CS 164 & CS 266: Computational Geometry

Lecture 3

Line segment intersection

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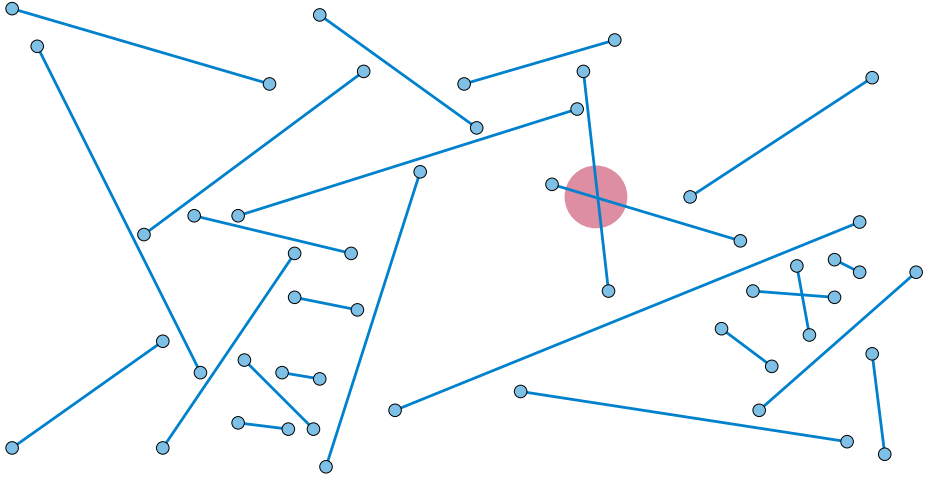
Intersection detection and crossing listing

Intersection detection problem

Input: A list of line segments (each one: 4 coords of 2 endpoints)



Output: Do any two cross? If so report a crossing (or maybe all)

Application: Test validity of geographic data or circuit designs

Naïve solution

For each pair of input segments:

test whether they cross

$O(n^2)$

Last time:

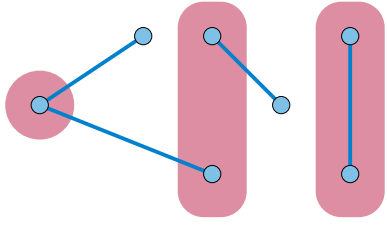
- Test for crossing using four left-right tests on triples of points
- Find line through two points, and point where lines cross, via projective geometry

$O(1)$

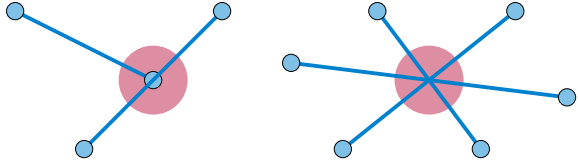
But can we do better?

(Unrealistic) general position assumptions

- No two segment endpoints have the same x-coordinate
  (Implies: No vertical segments)
- No endpoint lies on another segment
- No three segments cross at a single point



likely to occur
probably ok

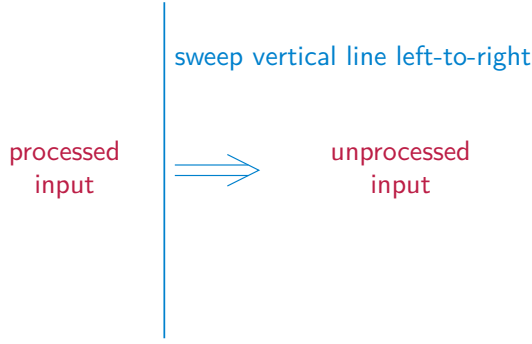


should be reported
as being crossings

Plane sweep algorithms

Plane sweep

General approach for designing geometric algorithms



The combinatorial structure of the result will only change at a finite set of discrete "event points" — process these in left-to-right order

Crossed segment data structure

As sweep line sweeps left-to-right across the input, maintain:

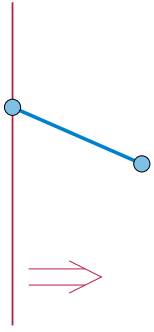
- Set of line segments that cross it
- Vertical ordering of these line segments in a binary search tree

Vertical ordering will help us find crossings when we sweep over them, because it changes at those points

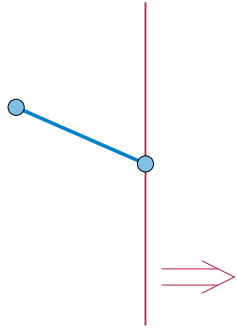
We can maintain it efficiently using a balanced binary search tree

Search tree operations only need to know the ordering of the segments, not the precise coordinates of the points where they cross the sweep line

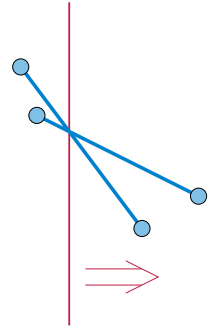
How the vertical ordering can change



cross left endpoint
of segment —
add to set of segments
crossed by sweep line



cross right endpoint
of segment —
remove from segments
crossed by sweep line



sweep over
crossing point —
swap in vertical order
of crossed segments

Event queue data structure

Keep track of future event points (where vertical ordering changes) in a priority queue, prioritized by x-coordinate (position in the left-to-right ordering used by the sweep line)

For crossing detection, we only need sorted list of segment endpoints

For listing all crossings, we also include the crossings we have found so far

In some algorithms we may also include "potential" events that we think might happen, but that can be removed from the event queue before they actually happen

Crossing detection pseudocode

- Initialize T to an empty binary search tree
- Initialize Q to be a sorted list of segment endpoints
- For each point p in Q:
  - If p is a left endpoint of a segment S: Add S to T, and check for crossings between S and its neighbors above and below it in the vertical crossing order (found using T)
  - Else p is a right endpoint of a segment S; remove S from T, and check for crossings between the two segments that were above and below it in the vertical crossing order

If we ever find a crossing, stop the whole algorithm and report it

If any two segments cross, they will be adjacent just before the sweep line sweeps over the crossing, and this algorithm will check them and discover the crossing

Listing all crossings

- Initialize T to an empty binary search tree
- Initialize Q to a priority queue of points, prioritized by x-coordinate, initially containing all segment endpoints
- While Q is non-empty:
  - Find the minimum-priority point p in Q and remove it from Q
  - If p is a left endpoint of a segment S: Add S to T, and check for crossings between S and its neighbors above and below it in the vertical crossing order (found using T)
  - Else if p is a right endpoint of a segment S; remove S from T, and check for crossings between the two segments that were above and below it in the vertical crossing order
  - Else p is a crossing point of two segments; swap the segments in T, and check for crossings between them and the two segments above and below them in T

Whenever we find a new crossing point, just insert it into Q

Analysis (of both algorithms)

Let n be the number of segments (so there are $2n$ endpoints) Let k be the number of crossing points; $0 \leq k \leq \binom{n}{2}$.

The detection algorithm has $2n$ events; the crossing listing algorithm has $2n + k$

Each event is performed using a constant number of operations in binary search trees and priority queues

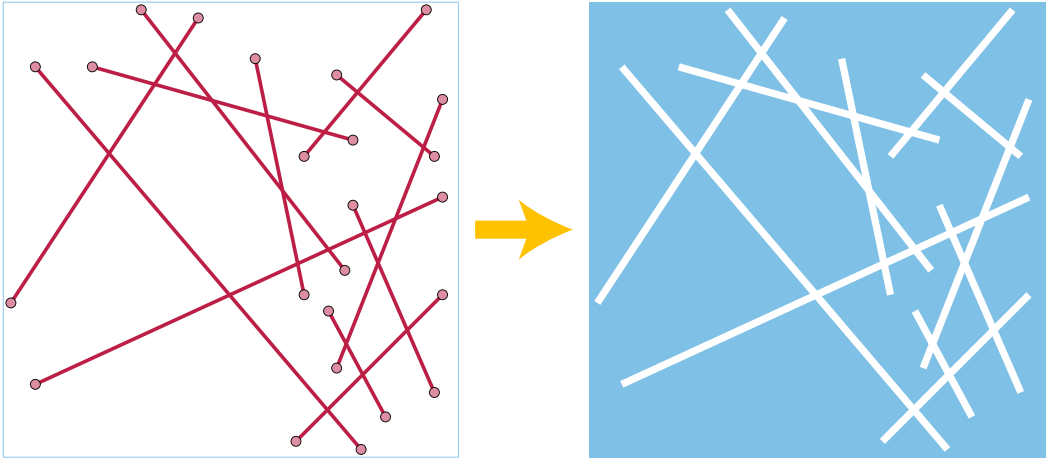
Total time: $O(n \log n)$ for detection, $O((n + k) \log n)$ for crossing listing

Arrangements and their representation

Arrangement

Think of any system of segments or curves as barriers to motion

"Face": 2d region within which you can get between any two points



How to find and represent this system of faces and their boundaries?

Some terminology

Face

2d connected region

Edge

1d boundary of two faces, separated by part of a segment

Same face can be on both sides

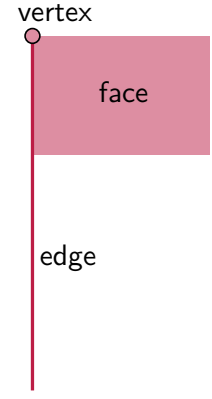
Vertex

An endpoint of a segment, or crossing point of segments

At an intersection point, multiple edges come together

Flag

A vertex, edge, and face that all touch each other



Representation issues

Most representations are centered on the edges of an arrangement

Each edge touches two vertices at its ends,
and two faces on its two sides

There are representations with:

one object per edge (pointing to all four of these things)

two objects per edge (one for each of its two sides)

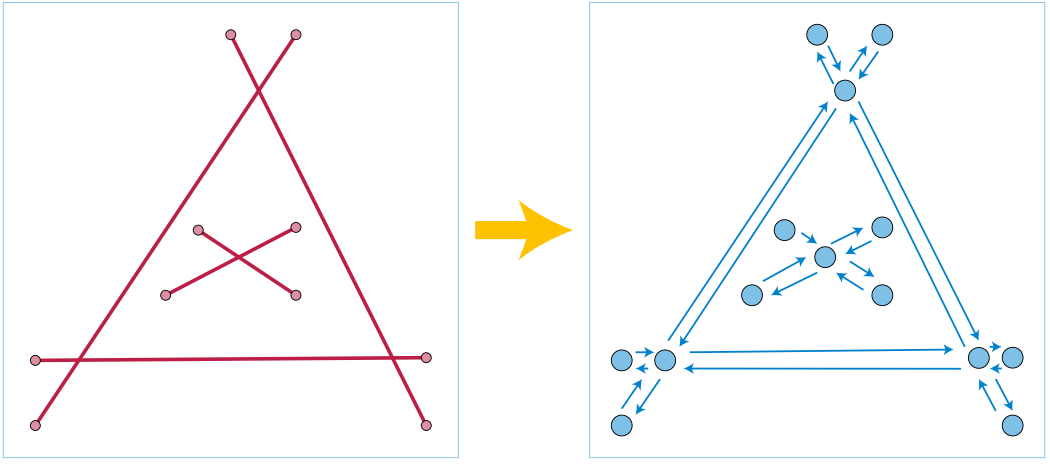
four objects per edge (one per flag)

Structure from our text: two objects per edge

Half-edges

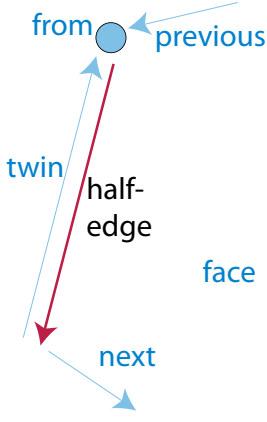
Represent each edge of the arrangement by two directed edges ("half-edges"), like the two directions of a two-way road

(But like in England or Japan where they drive on the left)



Doubly-connected edge list

Object-oriented, with objects for vertices, half-edges, and faces



Each half-edge stores:

- Pointer to twin half-edge from same edge
- The vertex it comes from (Can find other vertex from twin)
- The face on its side of the edge
- The next and previous half-edges in the cycle around its face

Each vertex stores:

- Its coordinates
- One of the half-edges it touches

Each face stores:

- A half-edge on its outer boundary
- A list of half-edges, one for each internal boundary

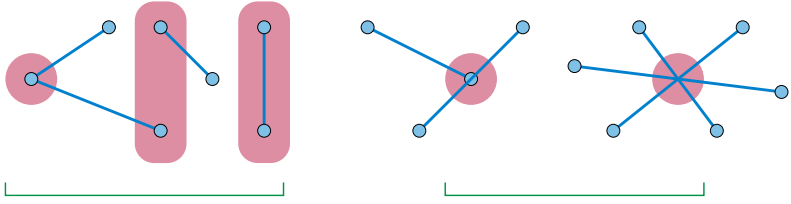
Constructing the arrangement of line segments

Same plane sweep algorithm, maintaining DCEL of the part of the arrangement to the left of the sweep line

Updates to DCEL at sweep events:

- When we sweep over an endpoint of one or more line segments:
  - Use search tree of segments crossed by sweep line to find its face(s)
  - If it is not a right endpoint, start new internal boundary in current face; otherwise, close off half-edges for which it is a right endpoint, and faces between them
  - If it is not a left endpoint, merge two faces or close off internal boundary; otherwise, start new half-edges for which it is a left endpoint, and faces between them
- When we sweep over a crossing, split its segments at that point and treat it as an endpoint of four line segments

Handling inputs that are not in general position



Event points have the same x-coordinate

- Break ties by y-coordinate
- Treat bottom endpoint of a vertical segment as left, and top endpoint as right

Endpoints on segments

- Report as a crossing?
- More cases for how to update DCEL

Multiple segments cross at one point

- Use only one event point, labeled by all the segments that cross there
- When processing event, reorder crossing segments in search tree