CS 164 & CS 266: Computational Geometry

Lecture 2

Coordinates and primitives

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Area

A toy example

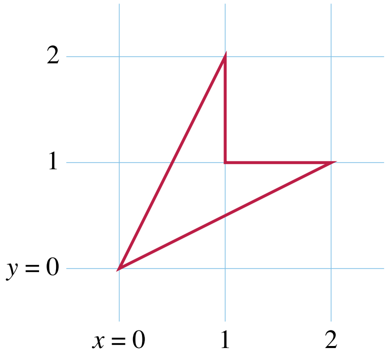
What is the area of this polygon?

Clever solution:

Cut along the grid lines

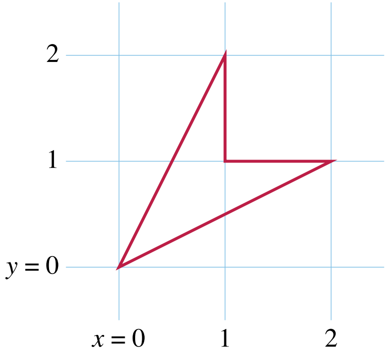
Flip the cut-off triangles into the lower left kite

The result exactly covers a square, area 1



Area of polygons

We want a solution without cleverness that a computer can follow



Vertex coordinates:

$[(0,0),\ (1,2),\ (1,1),\ (2,1)]$

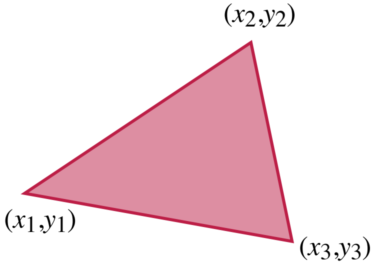
Area: 1

Input: Clockwise sequence of vertices
Each given by integer Cartesian coordinates

Output: A number, the area

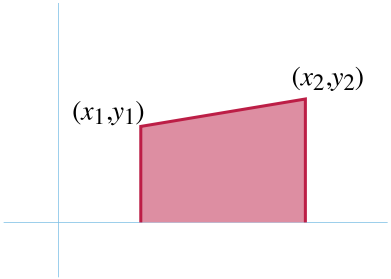
Idea: decompose into simpler shapes

Triangles



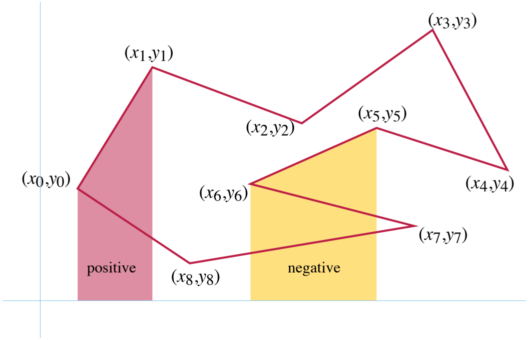
$$A = \frac{1}{2}\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$
$$= \frac{1}{2}(x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2)$$

Trapezoids



$$A = \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$$

Using trapezoids to compute area



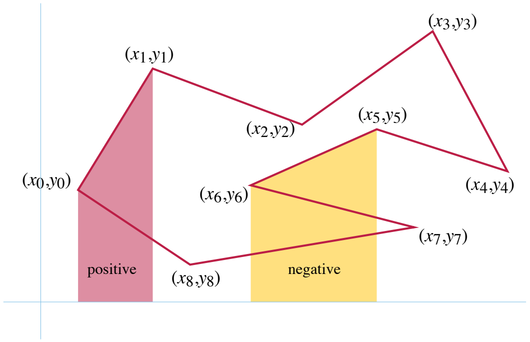
Add the areas of trapezoids below each upper edge of the polygon

Subtract the areas of trapezoids below each lower edge

Outside polygon, positive and negative areas cancel

Inside points all covered by one more positive than negative

Trapezoid formula for area



$$A = \sum_{i=0}^{n-1} \frac{1}{2}(x_{i+1} - x_i)(y_i + y_{i+1})$$

All indexes computed modulo n so $(x_n, y_n) = (x_0, y_0)$

Produces a positive number for positive trapezoids, negative for negative trapezoids

Trapezoid formula as an algorithm

Summation $\Rightarrow$ for-loop

```
def area(P):
    area = 0
    for i in 0, 1, 2, ... n-1:
        j = (i + 1) mod n
        xi,yi = P[i]
        xj,yj = P[j]
        area += (xj-xi)*(yj+yi)/2
    return area
```

Easy, time $= O(n)$

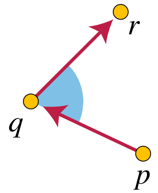
Geometric primitives

What is a primitive?

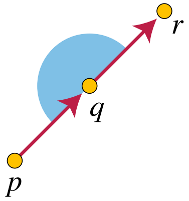
A constant-time formula/algorithm/subroutine for higher-level information than the input coordinates

Example: The area of a triangle

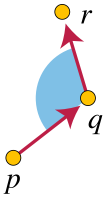
Another example (same subroutine, different interpretation):
If you travel from p to q, then turn and travel from q to r,
which way did you turn?



right turn:
$\text{area}(p,q,r) > 0$



straight:
$\text{area}(p,q,r) = 0$

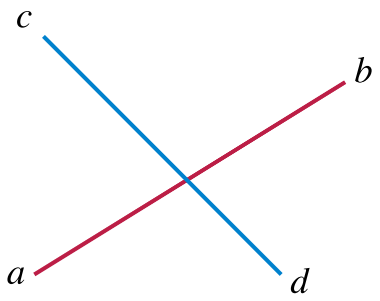


left turn:
$\text{area}(p,q,r) < 0$

Crossing test primitive

Idea: build more complicated primitives from simpler ones

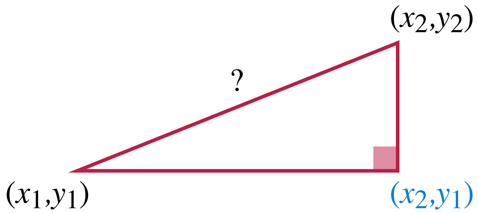
Does line segment *ab* cross line segment *cd*?



Yes, if: *abc* turns the opposite way from *abd*, and
cda turns the opposite way from *cdb*

Distance/length primitive

Distance from (x_1, y_1) to (x_2, y_2) (length of segment)?



$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

(apply Pythagoras to right triangle with sides $|x_1 - x_2|$, $|y_1 - y_2|$)

Coordinate systems

Motivation

Cartesian coordinates — (x, y) pairs — are familiar and work well for many purposes

Sometimes other coordinates are easier to use

Example:

> If we represent most lines as (m, b) where $y = mx + b$, finding the line through two points uses division, problematic if we want integers
>
> And what about vertical lines $x = c$?

Projective geometry eliminates both problems

Coordinate systems for points in the plane

Cartesian coordinates (x, y)

Simple, familiar

Generalize to higher dims

Widely used, familiar

Polar coordinates (r, θ)

Angle and distance from origin

Widely known, not as useful

Complex numbers

Built into some programming languages (Python)

Make certain transformations easy:

Translate by t:	$q \mapsto q + t$
Scale by s:	$q \mapsto qs$
Rotate by θ:	$q \mapsto q(\cos\theta + i\sin\theta)$

Not as easy to generalize to higher dimensions

Formulas for coordinate conversion

Cartesian to polar

$r = \sqrt{x^2 + y^2} = \mathrm{hypot}(x, y)$

$\theta = \mathrm{atan2}(y, x)$

Polar to Cartesian

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

Cartesian to complex

$q = x + i \cdot y = x + 1\mathrm{j} \; * \; y$

Complex to Cartesian

$x = \Re(q) = q.\mathrm{real}$

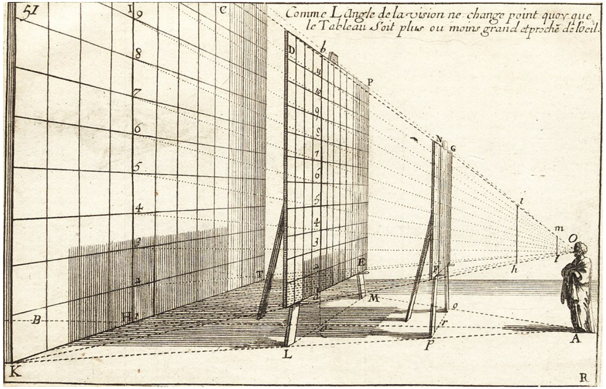
$y = \Im(q) = q.\mathrm{imag}$

Projective geometry

Equivalence between 2d points and 3d lines

Map plane into 3d by $(x, y) \mapsto (x, y, 1)$, view from $(0, 0, 0)$

Would get same view for all scaled embeddings (ax, ay, a)
All points (ax, ay, a) on a line through $(0, 0, 0)$ look the same!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Comme L'angle de la vision ne change point quoy que le Tableau soit plus ou moins grand et proché de l'oeil.
I
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Projective geometry

New meaning for the words "point" and "line":

- "Point": 3d line through $(0, 0, 0)$

  Represent by 3d coordinates of any nonzero point on the line

  Many different triples all represent the same "point":
  $(x, y, z) = (0.5x, 0.5y, 0.5z) = (3x, 3y, 3z) = \cdots$

- "Line": 3d plane through $(0, 0, 0)$

Contains all points and lines of the Euclidean plane

$$(x, y) \mapsto (x, y, 1) \qquad (x, y, z) \mapsto (x/z, y/z)$$

But it also contains extra "points" (x, y, z) with $z = 0$,
called "points at infinity" (although x, y, z are all finite numbers)

Lines in projective geometry

Line: set of points (x, y, z) obeying an equation $ax + by + cz = 0$

Coordinates of the line: the numbers a, b, c

Many different triples all represent the same line:
$(a, b, c) = (0.5a, 0.5b, 0.5c) = (3a, 3b, 3c) = \cdots$

Converting between coordinates

Euclidean to projective

Point $(x, y) \mapsto (x, y, 1)$

Line $(m, b) \mapsto (m, -1, b)$
$y = mx + b \mapsto mx - y + bz = 0$

Vertical line $c \mapsto (1, 0, -c)$
$x = c \mapsto x + 0y - cz = 0$

Projective to Euclidean

Point $(x, y, z) \mapsto (x/z, y/z)$

Line $(a, b, c) \mapsto (-a/b, -c/b)$
$ax + by + cz = 0$
$\mapsto y = -ax/b - c/b$

Vertical line $(a, 0, c) \mapsto -c/a$
$ax + cz = 0 \mapsto x = -c/a$

But points at infinity $(x, y, 0)$ and line at infinity $(0, 0, 1)$
do not correspond to anything in Euclidean geometry!

What are these extra points?

We can think of:

- The plane as a surface that we're viewing from slightly above it
- The line at infinity is the horizon
- The points at infinity are the vanishing points where parallel lines meet

Point-on-line primitive and projective duality

To test if point (x, y, z) is on line (a, b, c):
compute dot product $(x, y, z) \cdot (a, b, c)$, check if zero

Unaffected by multiplying the coordinates of either by a scalar

Unaffected by which triple is a “point” and which we call a “line”

So if we reinterpret all lines in projective geometry as being points, and all points as being lines, it doesn’t affect any properties defined using point–line intersection tests!

The space of points and lines formed by renaming the lines and points of projective geometry in this way is the projective dual

Algorithmic applications of projective duality

Whenever we have a valid mathematical statement about points, lines, and point-line incidence in projective geometry, we can change all points to lines and all lines to points in the statement and get another valid statement.

Example 1: every two points have a line that touches both of them
Dual 1: every two lines have a point that touches both of them

Not true of Euclidean parallel lines!

Example 2: f is a subroutine that takes as input two points and outputs the line that touches them
Dual 2: f is a subroutine that takes as input two lines and outputs the point that touches them

The same subroutine does two different things!

To find line through two points / point on two lines

Math version:

Line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is the set of points (x, y, z) for which

$$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{pmatrix} = 0$$

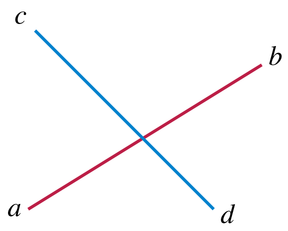
Because this equation is linear in x, y, z and is true of the two given points

Computer science version:

```
def thru2(p,q)
    x1,y1,z1 = p
    x2,y2,z2 = q
    x = y1*z2-y2*z1
    y = z1*x2-z2*x1
    z = x1*y2-x2*y1
    return (x,y,z)
```

Is the crossing test really that simple?

Does line segment *ab* cross line segment *cd*?



Yes, if: *abc* turns the opposite way from *abd*, and
cda turns the opposite way from *cdb*

But what if some of these turns are straight?

Simplifying assumption: no three input points are in a line

General position

"General position" = no unexpected numerical coincidences

E.g. when using turn direction primitive, never comes out zero (no three points are on a line)

This assumption simplifies algorithm design (and lecturing!)
but is a poor match for real-world inputs

Instead, we can:

- Do more analysis to carefully handle special cases
- Perturb inputs by small distances to make them general position
- Use numerical libraries that automatically simulate the results of infinitesimally small perturbations

The problem with distances

Recall the distance formula:

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Even when all coordinates are integers, square root is usually not!

Perimeter of a polygon = sum of square roots

How many bits of precision do we need to compare two perimeters and tell which one is shorter? Unknown!

Partial solution: when comparing distances, but not adding them, we can compare squared distance before taking its square root