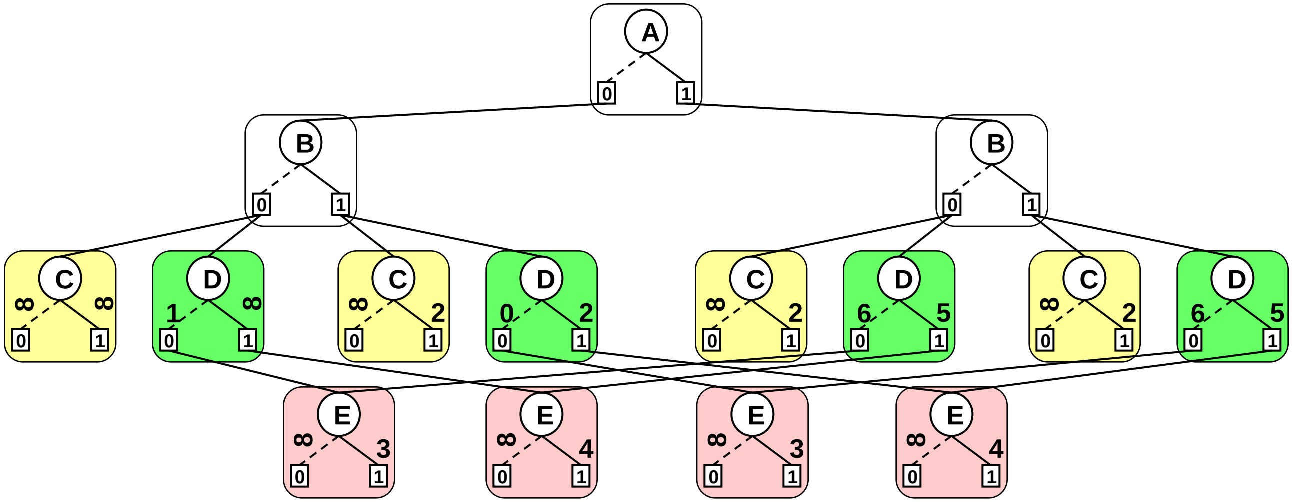
A	B	D	f_2(ABD)
0	0	0	1
0	0	1	∞
0	1	0	0
0	1	1	2
1	0	0	6
1	0	1	5
1	1	0	6
1	1	1	5

B	D	E	f_3(BDE)
0	0	0	∞
0	0	1	3
0	1	0	∞
0	1	1	4
1	0	0	∞
1	0	1	3
1	1	0	∞
1	1	1	4

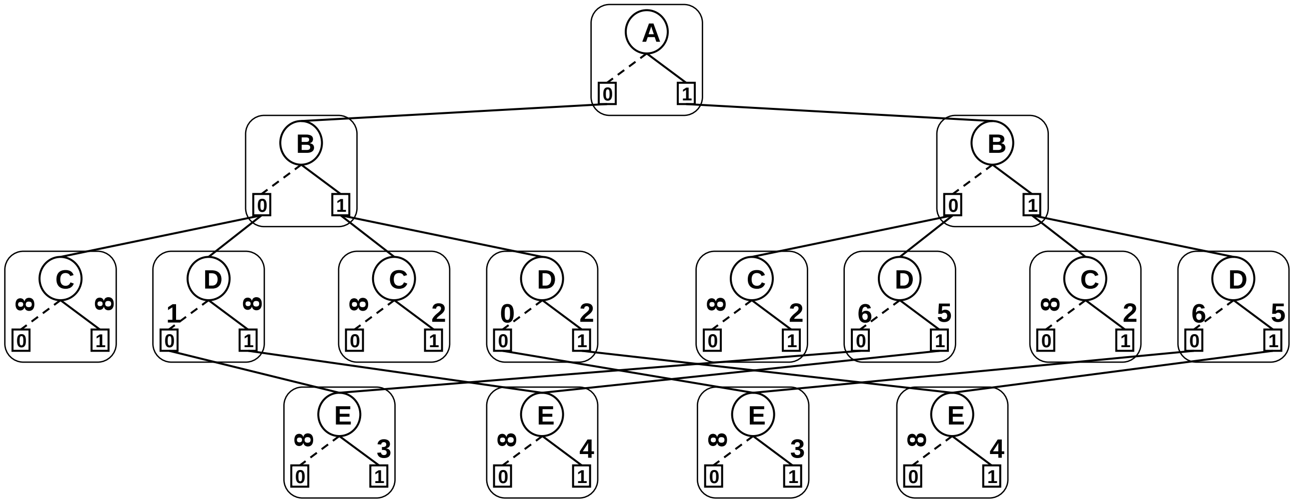
f_1(ABC)
f_2(ABD)
f_3(BDE)

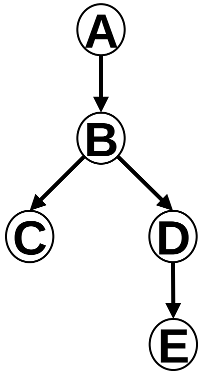


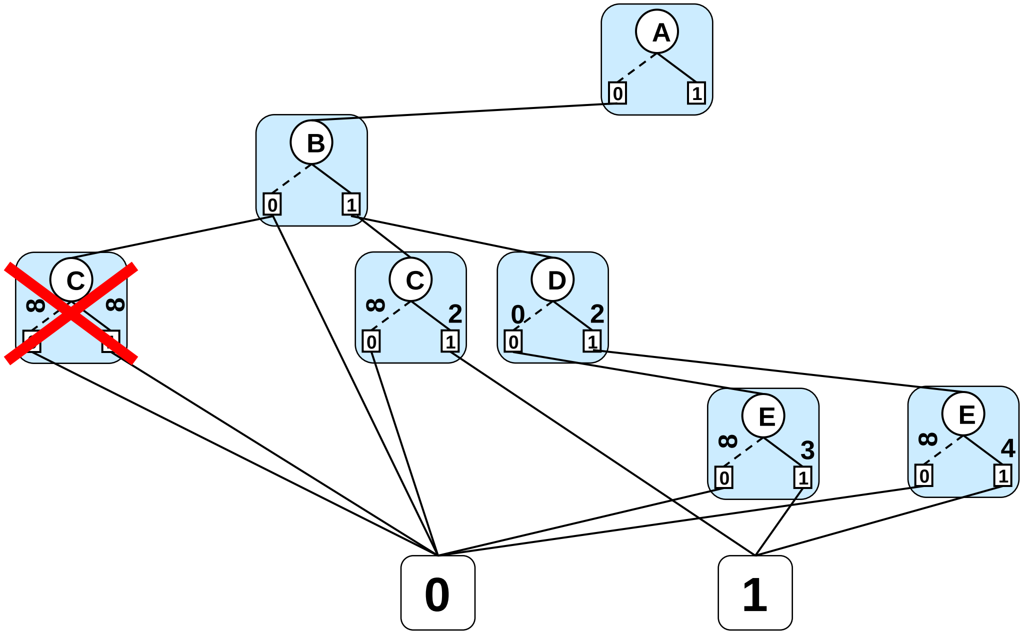




AOMDD – Compilation by Search

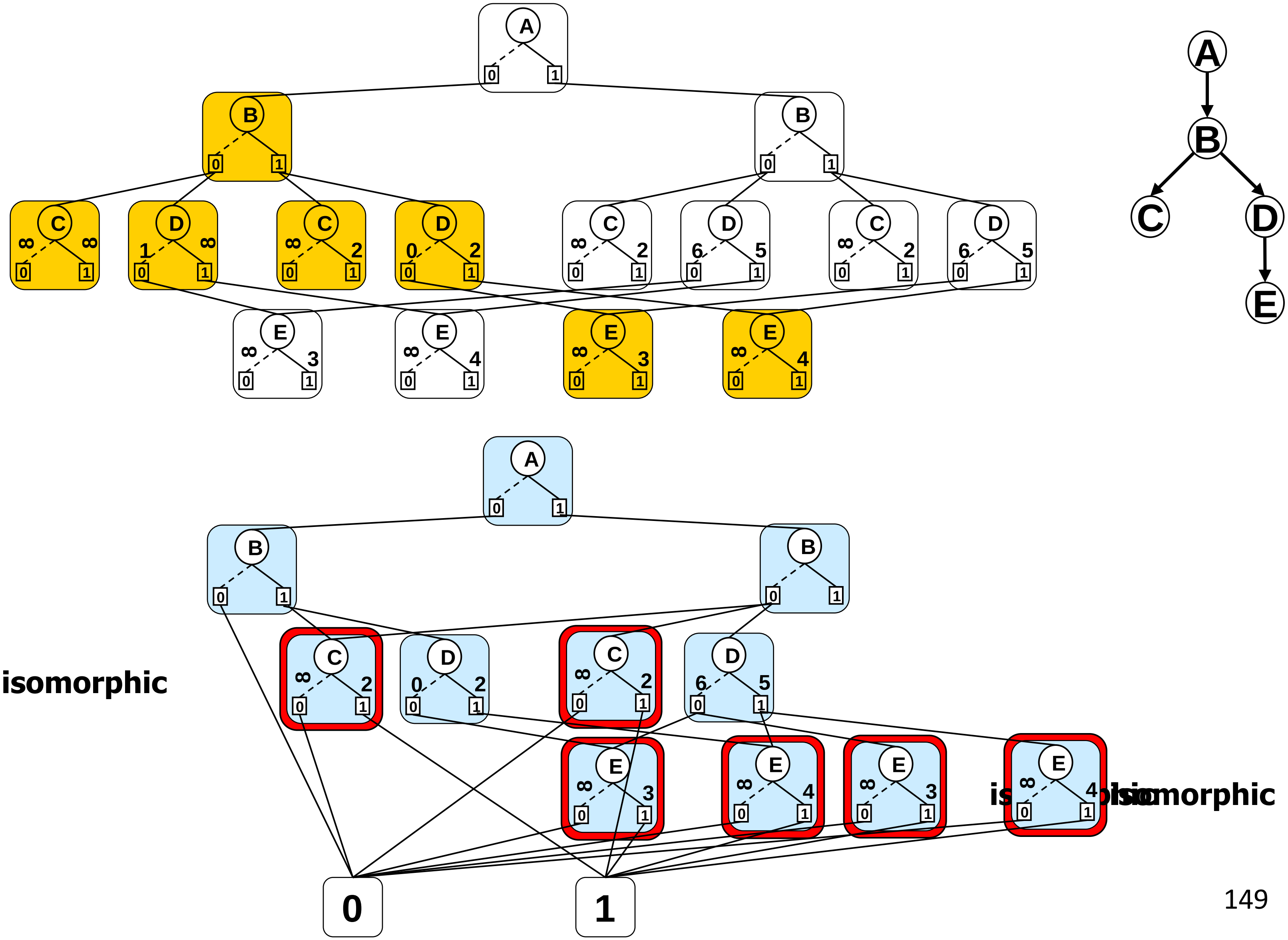




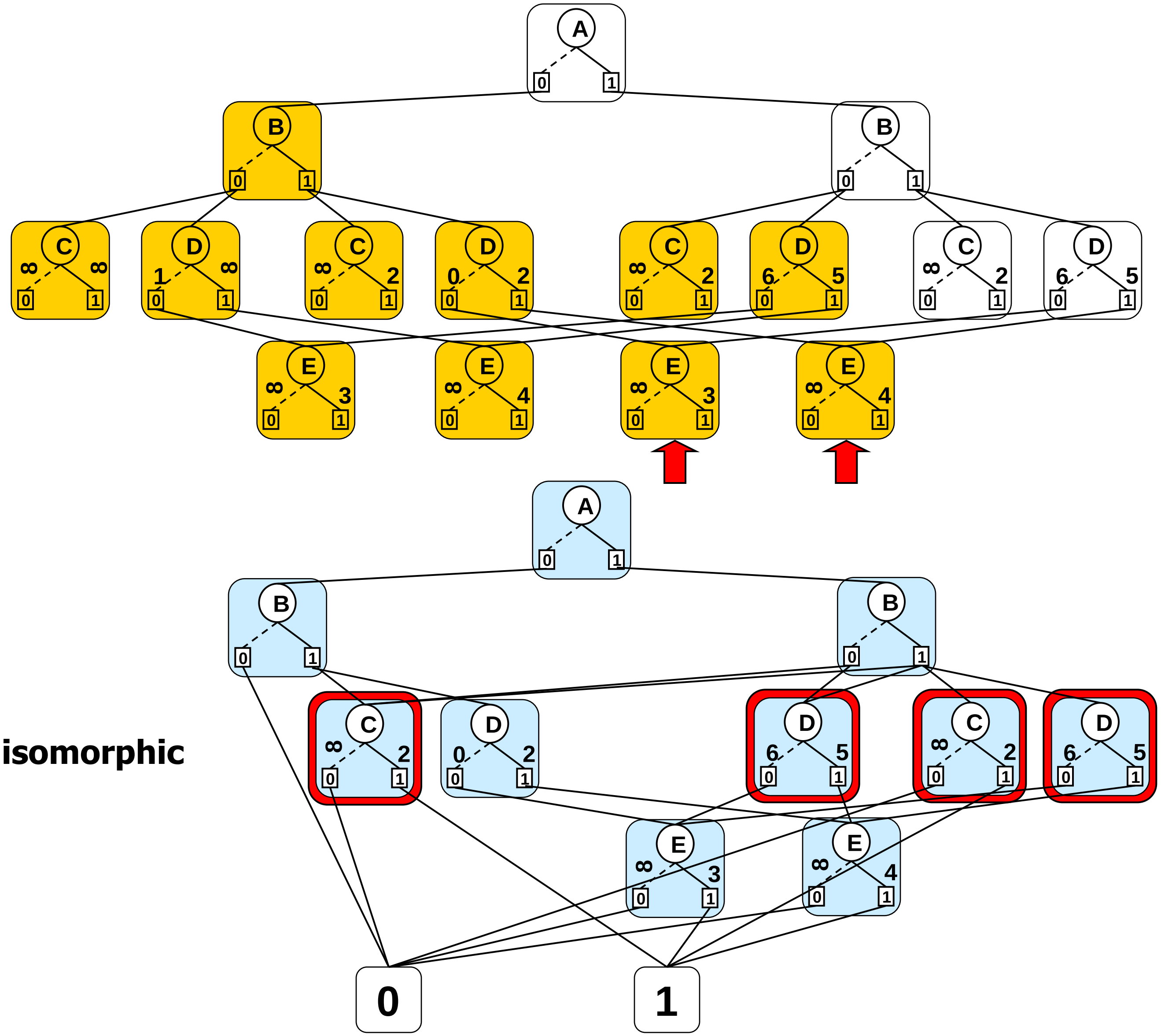


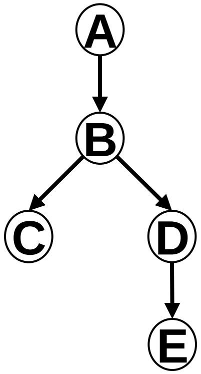
redundant

AOMDD – Compilation by Search

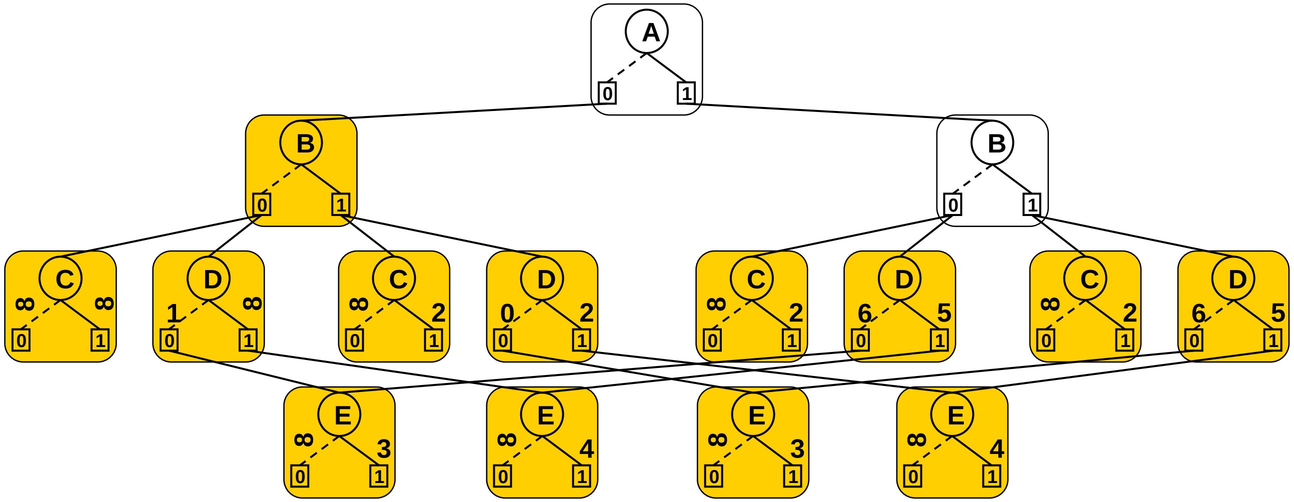


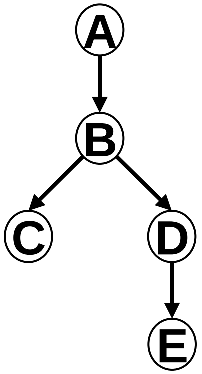
AOMDD – Compilation by Search

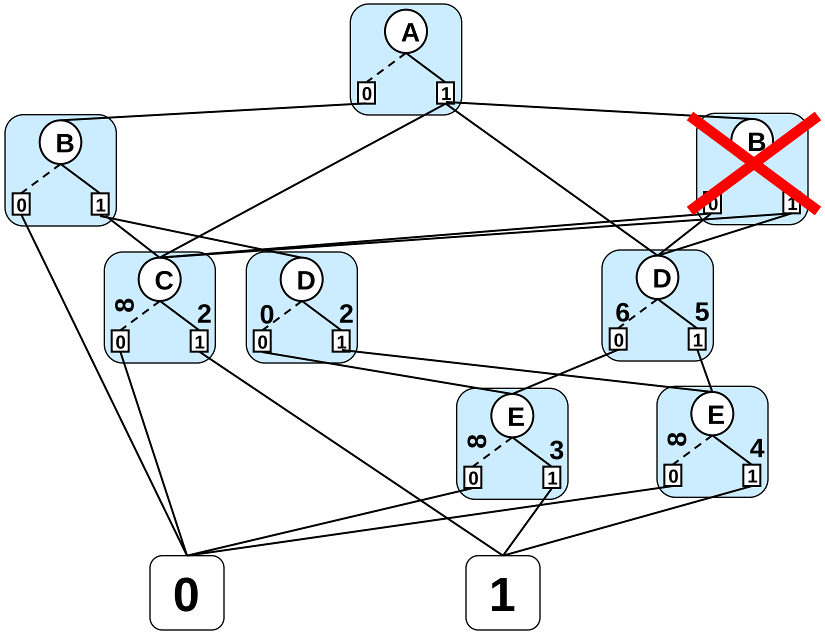




AOMDD – Compilation by Search

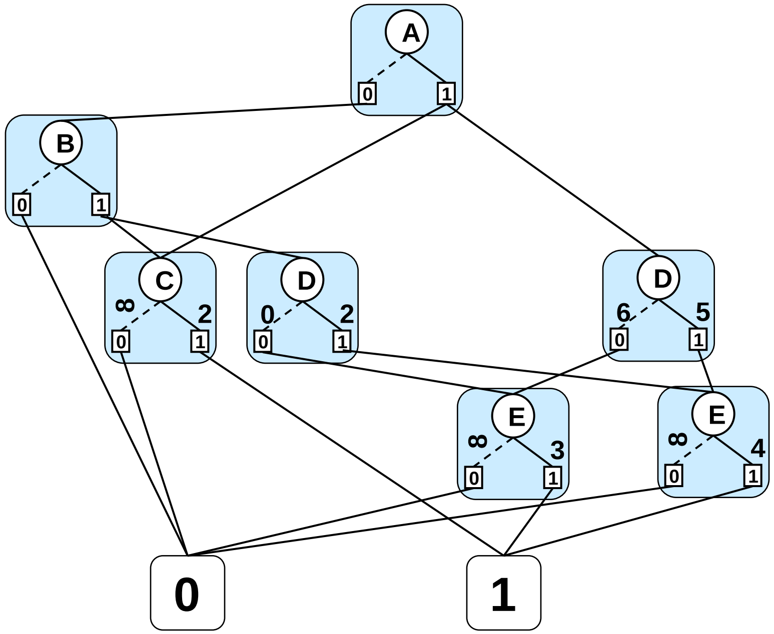




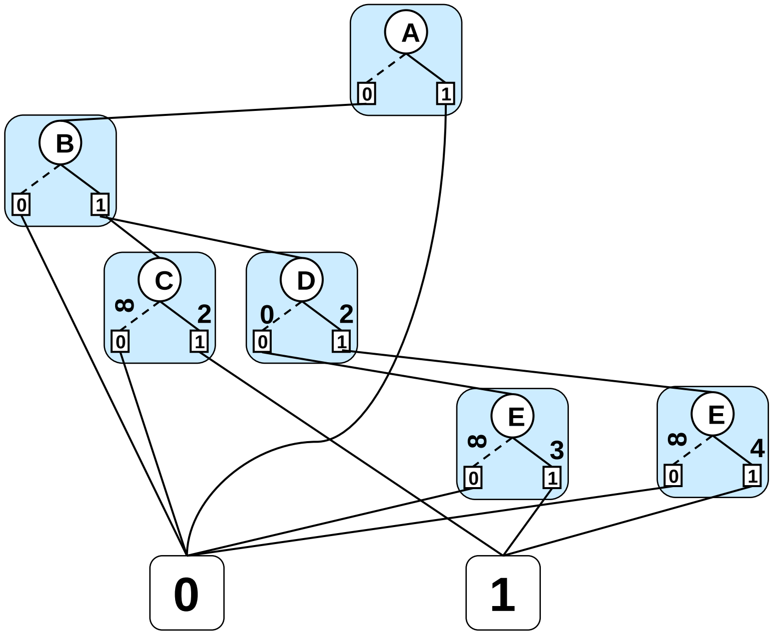


redundant

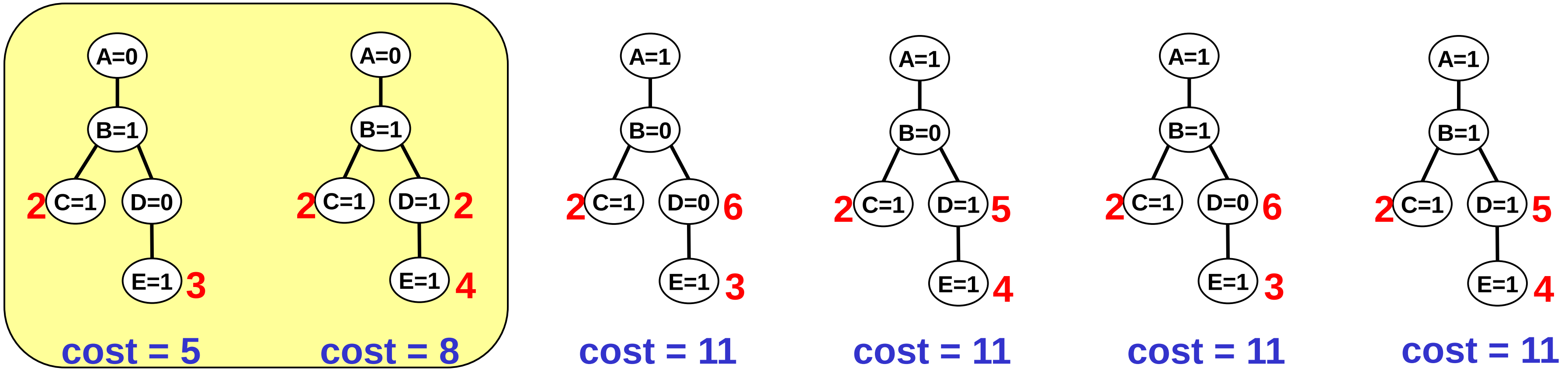
AOMDD for Constraint Optimization



AOMDD for all solutions



AOMDD for two best solutions



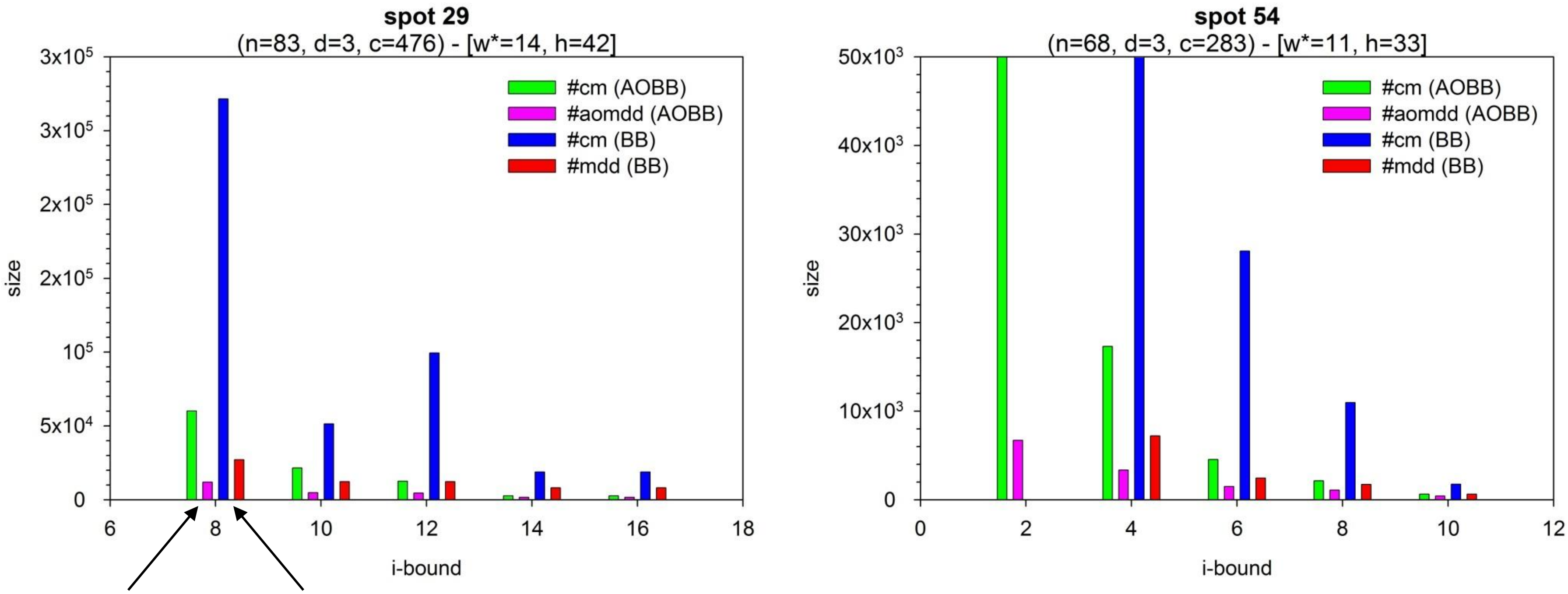
Complexity of Compilation

- The size of the AOMDD is $O(n\ k^{w^*})$

- The compilation time is also bounded by $O(n\ k^{w^*})$

k = domain size
n = number of variables
w^*= treewidth

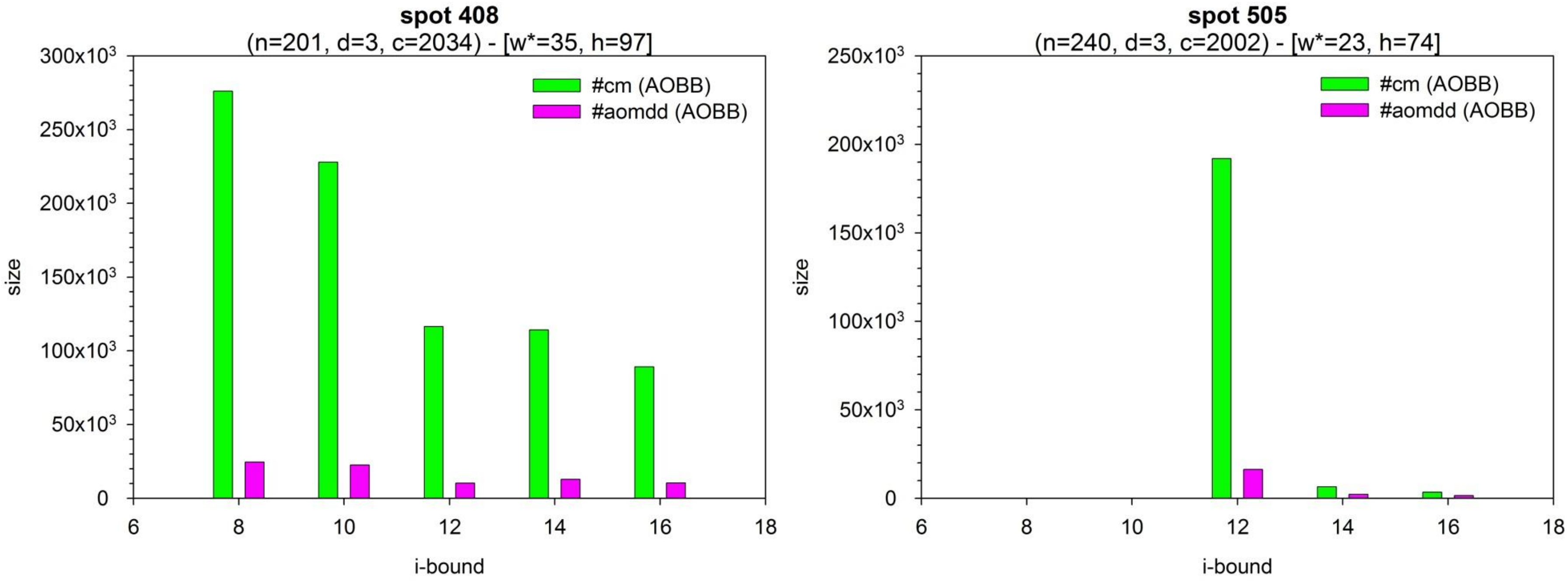
SPOT5 Benchmarks (WCSP)



AND/OR **OR**

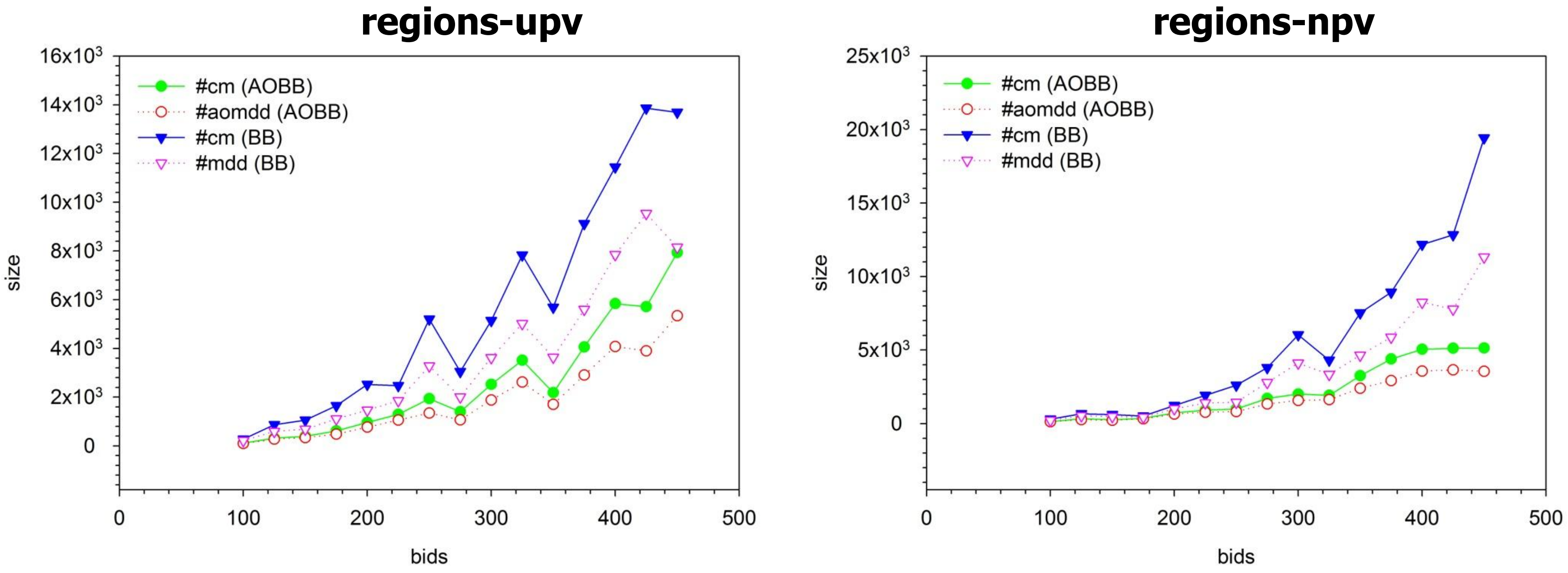
Results for Earth Observing Satellites benchmarks

SPOT5 Benchmarks (WCSP)



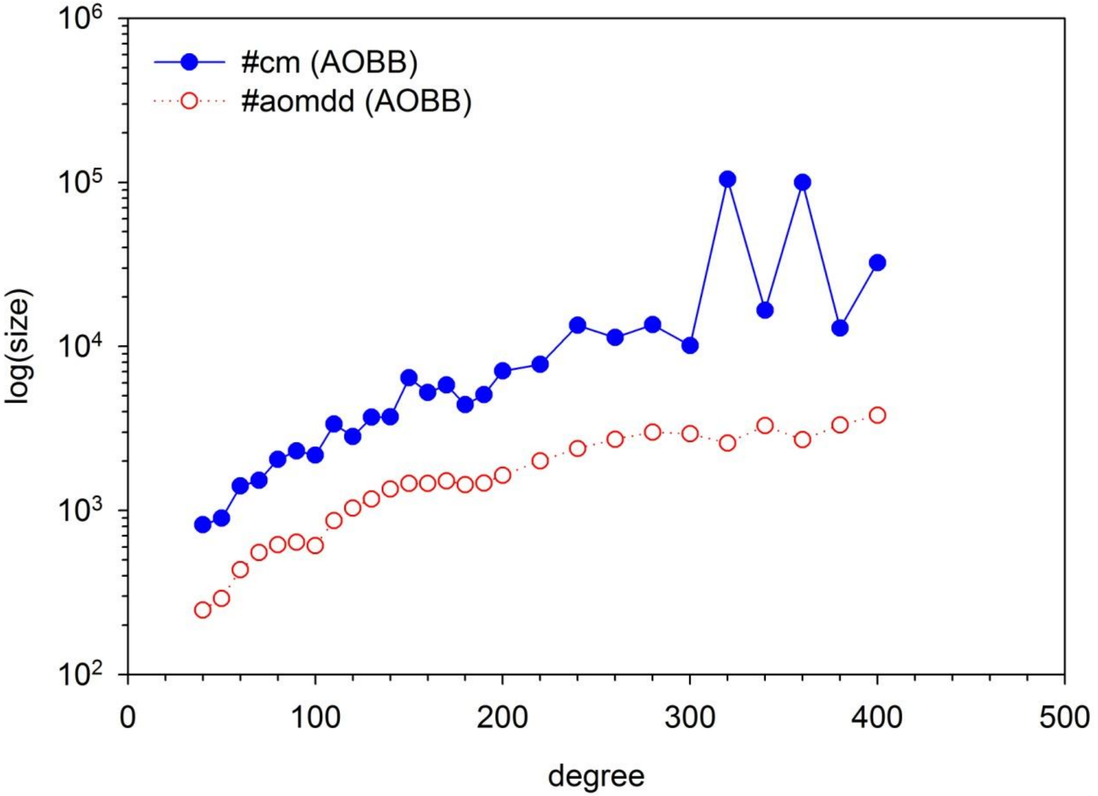
Results for Earth Observing Satellites benchmarks

Combinatorial Auctions (ILP)



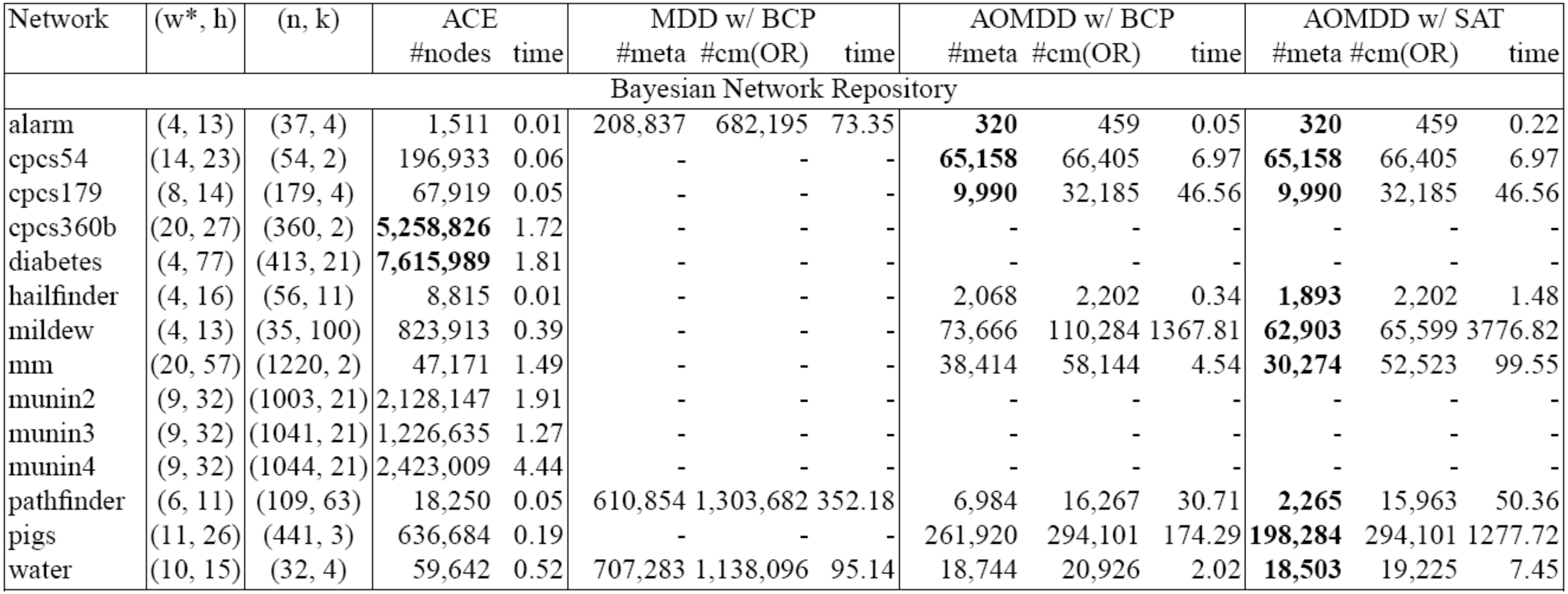
Results for combinatorial auctions from the CATS 2.0 distribution

MAX-SAT Instances (ILP)



Results for dubois MAX-SAT instances

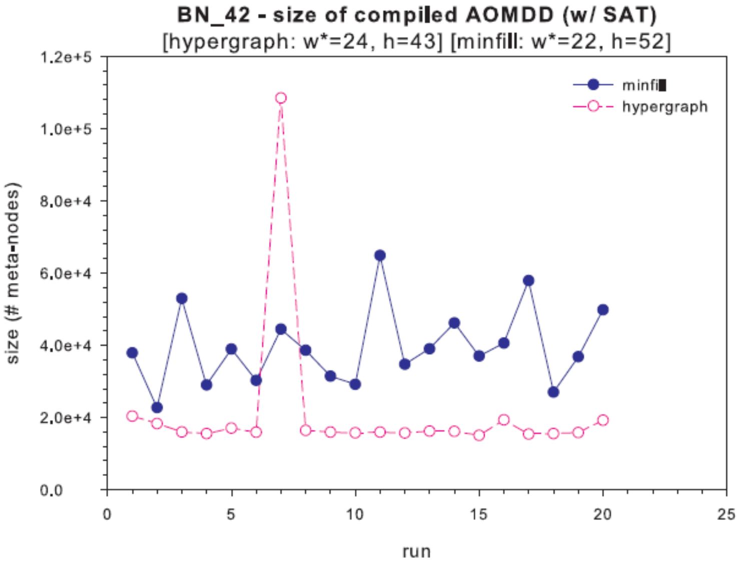
Bayesian Networks Repository

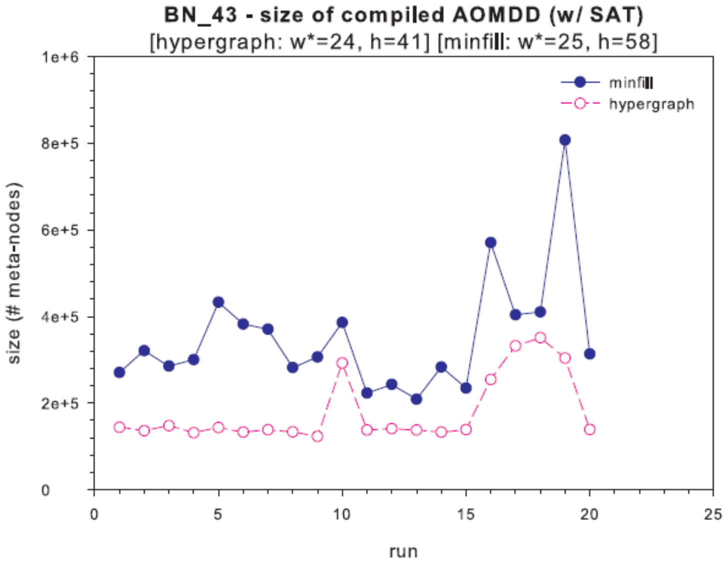


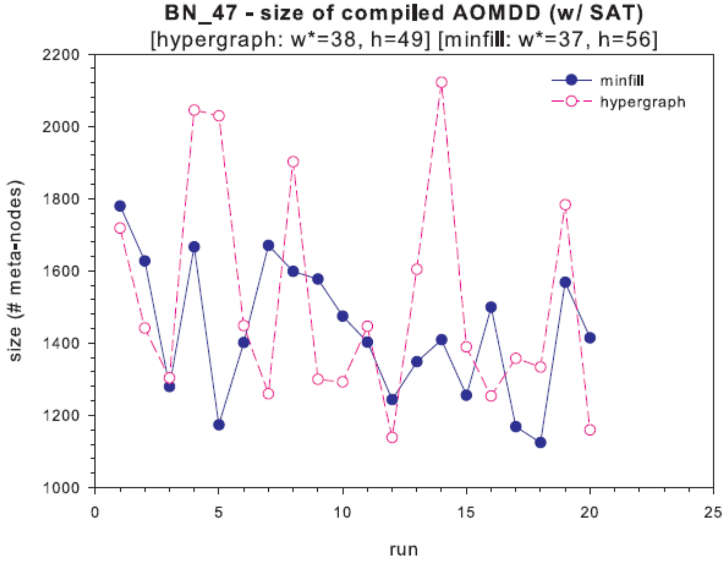
Network	(w*, h)	(n, k)	ACE		MDD w/ BCP			AOMDD w/ BCP			AOMDD w/ SAT		
			#nodes	time	#meta	#cm(OR)	time	#meta	#cm(OR)	time	#meta	#cm(OR)	time
Bayesian Network Repository													
alarm	(4, 13)	(37, 4)	1,511	0.01	208,837	682,195	73.35	**320**	459	0.05	**320**	459	0.22
cpcs54	(14, 23)	(54, 2)	196,933	0.06	-	-	-	**65,158**	66,405	6.97	**65,158**	66,405	6.97
cpcs179	(8, 14)	(179, 4)	67,919	0.05	-	-	-	**9,990**	32,185	46.56	**9,990**	32,185	46.56
cpcs360b	(20, 27)	(360, 2)	**5,258,826**	1.72	-	-	-	-	-	-	-	-	-
diabetes	(4, 77)	(413, 21)	**7,615,989**	1.81	-	-	-	-	-	-	-	-	-
hailfinder	(4, 16)	(56, 11)	8,815	0.01	-	-	-	2,068	2,202	0.34	**1,893**	2,202	1.48
mildew	(4, 13)	(35, 100)	823,913	0.39	-	-	-	73,666	110,284	1367.81	**62,903**	65,599	3776.82
mm	(20, 57)	(1220, 2)	47,171	1.49	-	-	-	38,414	58,144	4.54	**30,274**	52,523	99.55
munin2	(9, 32)	(1003, 21)	2,128,147	1.91	-	-	-	-	-	-	-	-	-
munin3	(9, 32)	(1041, 21)	1,226,635	1.27	-	-	-	-	-	-	-	-	-
munin4	(9, 32)	(1044, 21)	2,423,009	4.44	-	-	-	-	-	-	-	-	-
pathfinder	(6, 11)	(109, 63)	18,250	0.05	610,854	1,303,682	352.18	6,984	16,267	30.71	**2,265**	15,963	50.36
pigs	(11, 26)	(441, 3)	636,684	0.19	-	-	-	261,920	294,101	174.29	**198,284**	294,101	1277.72
water	(10, 15)	(32, 4)	59,642	0.52	707,283	1,138,096	95.14	18,744	20,926	2.02	**18,503**	19,225	7.45

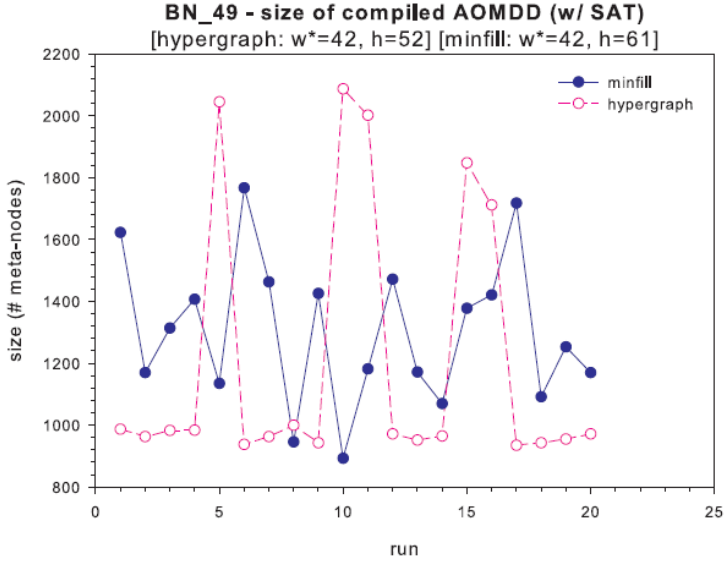
Size (number of nodes), time (seconds)

Effect of Variable Ordering


BN_42 - size of compiled AOMDD (w/ SAT)
[hypergraph: w*=24, h=43] [minfill: w*=22, h=52]


BN_43 - size of compiled AOMDD (w/ SAT)
[hypergraph: w*=24, h=41] [minfill: w*=25, h=58]


BN_47 - size of compiled AOMDD (w/ SAT)
[hypergraph: w*=38, h=49] [minfill: w*=37, h=56]


BN_49 - size of compiled AOMDD (w/ SAT)
[hypergraph: w*=42, h=52] [minfill: w*=42, h=61]

Outline

- Introduction
- Inference
- Search
- Compilation: AND/OR Decision Diagrams
- Software

Software & Competitions

- **How to use the software**
  - http://graphmod.ics.uci.edu/group/Software
  - http://mulcyber.toulouse.inra.fr/projects/toulbar2

- **Reports on competitions**
  - UAI-2006, 2008, 2010 Competitions
    - PE, MAR, MPE tasks
  - CP-2006 Competition
    - WCSP task

Toulbar2 and aolib

- toulbar2

  http://mulcyber.toulouse.inra.fr/gf/project/toulbar2

  (Open source WCSP, MPE solver in C++)

- aolib

  http://graphmod.ics.uci.edu/group/Software

  (WCSP, MPE, ILP solver in C++, inference and counting)

- Large set of benchmarks

  http://carlit.toulouse.inra.fr/cgi-bin/awki.cgi/SoftCSP

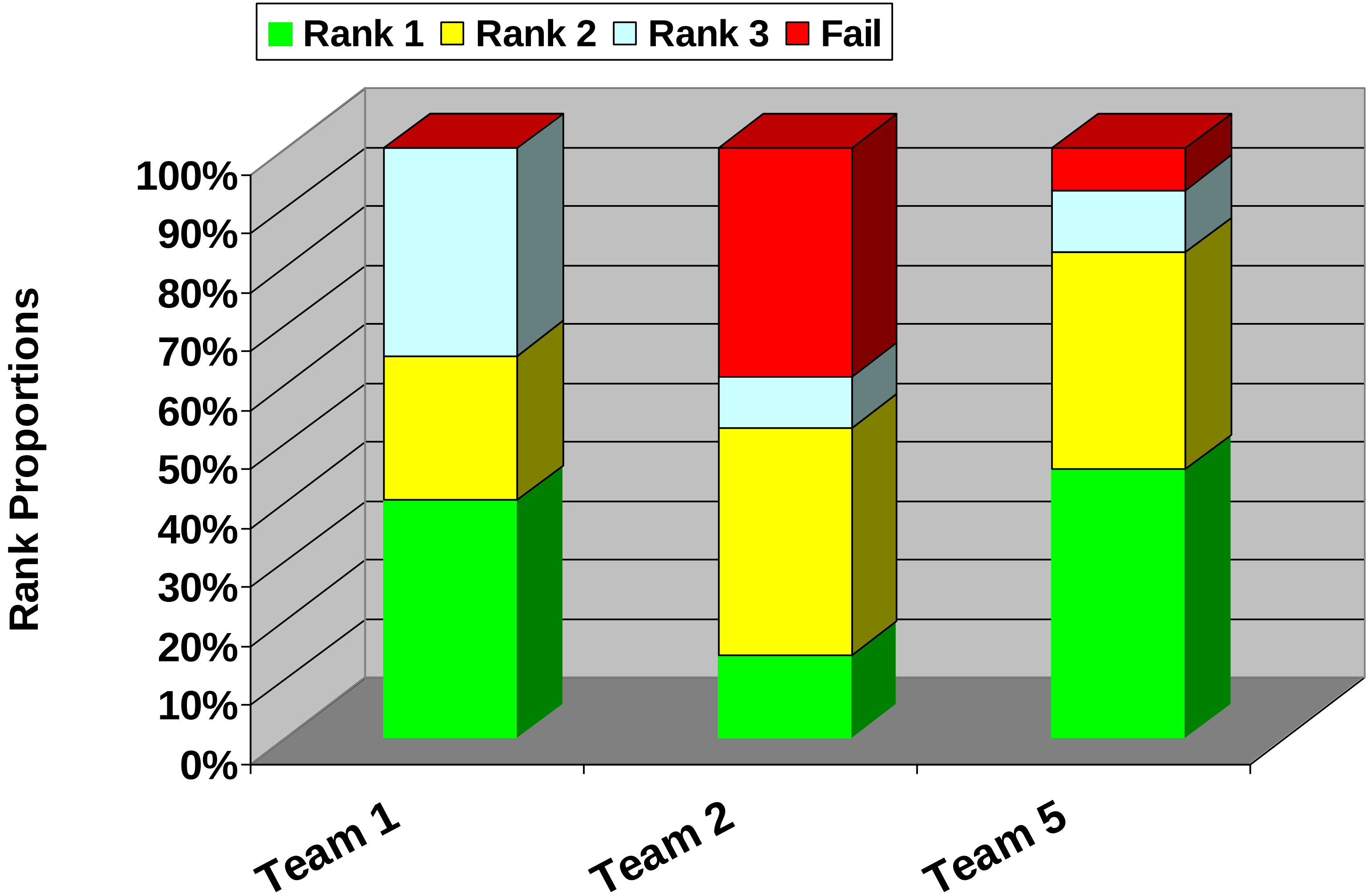
  http://graphmod.ics.uci.edu/group/Repository

UAI-2006 Competition

- **Team 1 (UCLA)**
    - David Allen, Mark Chavira, Arthur Choi, Adnan Darwiche
- **Team 2 (IET)**
    - Masami Takikawa, Hans Dettmar, Francis Fung, Rick Kissh
- **Team 5 (UCI)**
    - Radu Marinescu, Robert Mateescu, Rina Dechter
    - Used **AOBB-C+SMB(i)** solver for MPE

UAI-2006 Results

Rank Proportions (how often was each team a particular rank, rank 1 is best)

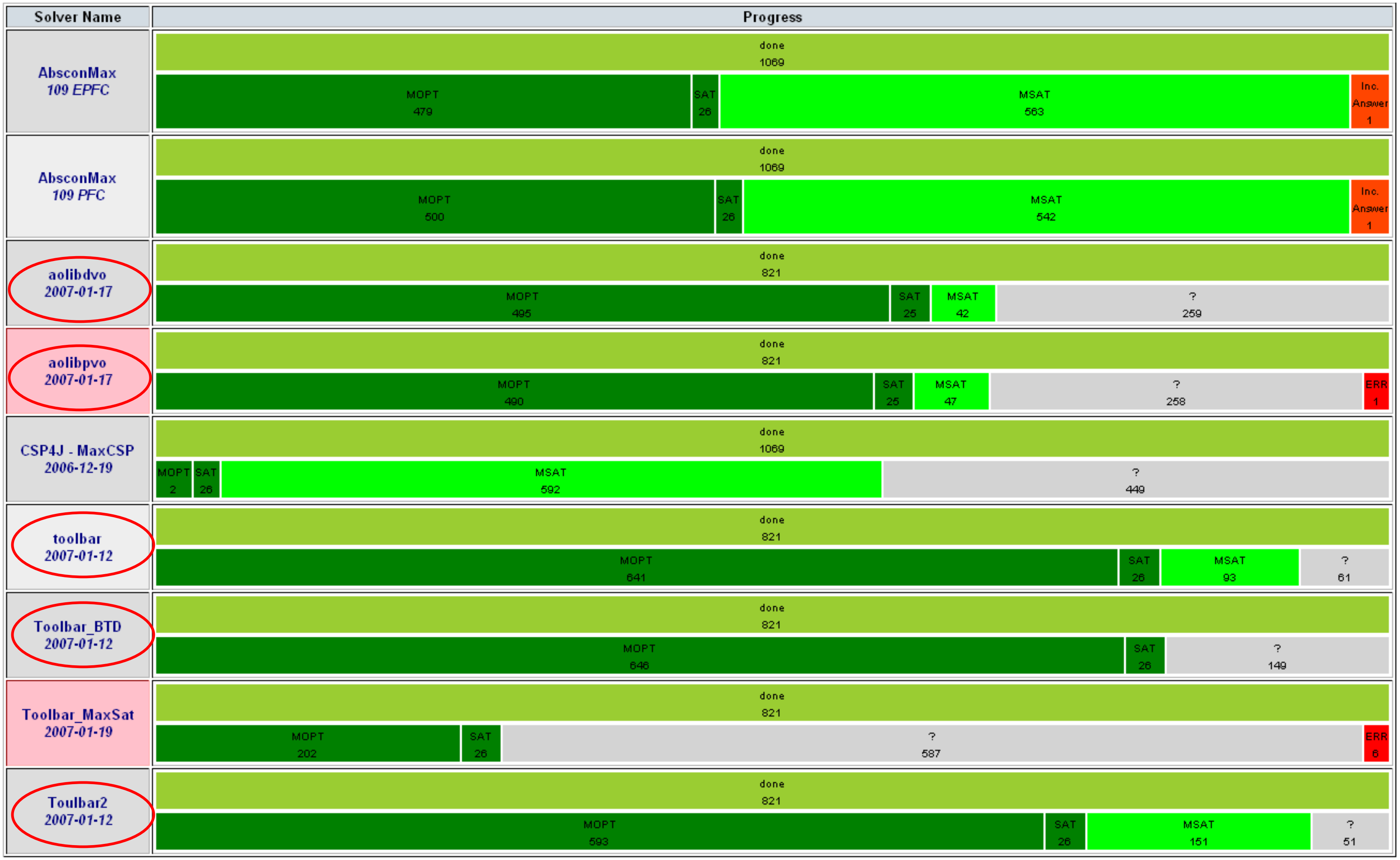


CP-2006 Competition (WCSP)

- Solvers
  - AbsconMax (ie, DFBB+MRDAC)
  - **aolibdvo** (ie, AOBB+EDAC+DVO solver)
  - **aolibpvo** (ie, AOBB+EDAC+PVO solver)
  - CSP4J-MaxCSP
  - **Toolbar** (ie, DFBB+EDAC)
  - **Toolbar_BTD** (ie, BTD+EDAC+VE)
  - **Toolbar_MaxSAT** (ie, DPLL+specific EPT rules)
  - **Toulbar2** (ie, DFBB+EDAC+VE+LDS)

CP-2006 Results

Overall ranking on all selected competition benchmarks



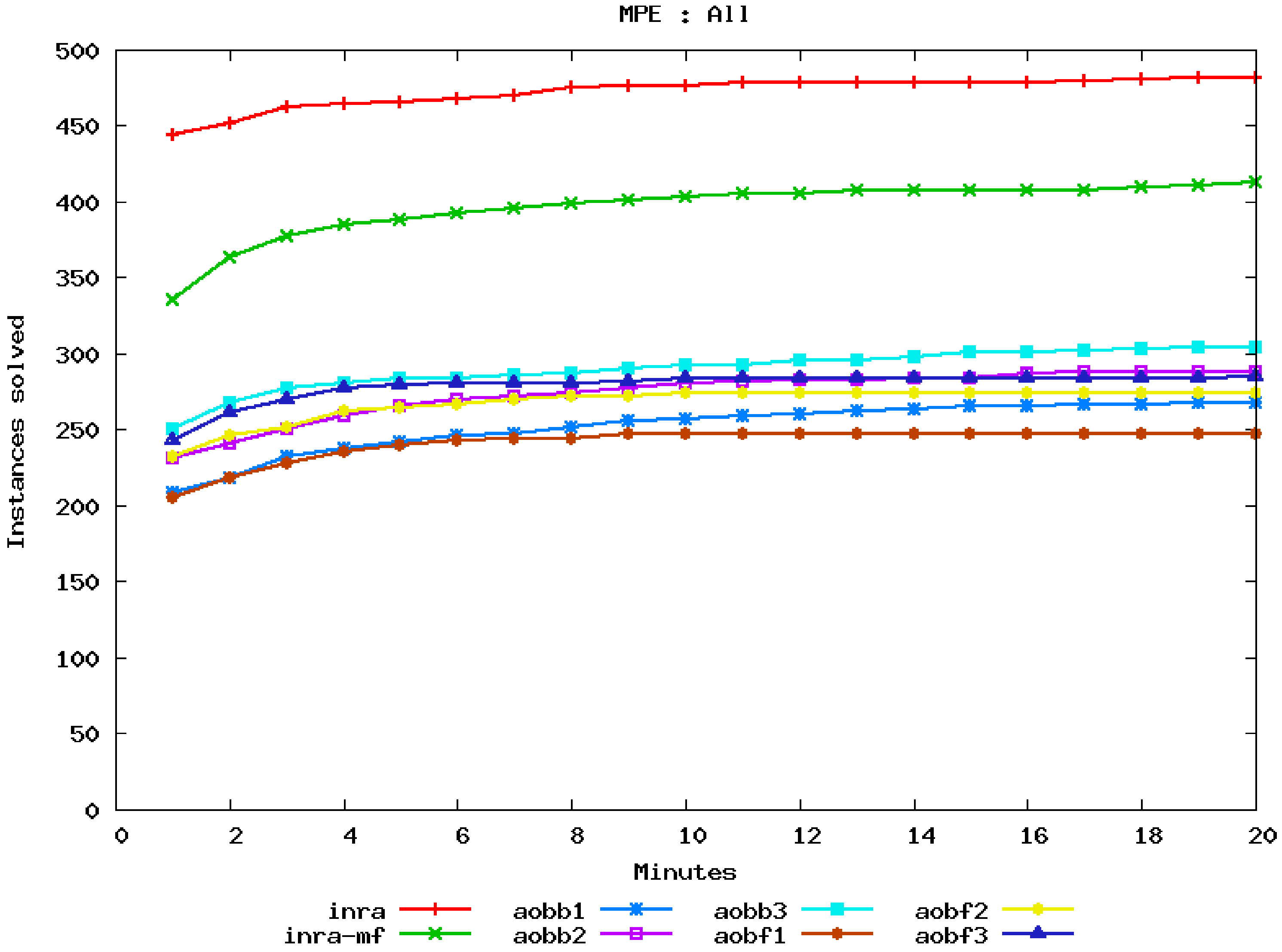
	Solver Name	Progress
	AbsconMax *109 EPFC*	done 1069 MOPT 479 SAT 26 MSAT 563 Inc. Answer 1
	AbsconMax *109 PFC*	done 1069 MOPT 500 SAT 26 MSAT 542 Inc. Answer 1
4	aolibdvo *2007-01-17*	done 821 MOPT 495 SAT 25 MSAT 42 ? 259
5	aolibpvo *2007-01-17*	done 821 MOPT 490 SAT 25 MSAT 47 ? 258 ERR 1
	CSP4J - MaxCSP *2006-12-19*	done 1069 MOPT 2 SAT 26 MSAT 592 ? 449
2	toolbar *2007-01-12*	done 821 MOPT 641 SAT 26 MSAT 93 ? 61
1	Toolbar_BTD *2007-01-12*	done 821 MOPT 646 SAT 26 ? 149
	Toolbar_MaxSat *2007-01-19*	done 821 MOPT 202 SAT 26 ? 587 ERR 6
3	Toulbar2 *2007-01-12*	done 821 MOPT 593 SAT 26 MSAT 151 ? 51

The longest dark green bar wins

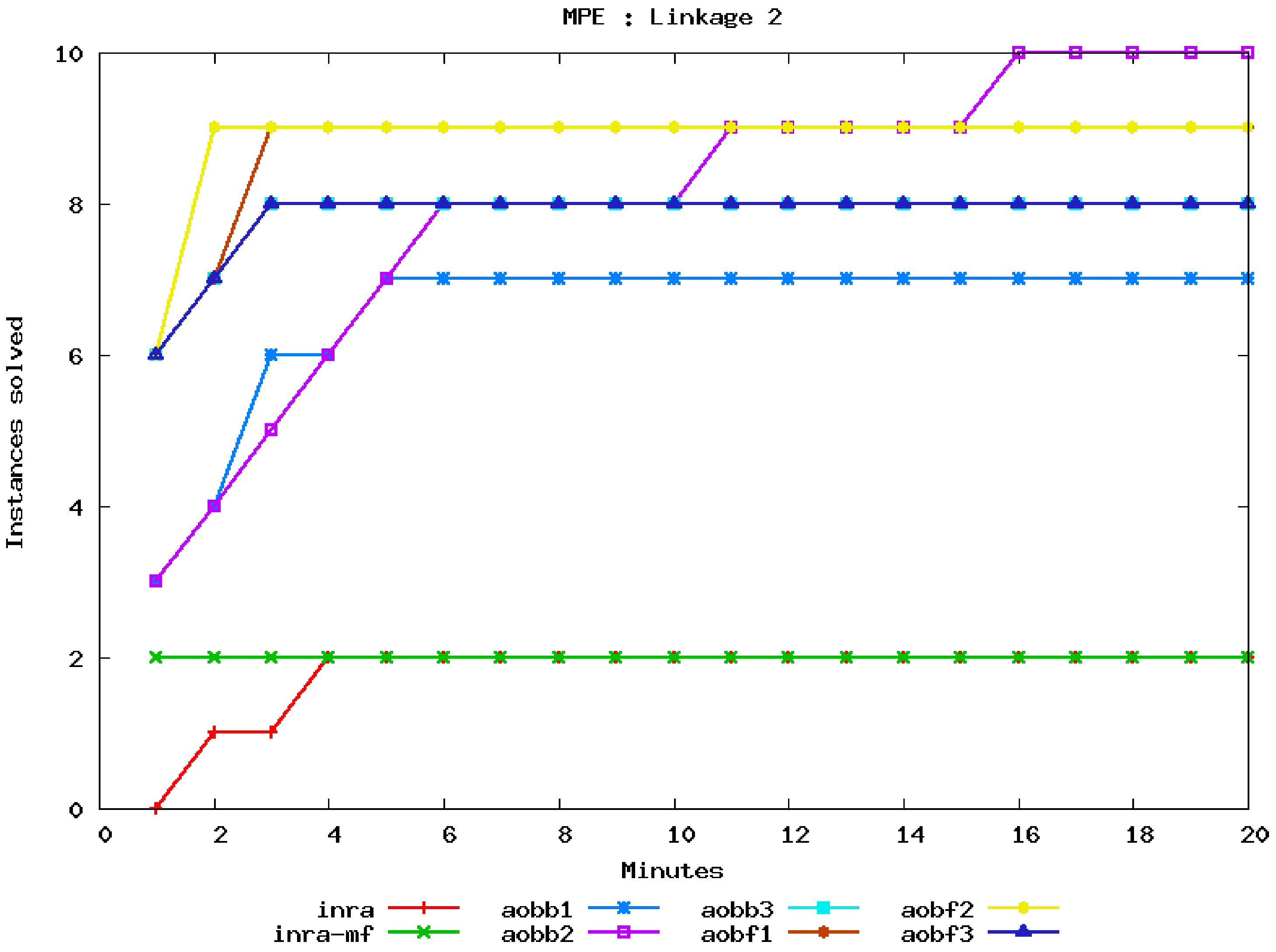
UAI-2008 Competition

- **AOBB-C+SMB(i) – (i = 18, 20, 22)**
  - AND/OR Branch-and-Bound with pre-compiled mini-bucket heuristics (i-bound), full caching, static pseudo-trees, constraint propagation
- **AOBF-C+SMB(i) – (i = 18, 20, 22)**
  - AND/OR Best-First search with pre-compiled mini-bucket heuristics (i-bound), full caching, static pseudo-trees, no constraint propagation
- **Toulbar2**
  - OR Branch-and-Bound, dynamic variable/value orderings, EDAC consistency for binary and ternary cost functions, variable elimination of small degree (2) during search
- **Toulbar2/BTD**
  - DFBB exploiting a tree decomposition (AND/OR), same search inside clusters as toulbar2, full caching (no cluster merging), combines RDS and EDAC, and caching lower bounds

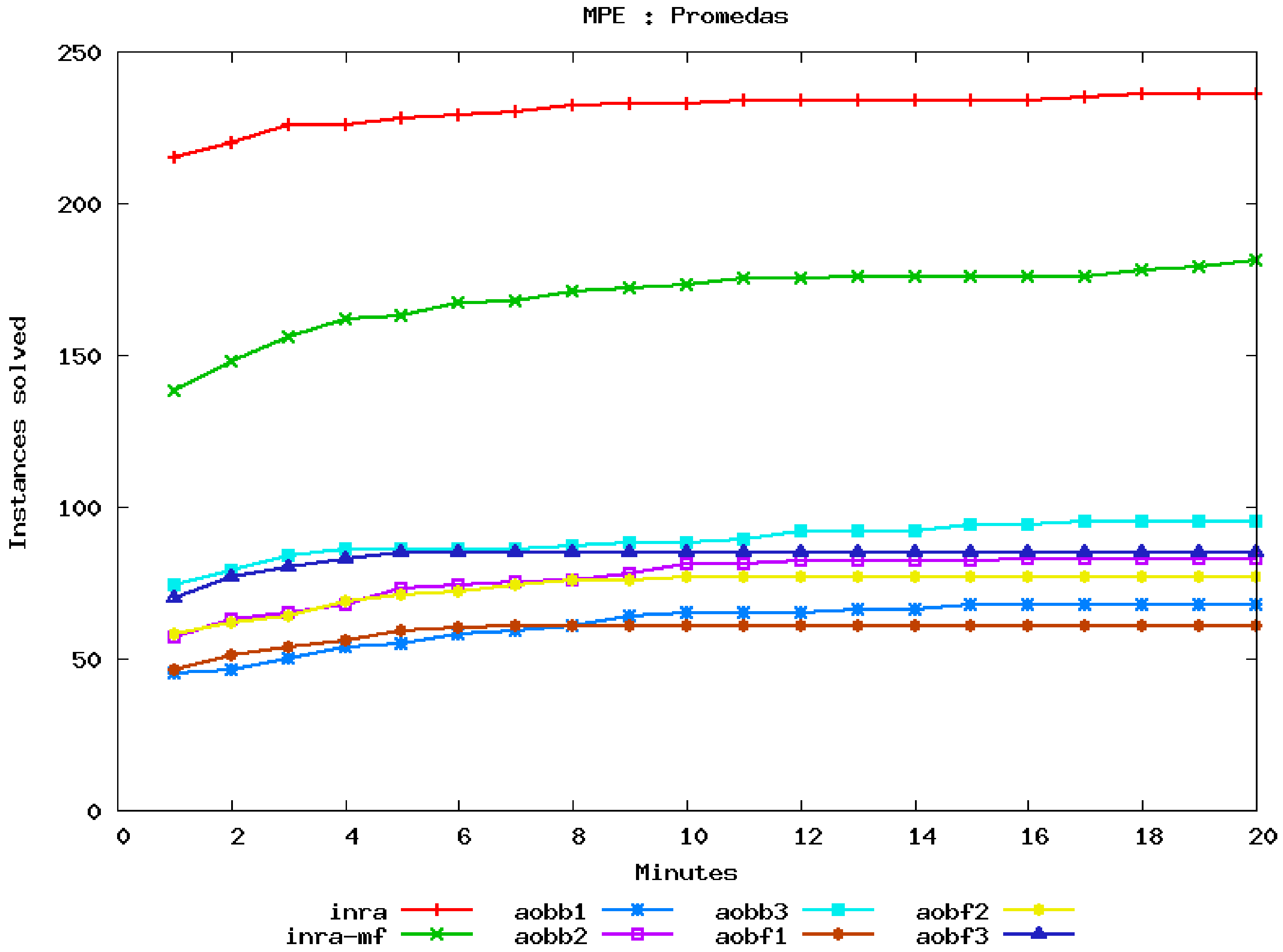
UAI-2008 Results



UAI-2008 Results (contd.)



UAI-2008 Results (contd.)



UAI-2010 Competition

- Tasks
  - PR: probability of evidence
  - MAR: posterior marginals
  - MPE: most probable explanation
- 3 tracks: 20 sec, 20 min, 1 hour
  - PR, MAR - 204 instances; MPE - 442 instances
    - CSP, grids, image alignment, medical diagnosis, object detection, pedigree, protein folding, protein-protein interaction, relational model, segmentation
- Exact and approximate solvers

UAI-2010 Results

- MAR task
  - **1st place** (20 min, 1 hour) – (impl. by Vibhav Gogate)
  - Anytime **IJGP(i)** with randomized orderings and SAT based domain pruning

(Mateescu et al, JAIR2010),
(Dechter et al, UAI2002)

- PR task
  - **1st place** (20 min, 1 hour) – (impl. by Vibhav Gogate)
  - Formula **SampleSearch** with IJGP(3) based importance distribution, w-cutset sampling, minisat based search, rejection control

(Gogate, Domingos and Dechter UAI2010)

- MPE task
  - **3rd place** (all tracks) – (impl. by Lars Otten)
  - **AND/OR BnB** with mini-buckets, randomized min-fill based pseudo tree, LDS based search for initial upper bound

(Marinescu and Dechter, AIJ2009),
(Otten and Dechter, ISAIM2010)