CS 164 & CS 266: Computational Geometry

Lecture 10

LP-type problems

David Eppstein
University of California, Irvine

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Minimum enclosing circle

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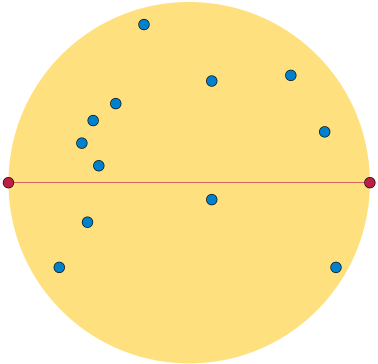
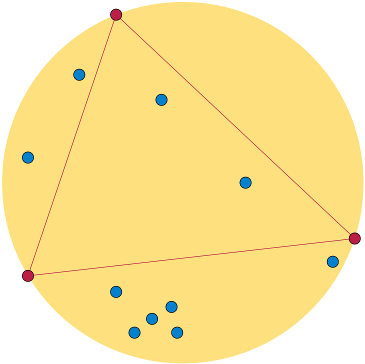
Problem: Given n points in the plane, find min-radius circle containing them

May be either of two cases:



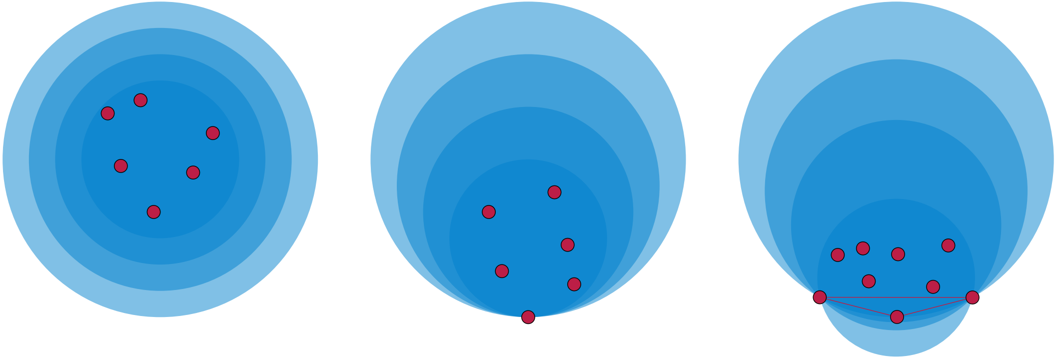
Two points form its diameter



Circle through an acute triangle

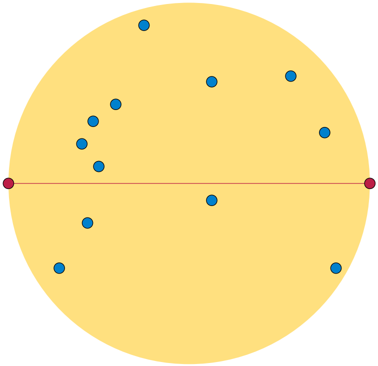
Why only those two cases?

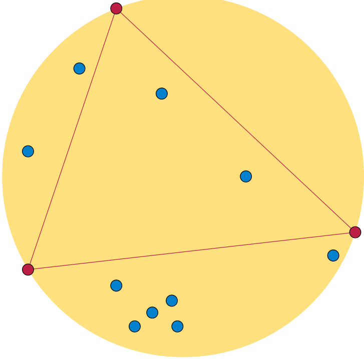
Any other circle can be shrunk



Cannot be a linear program

In a d-dimensional linear program, optimal solution is determined by exactly d constraints, but here solution sometimes comes from pairs of inputs and sometimes comes from triples





Circle primitives

Representation of circle

center, radius^2

Does it contain a given point?

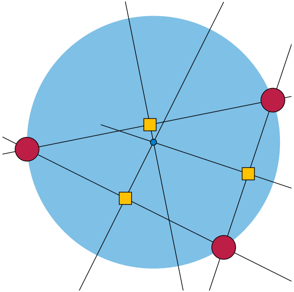
compare point-to-center dist^2
to radius^2

Circle from two points

center = average coords
$\text{radius}^2 = \text{dist}^2/4$

Circle from three points

Construct lines through pairs
Rotate 90° at midpoints
They meet at circle center
$\text{radius}^2 = \text{dist}^2$ to any point



As a nonlinear program

Given points x_i, y_i:

Find center X, Y
and squared radius R

Obey nonlinear constraints
$(x_i - X)^2 + (y_i - Y)^2 \leq R$

Minimize linear objective R

Define $S = R - X^2 - Y^2$

Find X, Y, S with linear constraints
$x_i^2 - 2x_iX + y_i^2 - 2y_iY \leq S$

Minimize nonlinear objective
$S + X^2 + Y^2$

LP-type problems

Key properties of circle problem

- We can describe it as a function mapping sets of points to their optimal solution (X, Y, R)
- Monotonic: We can compare solutions, and if $A \subset B$ are two sets of points then A's solution is at least as good as B's
- Local: If adding p or q separately to a set doesn't change its solution, then neither does adding both at once
- Low-dimensional: Every set has a basis of ≤ 3 points with same solution (diameter points or acute triangle)
- Primitives: Find circle defined by two or three points
  "Violation test": is point inside circle?

LP-type problem

Define a class of problems with the same properties:

- We can describe it as a function mapping sets of inputs to their optimal solution
- Monotonic: We can compare solutions, and if $A \subset B$ are two sets of inputs then A's solution is at least as good as B's
- Local: If adding p or q separately to a set doesn't change its solution, then neither does adding both at once
- Low-dimensional: Every set has a basis of $O(1)$ inputs with same solution ("dimension": maximum size of a basis)
- Primitives: Find solution for basis

  "Violation test": does adding p to B change its solution?

Goal: Find optimal solution using a small number of primitives

[Sharir and Welzl 1992]

Linear programming is LP-type

Fix objective function, and consider subsets of constraints. Then:

- We can describe it as a function mapping sets of constraints to their optimal solution points and its objective value
- Monotonic: We can compare solutions by their objective value, and if $A \subset B$ are two sets of constraints then A's solution is at least as good as B's
- Local: If a solution point obeys constraints p and q when one of them is added to the set of constraints, it still obeys them when both of them are added.
- Low-dimensional: Every set has a basis of d constraints with same solution
- Primitives: Solving a basis means turning those inequalities to equalities $\Rightarrow$ solve system of linear inequalities $\Rightarrow$ Gaussian elimination

  Violation test: check linear inequality on solution point

More example problems

Closest distance between disjoint convex hulls

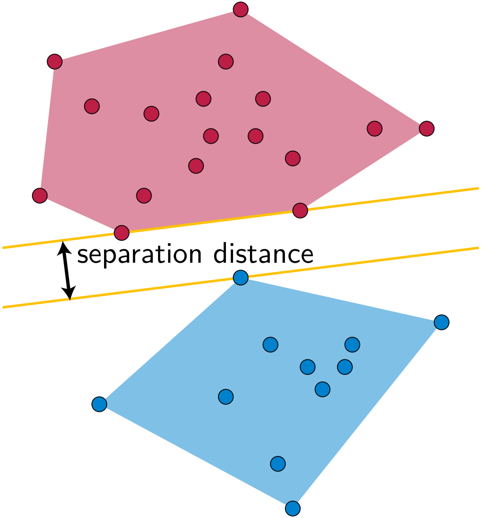
Map subsets to hull distance
= max separation of parallel lines
between hulls

Monotonic: More points $\Rightarrow$ closer hulls

Local: if two points stay outside
parallel lines when added separately,
they stay outside when added together

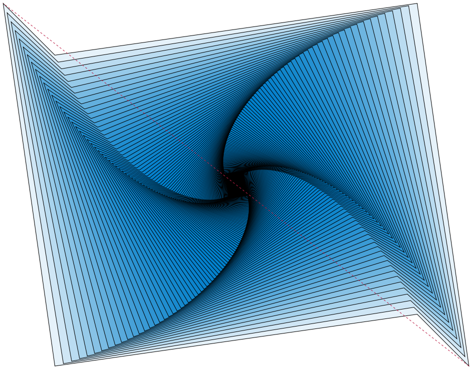
Low-dim: In $\mathbb{R}^d$, solution determined
by $\leq d + 1$ points

[Matoušek et al. 1996]



Like red-blue separation but Euclidean not vertical distance

Existence of continuous shrinking motion of polygon



Like star-shaped but more complicated

LP-type with dimension 3

Variables: center point and slope angle of logarithmic spirals traced by polygon vertices

Hard part: proving that dimension is 3

[Eppstein 2023]

Integer programming

Find best integer solution to a linear program

NP-complete when number of variables can be large,
even when all variables are restricted to $\{0, 1\}$

[Karp 1972]

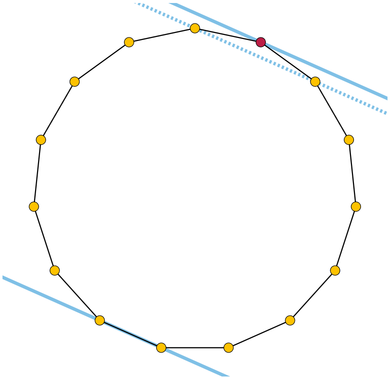
For k variables, LP-type with dimension 2^k

[Bell 1977; Scarf 1977]

A problem that is not LP-type

Enclose n points between parallel lines as close together as possible

Like L_∞ regression, but Euclidean distance not vertical distance



It is monotonic and local

For input = regular $(2n+1)$-sided polygon, optimal solutions use lines through one side and opposite vertex

If any point is removed, solution gets better

So all $2n+1$ points are needed to determine the solution

$\Rightarrow$ not low-dimensional

Algorithms

Seidel's algorithm

Instead of restricting recursive subproblems to a linear subspace, we pass them a subset of known basis elements, reducing the LP-type-dimension of the remaining problem by the number of these known elements

Call the following with Seidel(input set, empty set):

def Seidel(Inputs, Known):

  CurrentSolution = solution(Known)

  AlreadyProcessed = empty set

  For each element x of Inputs in random order:

    If x fails violation test for CurrentSolution:

      CurrentSolution = Seidel(AlreadyProcessed, Known $\cup \{x\}$)

    Add x to AlreadyProcessed

  Return CurrentSolution

Random sampling

Choose and solve a random subset R of the items

Let $V(R)$ = items that fail violation test for solution

- Each x in V must be part of basis for $V \cup \{x\}$
- Probability that this is true for x is $\leq d/(|R|+1)$
- So expected size of $V(R)$ is $O(nd/|R|)$
- If $V(R)$ is non-empty, it includes at least one member of basis of whole problem

[Clarkson 1995]

Recursive sampling

V = empty set

repeat d times or until no more violators are found:

  Choose sample R of size $d\sqrt{n}$

  Compute solution for $R \cup V$ (recursively using this or another algorithm)

  Add its violators to V

Each time through the loop, adds $O(\sqrt{n})$ elements to V including at least one more basis element

Solves whole problem in $O(dn)$ violation tests and $O(d)$ recursive calls on problems of size $O(d\sqrt{n})$

When $n = O(d^2)$, sample size is already n, does not make progress

Branching factor of d quickly blows up; better to use this only once and solve subproblems using a different algorithm

Iterated reweighting

Give all elements weight 1

Repeat:

> Select a random subset of $2d^2$ elements,
> with probability proportional to their weights
>
> Compute solution for subset and its set V of violated elements
>
> Double the weights of the members of V

Total expected weight of all items increases by $(1 + 1/2d)$ factor

Weight of optimal basis increases by larger $(1 + 1/d)$ factor

Can only happen $O(d \log n)$ times until subset has bigger weight than whole set (impossible) or we find optimal solution

Solves whole problem in $O(dn \log n)$ violation tests and $O(d \log n)$ recursive calls on problems of size $O(d^2)$

Putting it together

Outer level of algorithm: use random sampling

- $O(dn)$ violation tests
- $O(d)$ second-level calls on subproblems of size $O(d\sqrt{n})$

Second-level calls: use iterated reweighting

- Each subproblem has size $O(d\sqrt{n})$
- It does $O(d^2\sqrt{n}\log n)$ violation tests
- $O(d\log n)$ third-level calls on subproblems of size $O(d^2)$

Third-level calls: use Seidel's algorithm

- Each subproblem has size $O(d^2)$
- $O(d!\,d^2) \Rightarrow d^{O(d)}$ solution primitives

Total: $O(dn)$ violation tests, $d^{O(d)}\log n$ solution primitives
Can reduce $d^{O(d)}$ to $d^{O(\sqrt{d})}$: use algorithm of Matoušek et al. [1996] in place of Seidel

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