CS 164 & CS 266:
Computational Geometry

Lecture 16

Range reporting, onion layers, and fractional cascading

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Half-plane range search

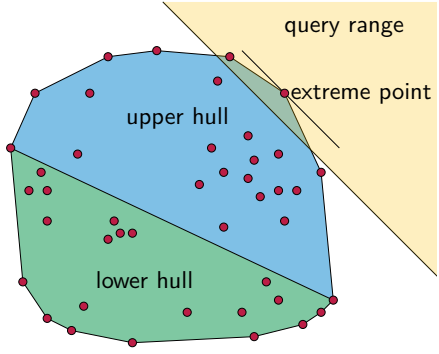
Warm-up: Range emptiness

Query: Is a given half-plane (the set above or below a line) empty?

Store upper and lower convex hull edges in arrays, sorted by slope

For a half-plane above line, binary search for line's slope in upper hull

For half-plane below, search lower hull



Edges with slopes closest to query slope are the endpoints of the extreme point in a direction perpendicular to the given line

It is the deepest point in the range (if there is any point in the range) and the closest point to the range otherwise

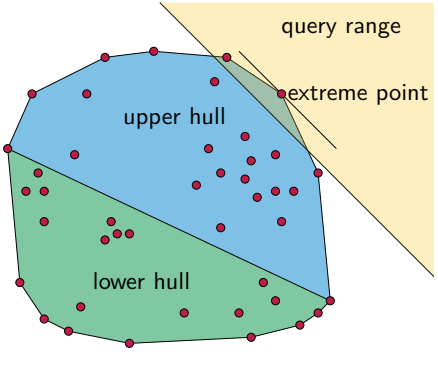
Listing all convex hull vertices in the range

Find extreme point

Walk left and right from it listing points in range until finding a point that is not in range (or looping back to start)

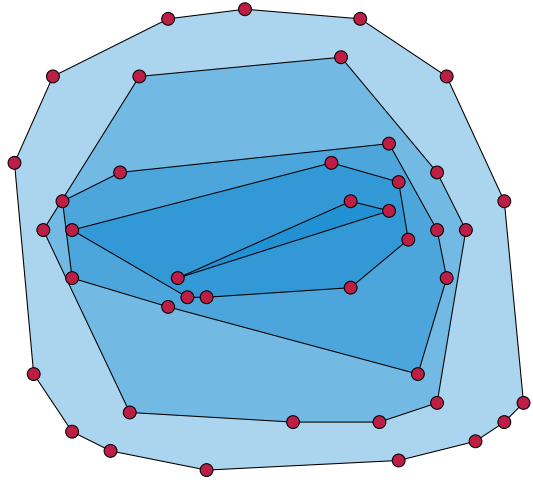
Time $O(\log n + k)$

But only finds hull vertices!



Nested hulls

Find hull of given points, remove its vertices, and repeat



Result: A sequence of nested convex polygons

Can be constructed in time $O(n \log n)$ (but complicated)

Using nested convex hulls for range reporting

To list all points in a query halfplane:

Start at the outermost hull

While we have not yet found a hull that is separate from the query range:

- Do a binary search to find the extreme point
- If it is inside the range, walk around the hull to find all other hull vertices in range, and move on to the next hull
- Otherwise, stop: this level has no results and none of the hulls nested inside it can have any results

Analysis of the nested binary search algorithm

Each binary search except for the last one finds a point to report

Each walk step except for the last in a walk finds a point to report

Worst case: one point in range in each hull, $k + 1$ binary searches

Time: $O(k \log n)$

Can we do better?

Insight: all binary searches use the same key (the slope)

Fractional cascading

What is fractional cascading?

Goal: Speed up related binary searches without too much space

A simplified version:

Data: k sorted lists of numbers $S_0, S_1, \ldots S_{k-1}$

Total length: $n = |S_0| + |S_1| + \cdots + |S_{k-1}|$

No repeated values, even in different lists

Query: find the successors of a given number q in each list

(s_i = successor of q in list S_i)

Example

Data:

- $S_0 = [0, 10, 20, 30, 40, 50, 60, 70]$
- $S_1 = [1, 2, 13, 25, 27, 51, 57]$
- $S_2 = [21, 22, 31, 32, 33, 41, 99]$
- $S_3 = [67, 68, 69]$

Total length $n = 8 + 7 + 7 + 3 = 25$

Query for $q = 24$ would find
$s_0 = 30 \quad s_1 = 25 \quad s_2 = 31 \quad s_3 = 67$

Naïve solutions

Do the binary searches separately

Space $= O(n)$ for storing each S_i as a sorted list

Query time $= O(k \log n)$ for k binary searches

Merge into one list

For each value x, store k-tuple of successors
for queries that return x as their smallest value

0:(0,1,21,67), 1:(10,1,21,67), 2:(10,2,21,67), 10:(10,13,21,67), 13:(20,13,21,67),
20:(20,25,21,67), 21:(30,25,21,67), ...

Binary search in merged sorted array + look up k-tuple

Space $O(kn)$, query time $O(k + \log n)$

Fractional cascading

Working backwards through the sequence of lists S_i,
construct T_i: merged structure for (S_i + half the elements of T_{i+1})

Choosing the half of the elements that are in odd-numbered positions e.g. if $T = 1, 2, 3, 5, 7, 11, 20$ then $\frac{1}{2}T = 2, 5, 11$

So T_i consists of:

- A sorted array of the merged items from $S_i + \frac{1}{2}T_{i+1}$
- A dictionary mapping each merged item x to a pair (a, b) where one of a or b is x, and the other one is the successor of x in the other merged list
- When there is no successor in the other list, use $+\infty$

Example

- $S_3 = 67, 68, 69$ $T_3 = S_3$ (nothing to merge) Half elements: 68
- $S_2 = 21, 22, 31, 32, 33, 41, 99$
- $T_2 = 21{:}(21,68), 22{:}(22,68), 31{:}(31,68), 32{:}(32,68), 33{:}(33,68), 41{:}(41,68), 68{:}(99,68), 99{:}(99,+\infty)$
- Half the elements of T_2: 22, 32, 41, 99
- $S_1 = 1, 2, 13, 25, 27, 51, 57$
- $T_1 = 1{:}(1,22), 2{:}(2,22), 13{:}(13,22), 22{:}(25,22), 25{:}(25,32), 27{:}(27,32), 32{:}(51,32), 41{:}(51,41), 51{:}(51,99), 57{:}(57,99), 99{:}(+\infty,99)$
- Half the elements of T_1: 2, 22, 27, 41, 57
- $S_0 = 0, 10, 20, 30, 40, 50, 60, 70$
- $T_0 = 0{:}(0,2), 2{:}(10,2), 10{:}(10,22), 20{:}(20,22), 22{:}(30,22), 27{:}(30,27), 30{:}(30,41), 40{:}(40,41), 41{:}(50,41), 50{:}(50,57), 57{:}(60,57), 60{:}(60,+\infty), 70{:}(70,+\infty)$

Searching fractionally cascaded lists

To find the successors of q:

- Binary search for successor t_0 in merged list T_0
- Set $i = 0$
- Then, repeat:
  - Use dictionary for T_i to find the pair (a, b)
    where $a = s_i =$ successor in S_i
    and b is successor in $\frac{1}{2} T_{i+1}$
  - Output s_i
  - Let c be the (skipped) element of T_{i+1} just before b
  - If $q < c$ then $t_{i+1} = c$ else $t_{i+1} = b$
  - Set $i = i + 1$

Example (continued)

To search for the successor of $q = 24$:

- Binary search in T_0 finds successor t_0: 27:(30,27)
- Output $s_0 = 30$, successor in S_0
- Successor in T_1 might be either 27 or previous item, 25
- Because $q < 25$, successor in T_1 is 25:(25,32)
- Output $s_1 = 25$, successor in S_1
- Successor in T_2 might be either 32 or previous item, 31
- Because $q < 31$, successor in T_2 is 31:(31,68)
- Output $s_2 = 31$, successor in S_2
- Successor in T_3 might be either 68 or previous item, 67
- Because $q < 67$, successor in T_3 is 67
- Output $s_3 = 67$, successor in S_3

Fractional cascading analysis

Query time

One binary search + $O(1)$ for each list after the first

Total $O(k + \log n)$

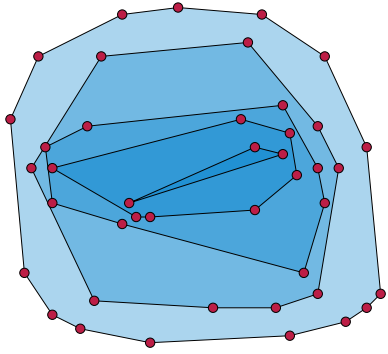
Space and set-up time

Each element of S_i contributes 1 to the length of T_i, $\frac{1}{2}$ to the length of T_{i-1}, $\frac{1}{4}$ to the length of T_{i-2}, ...

So the total space and total set-up time is $O(n)$

Best combination of time and space from naïve solutions

Half-plane range reporting, revisited



Structure:

- Nested hulls
- Sorted lists of slopes of upper and lower hull edges, ordered from outer hull to inner
- Fractional cascading

Query: One binary search to start fractional cascading, after which we can find extreme point in each level in constant time

List vertices on each hull, starting from its extreme point, stopping when we reach a hull that is entirely outside the range

Total $O(\log n + k)$