CS 164 & CS 266: Computational Geometry

Lecture 4

Arrangements of lines

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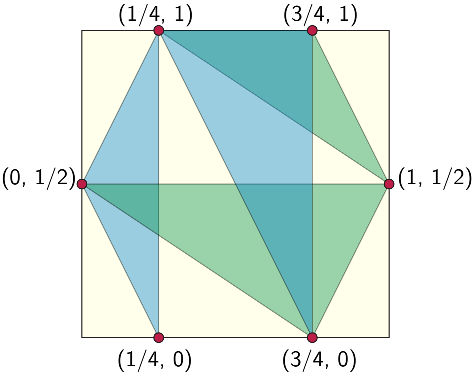
Minimum-area triangles

The Heilbronn triangle problem

Find n points in unit square maximizing the minimum area triangle

3 in line $\Rightarrow$ area $= 0$, so we want as far out of line as possible



Example: $n = 6$. For these points, triangle areas are $1/8$, $1/4$, and $3/8$, so its minimum triangle area is $1/8$.

This point set is optimum: no other 6-point set has minimum $> 1/8$.

Known bounds on the Heilbronn problem

Sort points by x-coordinates $\Rightarrow$ some three points lie in a thin vertical strip $\Rightarrow$ their triangle has area $O(1/n)$

Every sufficiently large point set has a triangle with area

$$\Delta(n) \leq \left(\frac{1}{n}\right)^{\frac{8}{7}+\frac{1}{2000}}$$

[Cohen et al. 2023] slightly improving [Komlós et al. 1981]

Some n-point sets have smallest triangle area

$$\Omega\left(\frac{\log n}{n^2}\right)$$

[Komlós et al. 1982]

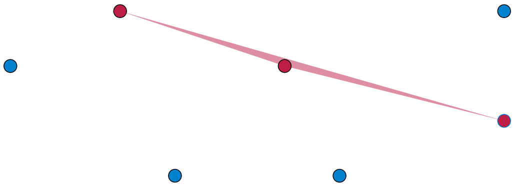
Unsolved problem: close this big gap!

How to test quality of Heilbronn triangle solution?

Algorithmic problem:

Given n points,
find their minimum-area triangle

Special case: detect an area-zero triangle – is this input in general position?



Naïve algorithm:

Use the triangle-area primitive on all triples

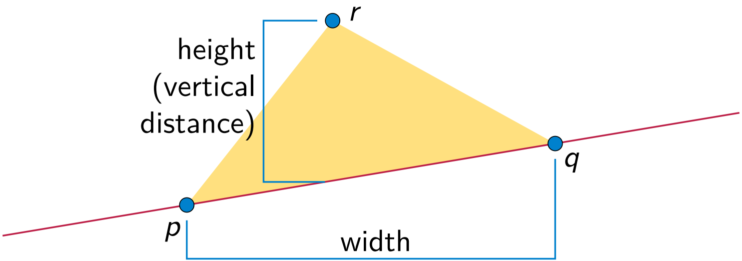
Time $= O(n^3)$

We can do much better!

From triangle area to vertical distance

For each two (non-vertically-aligned) points p, and q:

- We can find the line $\ell : y = mx + b$ through p and q
- Area of pqr is $\frac{1}{2}$width·height where width is $|x_p - x_q|$ and height (vertical distance of r from ℓ) is $|y_r - (mx_r + b)|$, the difference between the actual y-coordinate of r and the y-coordinate predicted by the line given x_r
- Do this for all $r \Rightarrow$ find the smallest triangle using p and q
- But this is still $O(n^3)$ because there are $O(n^2)$ lines to try



Duality of vertical distance

Given points with coordinates (x, y)
and non-vertical lines with coordinates (m, b)
$\Rightarrow$ (signed) vertical distance is $y - mx - b$

Given points with coordinates $(m, -b)$
and non-vertical lines with coordinates $(x, -y)$
$\Rightarrow$ (signed) vertical distance is $-b - mx + y$

Using point coordinates for lines and vice versa
(and negating the second coordinate)
doesn't change our calculation!

Original and dual problems

Original problem:

Given n points, no two x-coordinates equal

Find the non-vertical line through each pair

Find third given point with minimum vertical distance to it

(Min-area triangle will be one of the triples we find)

Equivalent dual problem:

Given n non-vertical lines, no two slopes equal

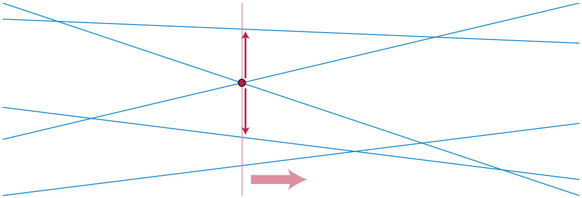
Find the crossing point of each two lines

Find third given line with minimum vertical distance to it

(Min-area triangle will still be one of the triples we find)

Plane sweep

Use plane sweep for finding crossings of line segments, on the n given lines



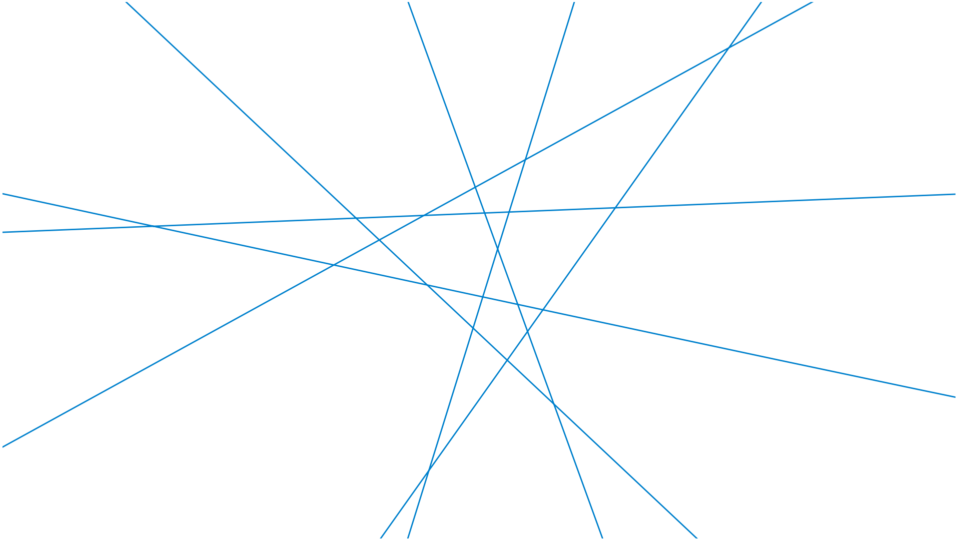
Each time it sweeps over a crossing, use binary search tree (of lines meeting vertical sweep line) to find the two lines directly above and directly below the crossing; one of the two will have minimum vertical distance

$O(n^2)$ crossings
$\Rightarrow$ $O(n^2)$ priority queue and binary search tree operations
$\Rightarrow$ total time $O(n^2 \log n)$

We can do better!

Arrangements of lines

An arrangement of lines

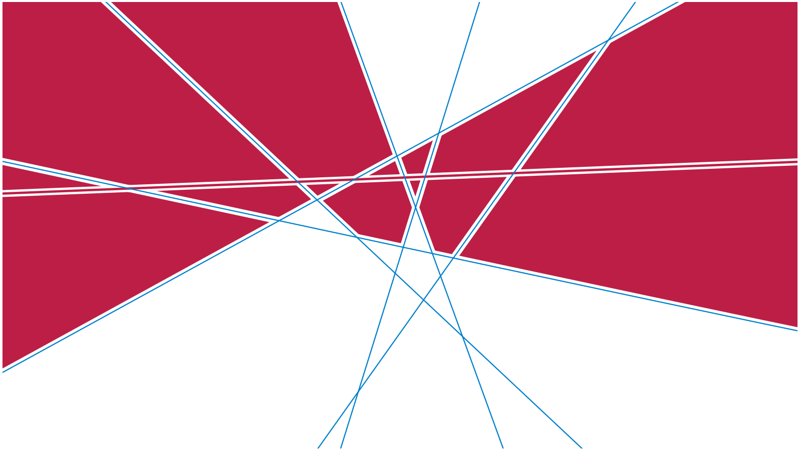


With no two parallel lines, every two lines cross

Convenient to include one more line at infinity, proj. coords $(0, 0, 1)$

Incremental construction algorithm

- Start with DCEL describing only the line at infinity
- For each line ℓ_i in arbitrary order:
  - Walk along line at infinity to find start point of new line
  - For each face crossed by the new line, walk around the face to find where it exits the face, and use that information to split the face into two smaller faces



Combinatorial complexity of arrangements of lines

By understanding how many things are in the arrangement
we can analyze algorithms that do something for each thing

$$\#\text{ vertices (crossings)} \leq \binom{n}{2} = O(n^2)$$

Each line is divided into 2 rays and $\leq n-2$ segments $\Rightarrow$

$$\#\text{ rays} = 2n$$

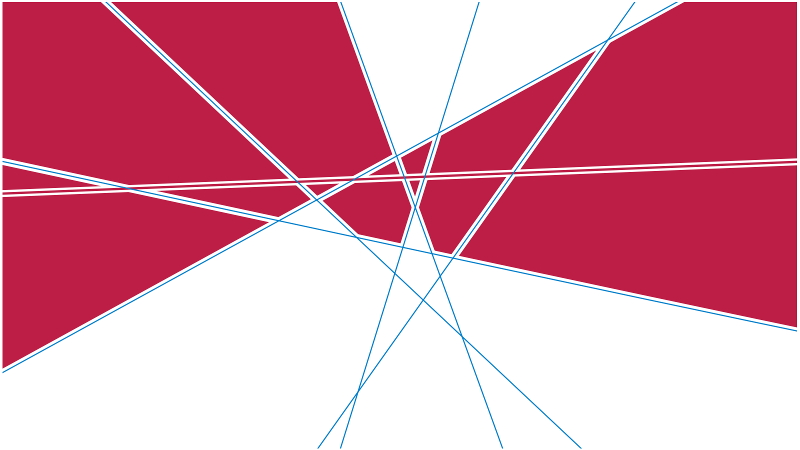
$$\#\text{ segments} \leq n(n-2) = O(n^2)$$

Each face has a bottom vertex or extends infinitely far down
Infinite-downward faces are separated by n lines $\Rightarrow$

$$\#\text{ faces} \leq \binom{n}{2} + n + 1 = O(n^2)$$

Zones

Zone = faces touched by a line = new faces after inserting the line



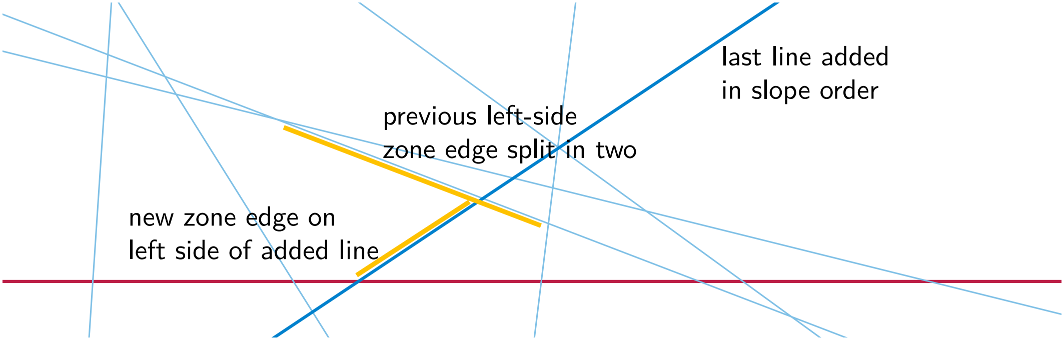
Its complexity (sum of # edges in each face) controls the time taken for walks across faces when adding the line to the arrangement

Zone theorem: For every zone, the complexity is $O(n)$

Ideas for proof of zone theorem

Rotate so the given line is horizontal
Add other lines to zone, sorted by slope (negative to positive)

Each line creates ≤ 2 edges on left sides of lines, above given line:
One on new line, one formed by splitting an existing edge



Summing up zones

Zone has:

- $\leq 2n$ half-edges above the horizontal line, on the left side of each line
- $\leq 2n$ half-edges above the horizontal line, on the right side of each line
- $\leq 2n$ half-edges below the horizontal line, on the left side of each line
- $\leq 2n$ half-edges right the horizontal line, on the right side of each line
- $\leq 2n$ half-edges on the horizontal line itself

Total: $\leq 10n$ by this analysis

Actual max complexity is $\lfloor 9.5n \rfloor - 3$ [Bern et al. 1991; Pinchasi 2011]

Analysis of incremental arrangement construction

Each zone has complexity $O(n)$

Adding each line to the arrangement takes time proportional to its zone (in the arrangement so far), $O(n)$

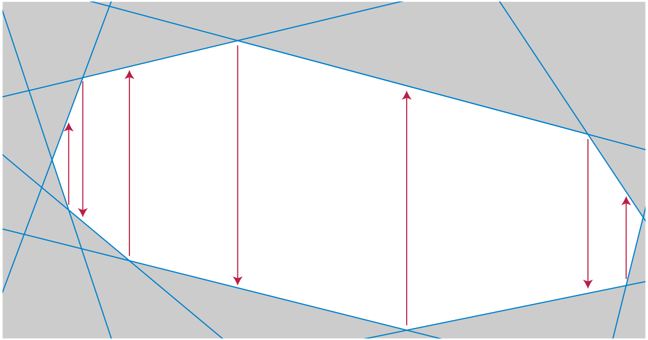
Constructing the whole arrangement takes time $O(n^2)$

To find the minimum-area triangle

For each face of arrangement:

- Merge top and bottom left-to-right sequences of vertices to find top edge above each bottom vertex, and bottom edge below each top vertex
- Each edge-vertex pair gives you a candidate triple of input lines

Return the smallest triangle among the candidate triples



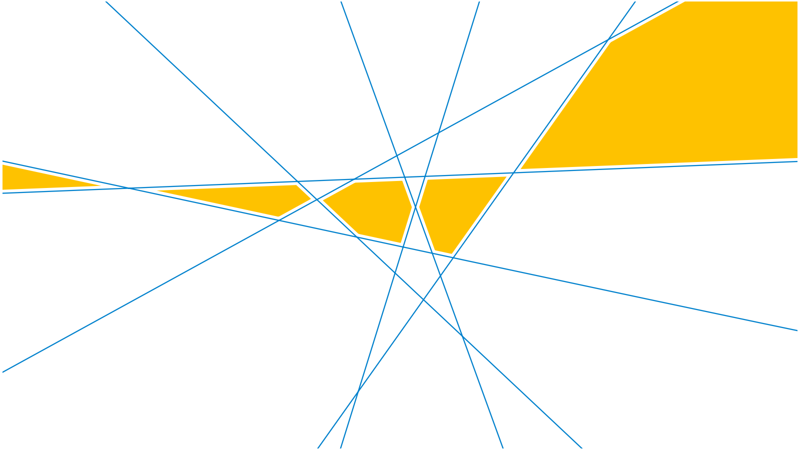
Time = linear in DCEL = $O(n^2)$

The middle level problem

Middle level = faces with $\lfloor n/2 \rfloor$ lines above and $\lceil n/2 \rceil$ lines below

Big unsolved problem: What is its complexity?

Dual: In how many ways can we bisect n points by a line?



$O(n^{4/3})$ [Dey 1998] $\quad \Omega\left(ne^{c(\log n)^{1/2}}\right)$ [Tóth 2001]

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