Flipping Cubical Meshes

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Problem:

Understand local improvement operations in quadrilateral and hexahedral meshing

Goals:

Find set of operations sufficient for all desired mesh topology changes
Relate local improvement to problems of mesh existence

Approach:

Define and analyze natural set of “flips”
by analogy to standard flips for triangle and tetrahedron meshes

Types of hexahedral mesh

Topological – abstract cell complex

Main focus of work on mesh existence
Not much use in practice

Warped – complex w/vertex locations

Cell facets are reguli bounded by warped quadrilaterals
Preferably non-self-intersecting
Main type of mesh used in practice

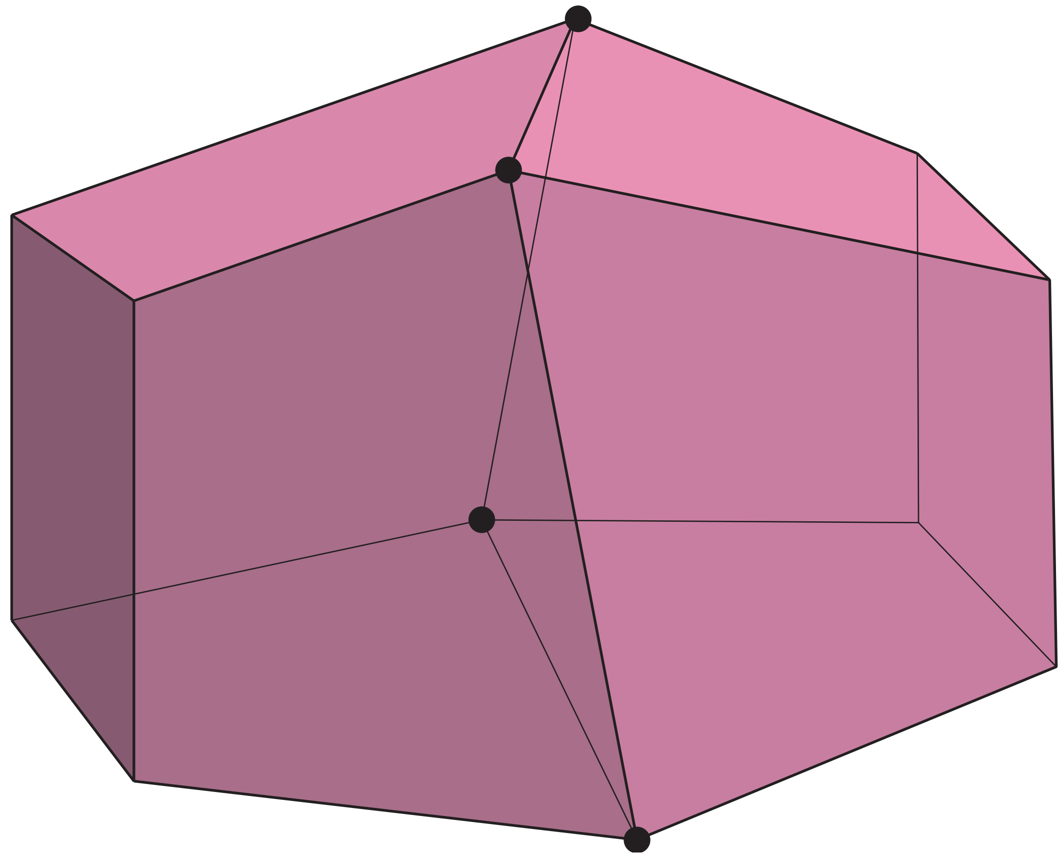
Geometric – polyhedral subdivision

Main focus of work in computational geometry
Not used much in practice because difficult or impossible to generate,
other quality criteria more important than flat facets

Open question:

Understand geometric hex mesh existence

E.g. does bicuboid with warped equator have a geometric mesh?



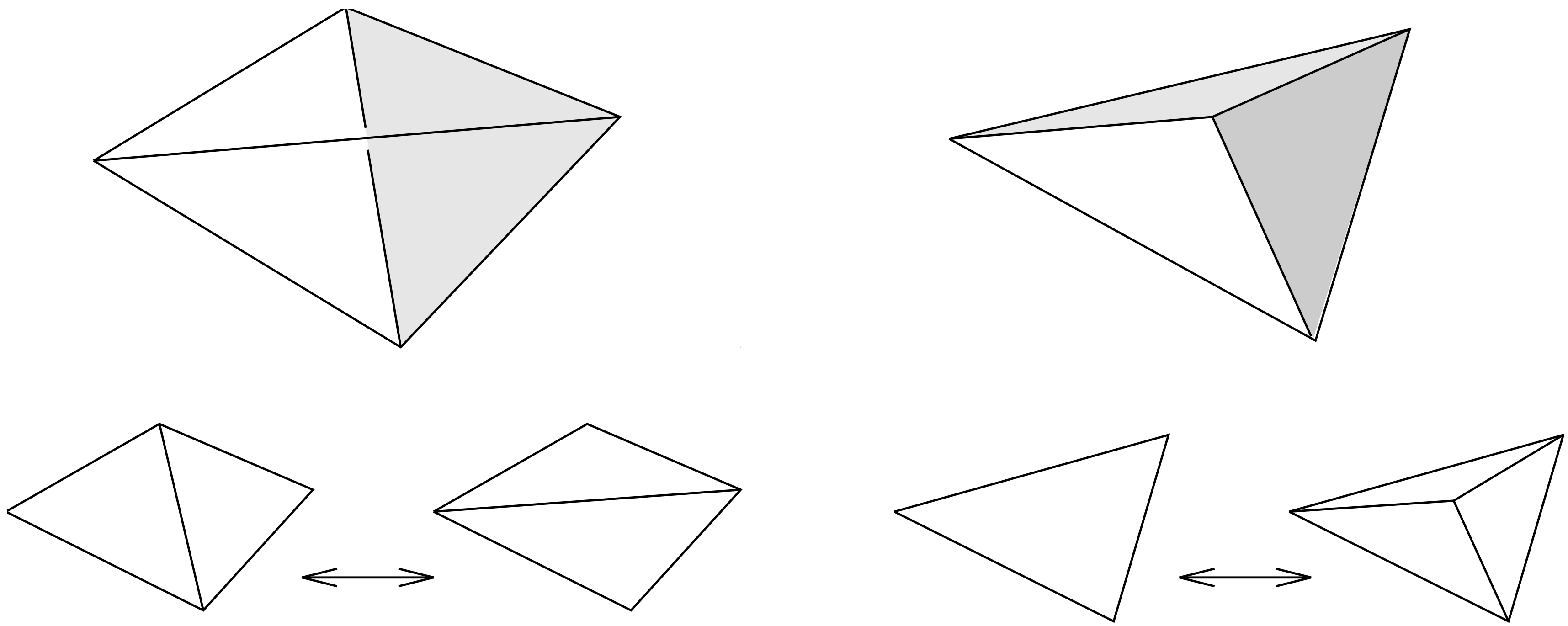
Seems to be the hard case for geometric hex meshing more generally
(other domains can be decomposed into bicuboids)

Flipping

Small set of local connectivity-changing operations
Applied in greedy fashion to improve mesh

Simplest case: triangle mesh. Two types of flip:

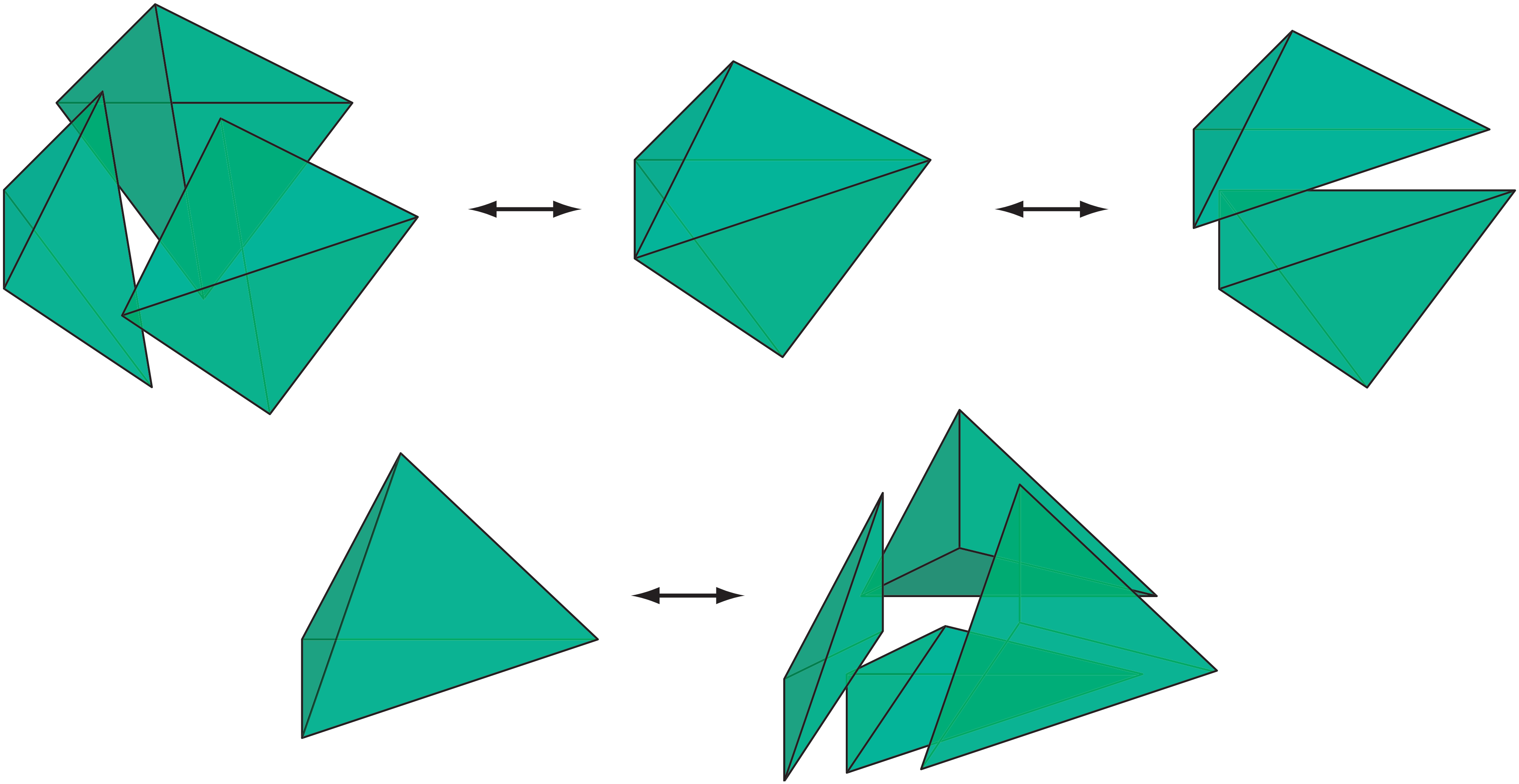
switch diagonal of quadrilateral (2-2)
add/remove degree three vertex (1-3 or 3-1)



Initial and final configurations of a flip
can be viewed as projections of the bottom and top faces of a tetrahedron

so flipping = gluing tetrahedron onto top of 3d “history mesh”
having the desired 2d mesh as its top surface

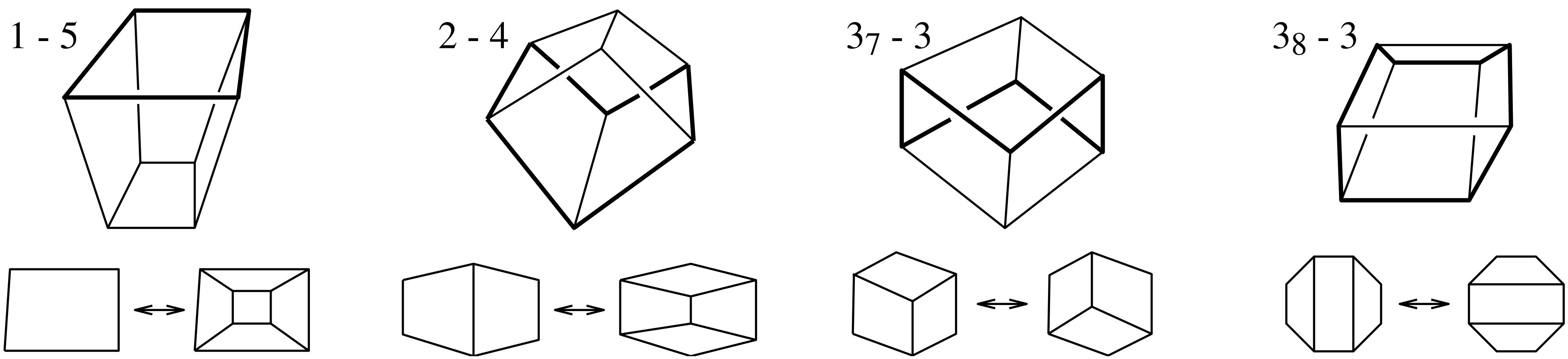
Tetrahedron mesh flips: 2-3 or 1-4



Can be viewed as swapping top/bottom views of 4d simplex
Similar sets of flips generalize to any dimension

Quadrilateral flips

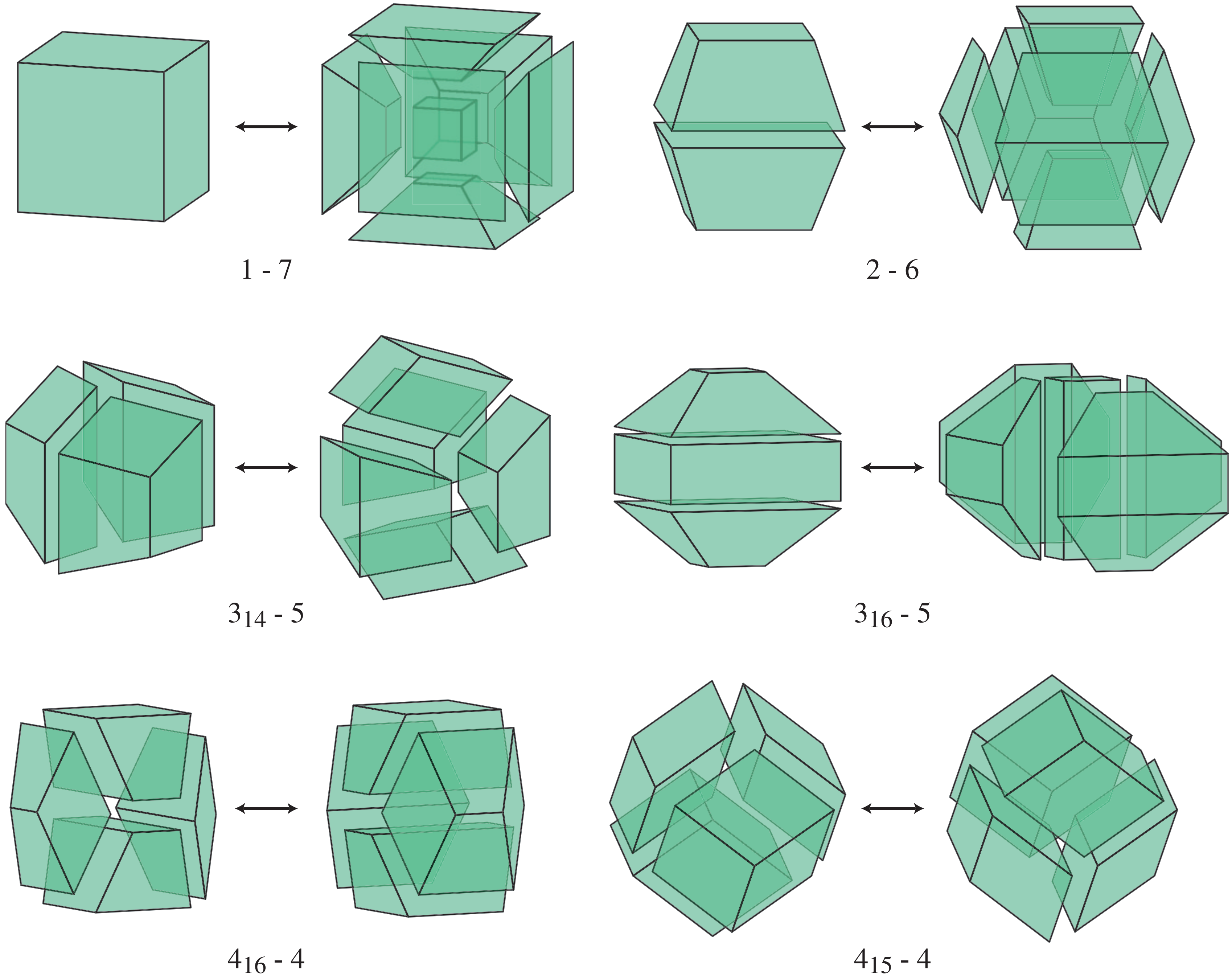
By analogy to triangle/tetrahedron flips,
define as swapping top/bottom views of a cube



Possibilities:

One quad split into five or vice versa
Two quads replaced by four or vice versa
Three quads turned into three rotated quads (two different ways)

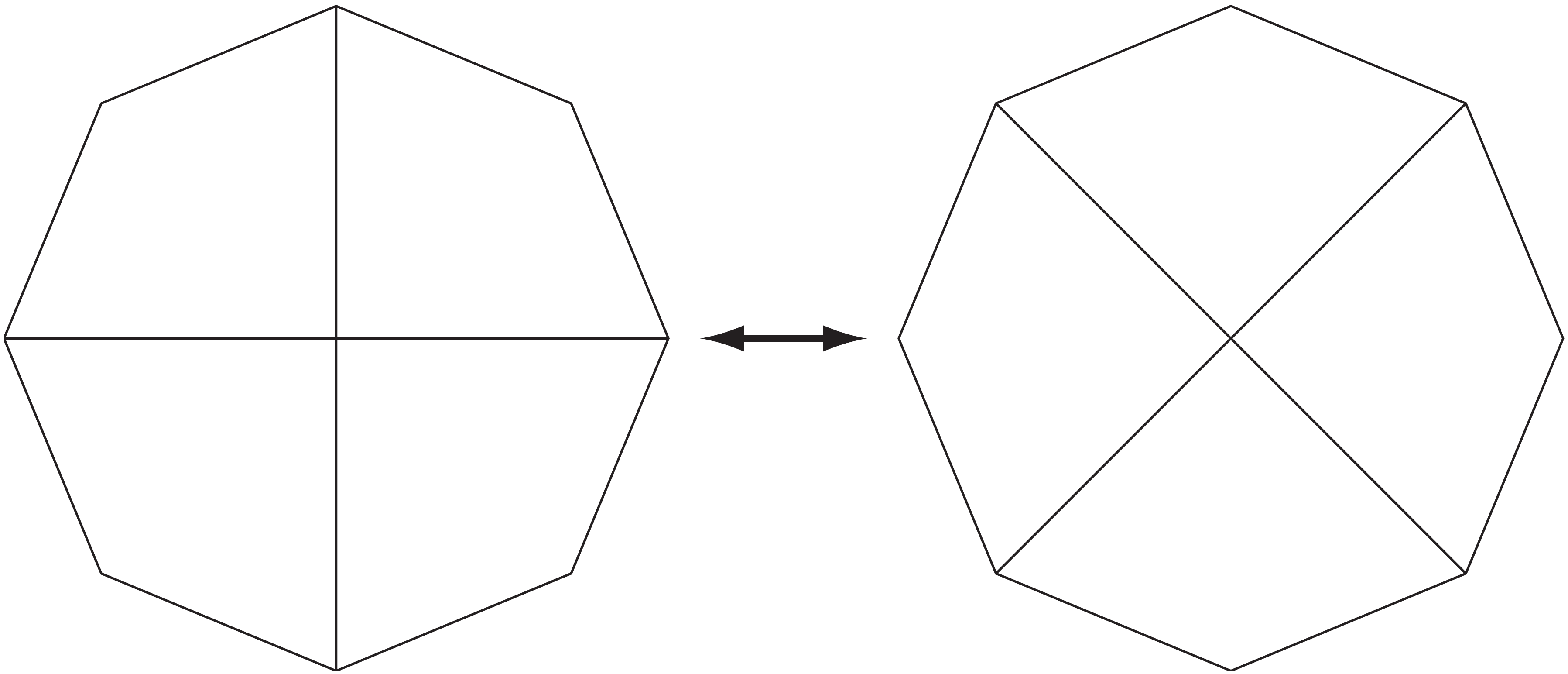
Hexahedral flips



Are flips enough?

I.e. can they substitute for any other local replacement?

A difficult example:



Some useful tools:

Duality

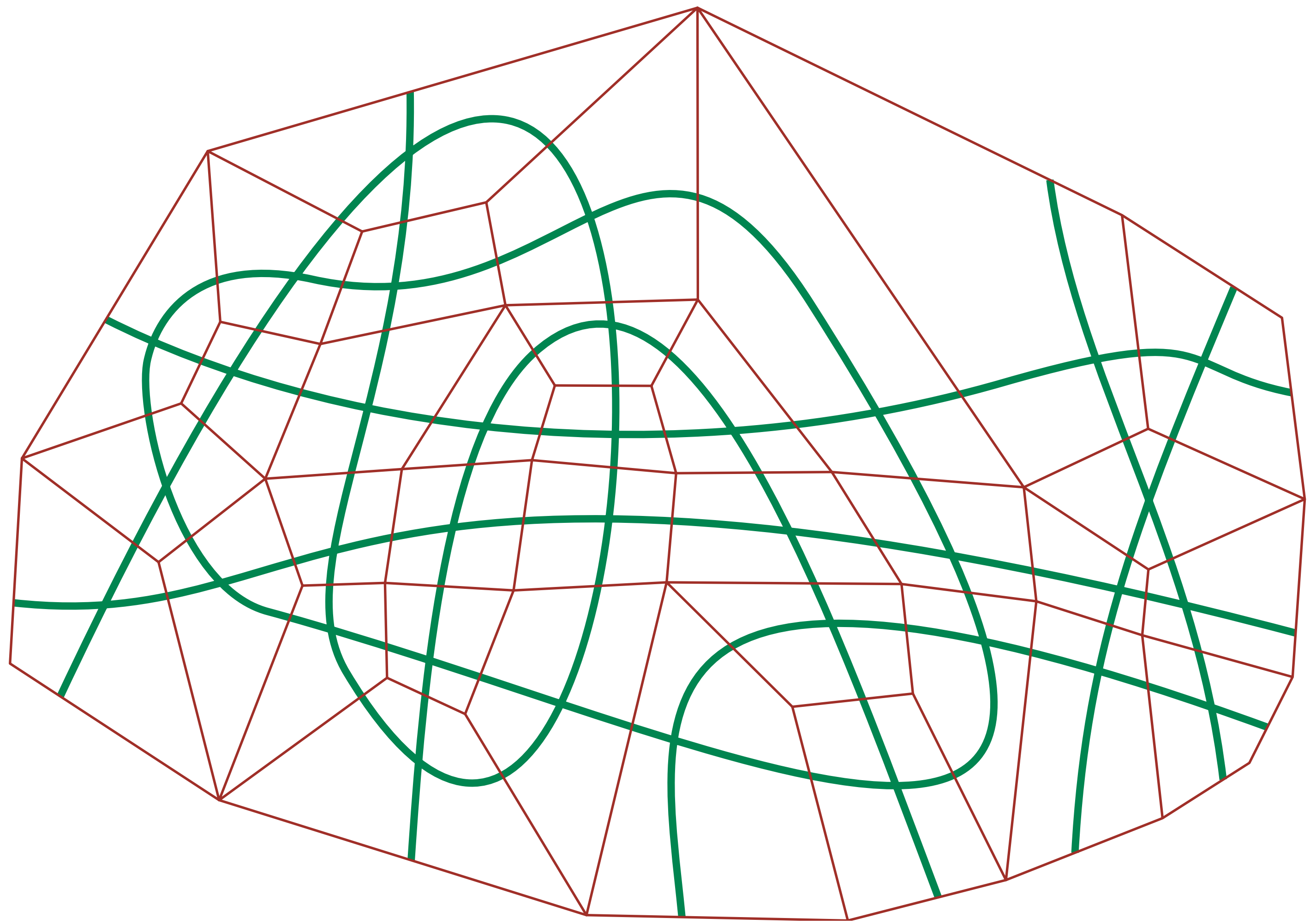
Equivalence between meshes and
arrangements of curves and surfaces

Topological mesh existence results

Sufficient conditions for mesh to exist
Necessary conditions without which mesh cannot exist

Unfortunately, don't always match...

Duality for Quadrilateral Meshes



Draw curves connecting opposite edges of each quadrilateral
Subdivides quadrilateral into four pieces

Mesh corresponds to curve arrangement connecting midpoints of boundary edges
(connected, with no multiple adjacencies among arrangement vertices)

May possibly include curves nonadjacent to boundary

Which 2d domains can we mesh?

Simple necessary and sufficient condition:
even number of edges

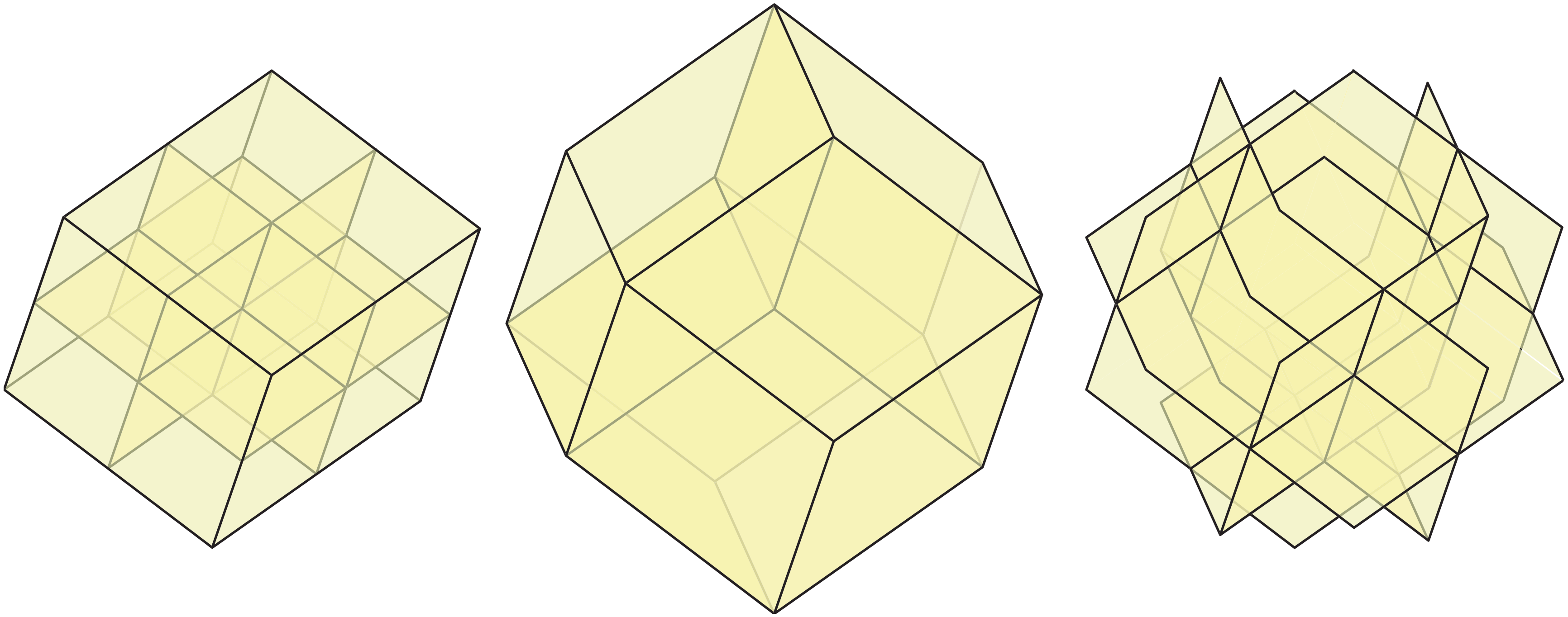
Why necessary? Each quad has an even number of edges,
and interior edges match up in pairs

Equivalently, each dual curve has an even number of endpoints

Why sufficient? Choose points at each edge midpoint,
form curves connecting pairs of points
Use duality to turn curve arrangement into topological mesh

More complicated techniques can be used
to construct a geometric mesh of convex quadrilaterals

Duality for hexahedral meshes



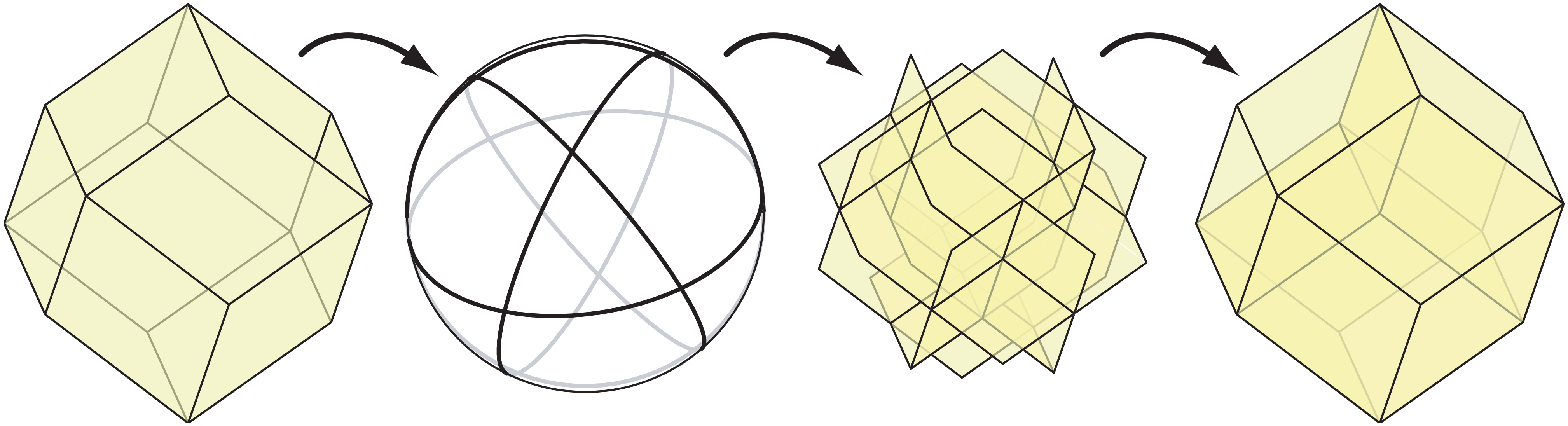
Left: cuboid subdivided by three surfaces into eight pieces
Center: four-cuboid mesh of rhombic dodecahedron
Right: dual surface arrangement

Hex mesh corresponds to arrangement of surfaces
meeting domain boundary in dual of boundary quad mesh
connected skeleton, no pinch points, no multiple adjacencies

May possibly include surfaces nonadjacent to boundary
surface can self-intersect, no requirement of orientability

Which 3d domains can we mesh? [Mitchell and Thurston, 1996]

Even # facets sufficient for topological mesh of simply connected 3d domains



Dualize boundary mesh to curve arrangement on sphere

Extend curves with even # self-intersections to surfaces [Smale]
pair up odd curves and similarly extend to surfaces

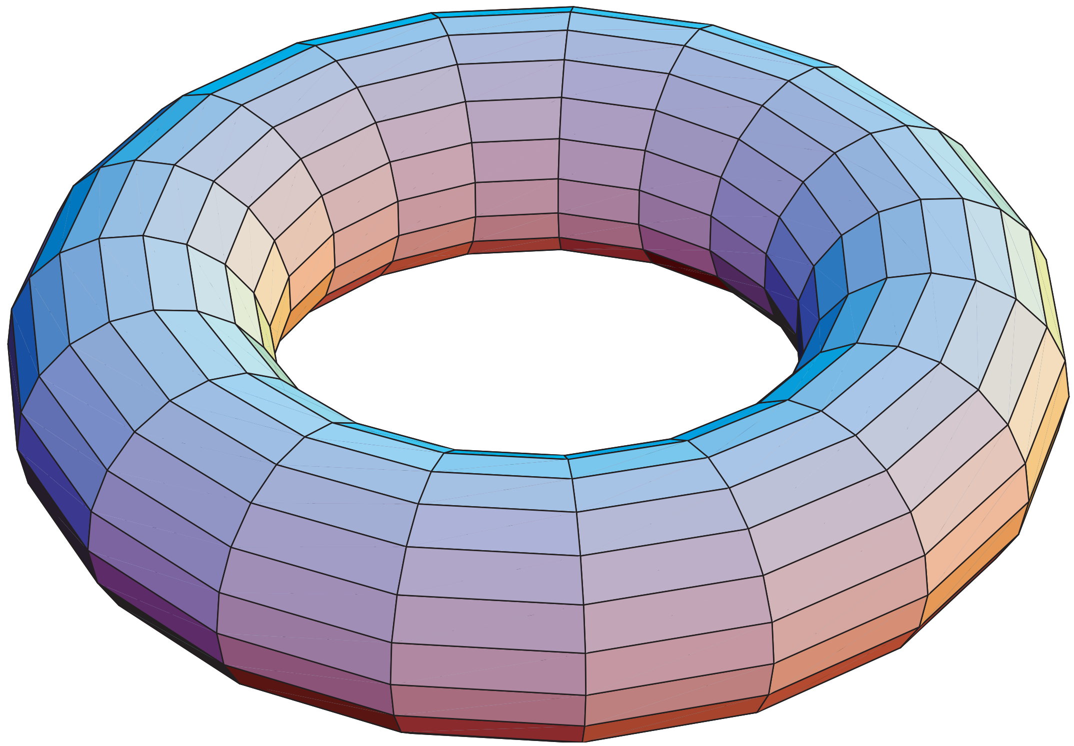
Add extra surfaces to enforce no-multiple-adjacency rules

Dualize surfaces back to hexahedral mesh

Extensions to non-simply-connected domains? [Mitchell & Thurston]

Necessary: no odd cycle of skeleton bounds a surface in domain

Because intersection with mesh's dual surfaces
would form curves with an even number of endpoints



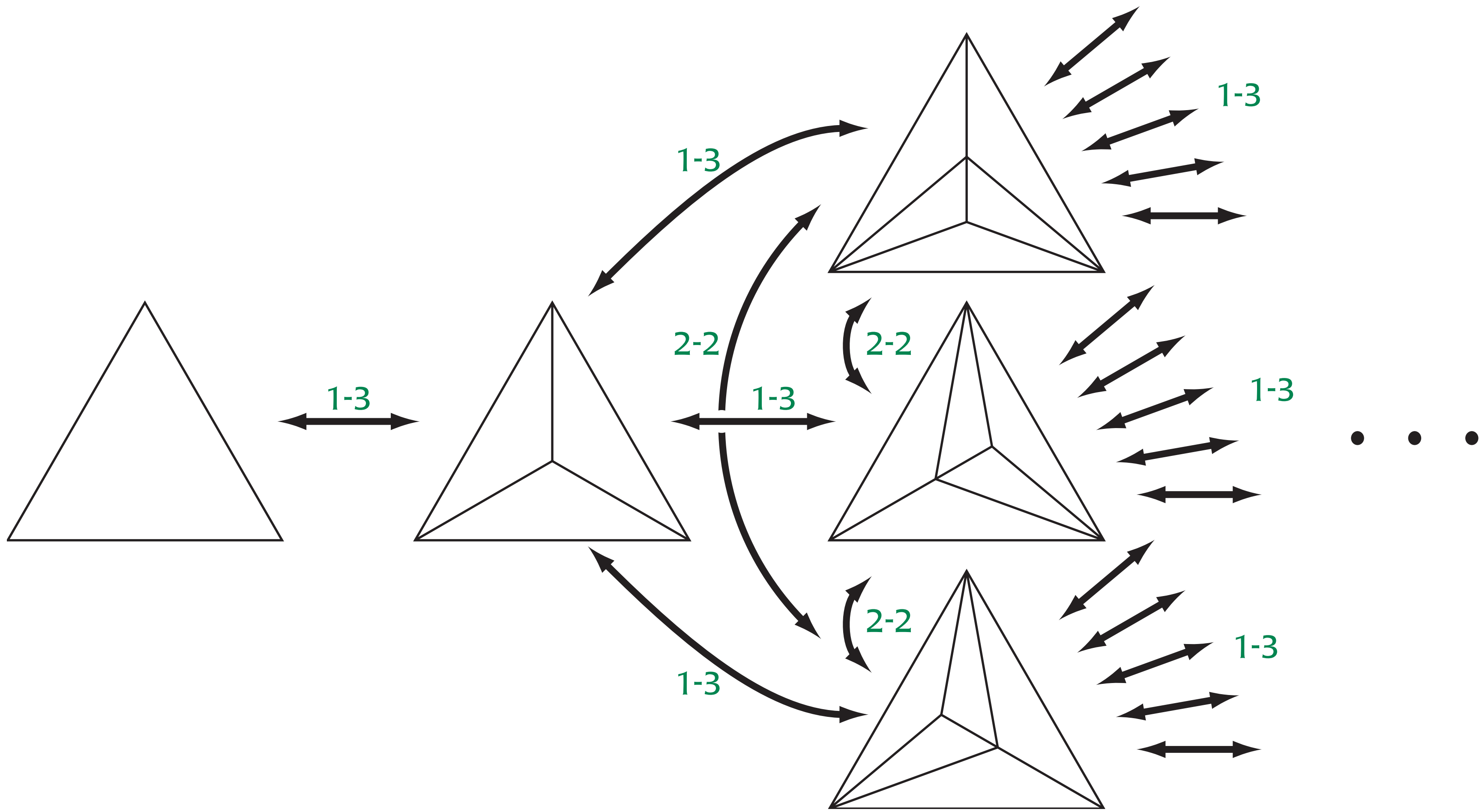
Sufficient: handlebody, each handle can be cut by an even cycle

Cut the handles by disks
Form quad mesh on each disk
Mesh the resulting simply-connected domain

Flip Graph

vertices = meshes on some domain, edges = flips between meshes

Always connected for triangles, (topological) tetrahedra
open whether connected for geometric tetrahedra



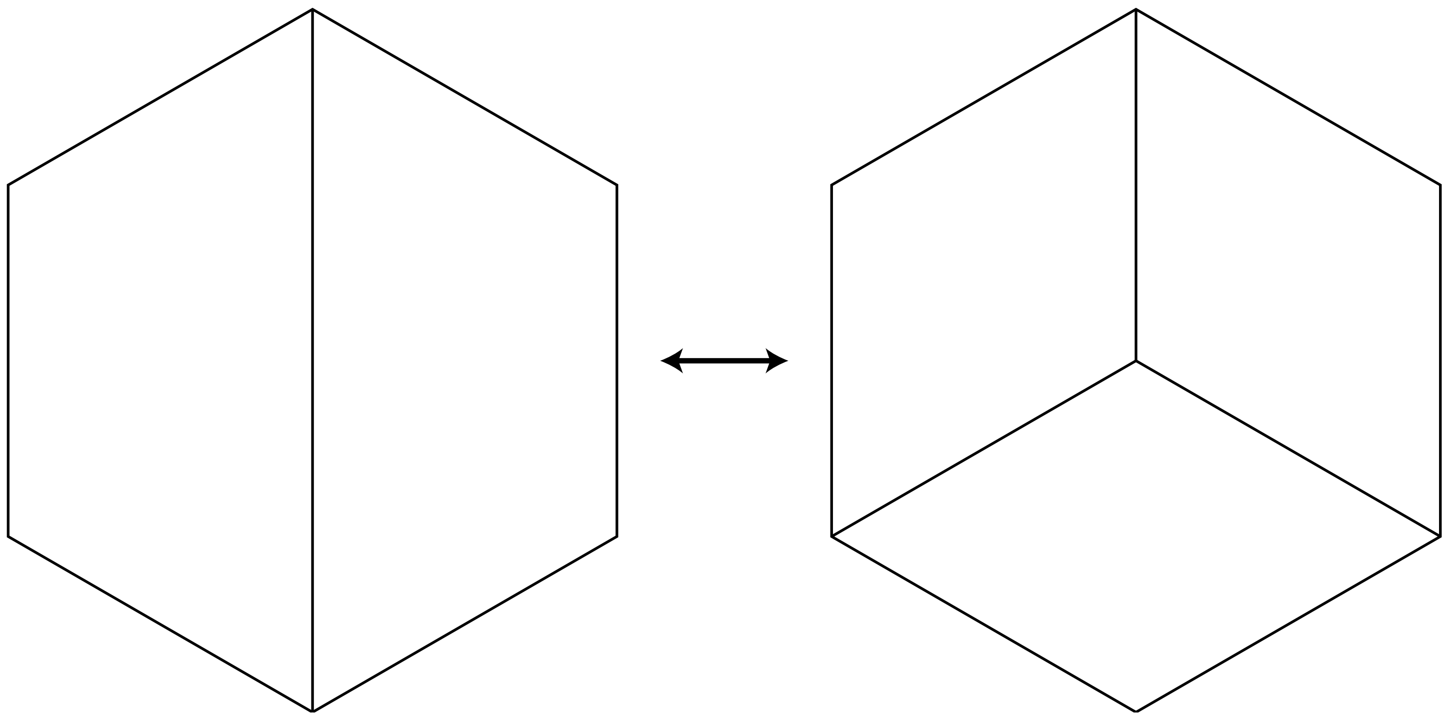
Is flip graph connected for quadrilaterals, hexahedra?

Possibly different answers for different domains, topological vs geometric meshes

Flips preserve parity

Cube and hypercube have even numbers of facets
so quadrilateral and hexahedral flips always replace odd-odd or even-even

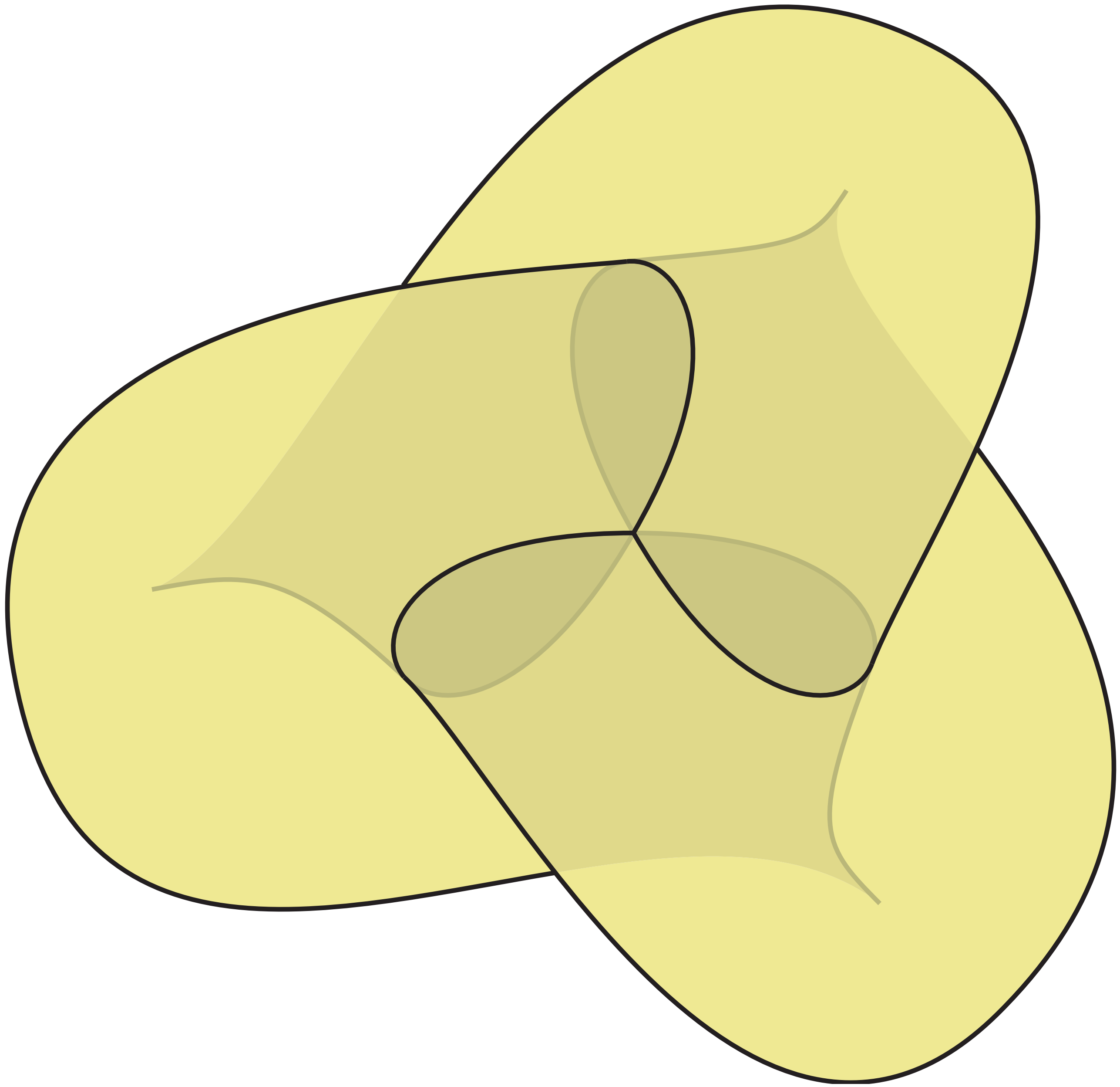
But same domain can have both odd and even meshes:



So flip graph is not connected

How to change parity in hexahedral meshes

Add a copy of Boy's surface to dual surface arrangement



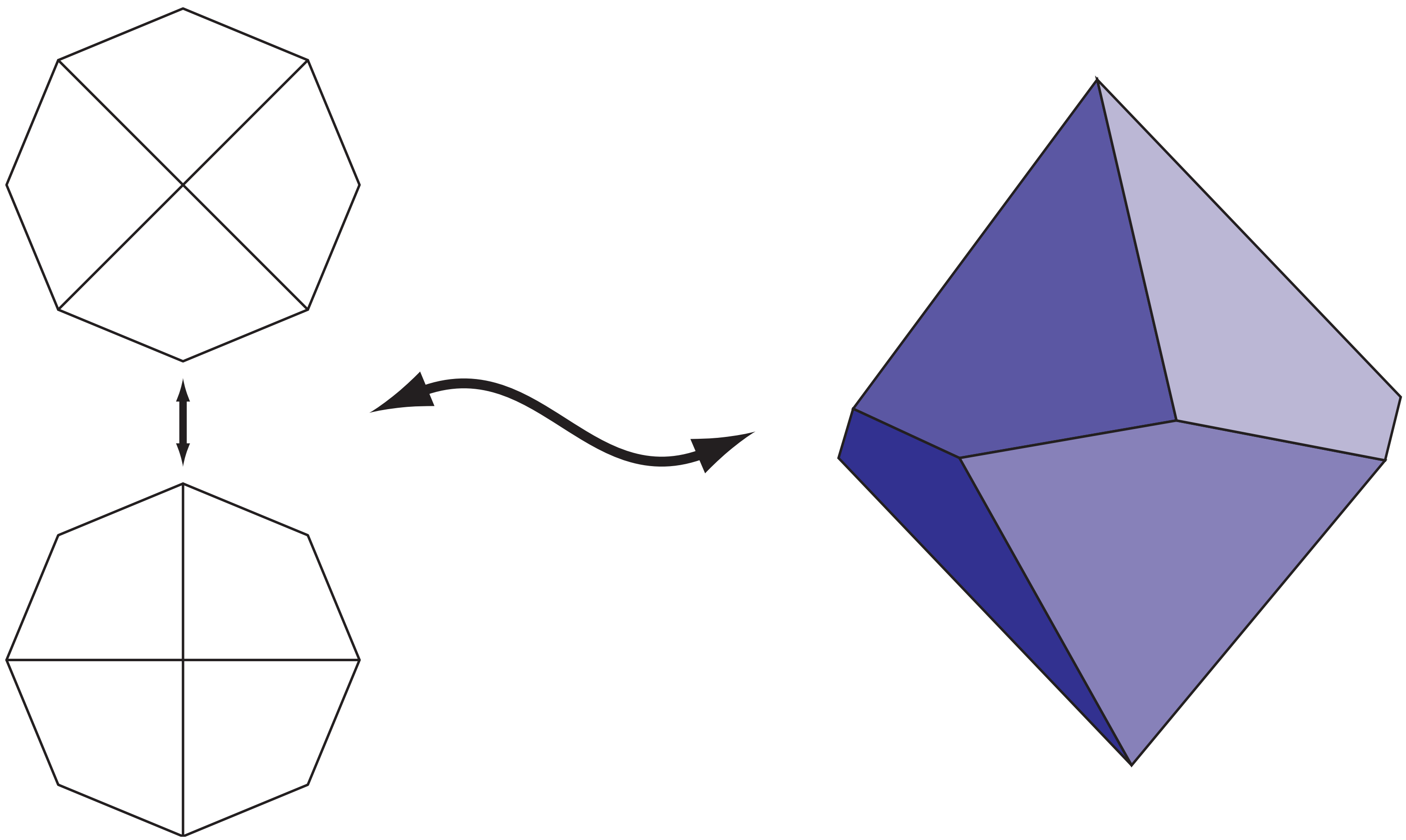
One new hex from self-triple-intersection, even number from intersections w/other surfaces

...but parity is the only obstacle to flipping!

Theorem:

Any equal parity quadrilateral meshes of a topological disk can be connected by a sequence of flips

Proof idea: View two meshes as top and bottom surfaces of a 3d domain



Use a hexahedral mesh to determine set of flips
BUT flip sequence ~ shelling, so need shellable mesh

More details of connectivity proof

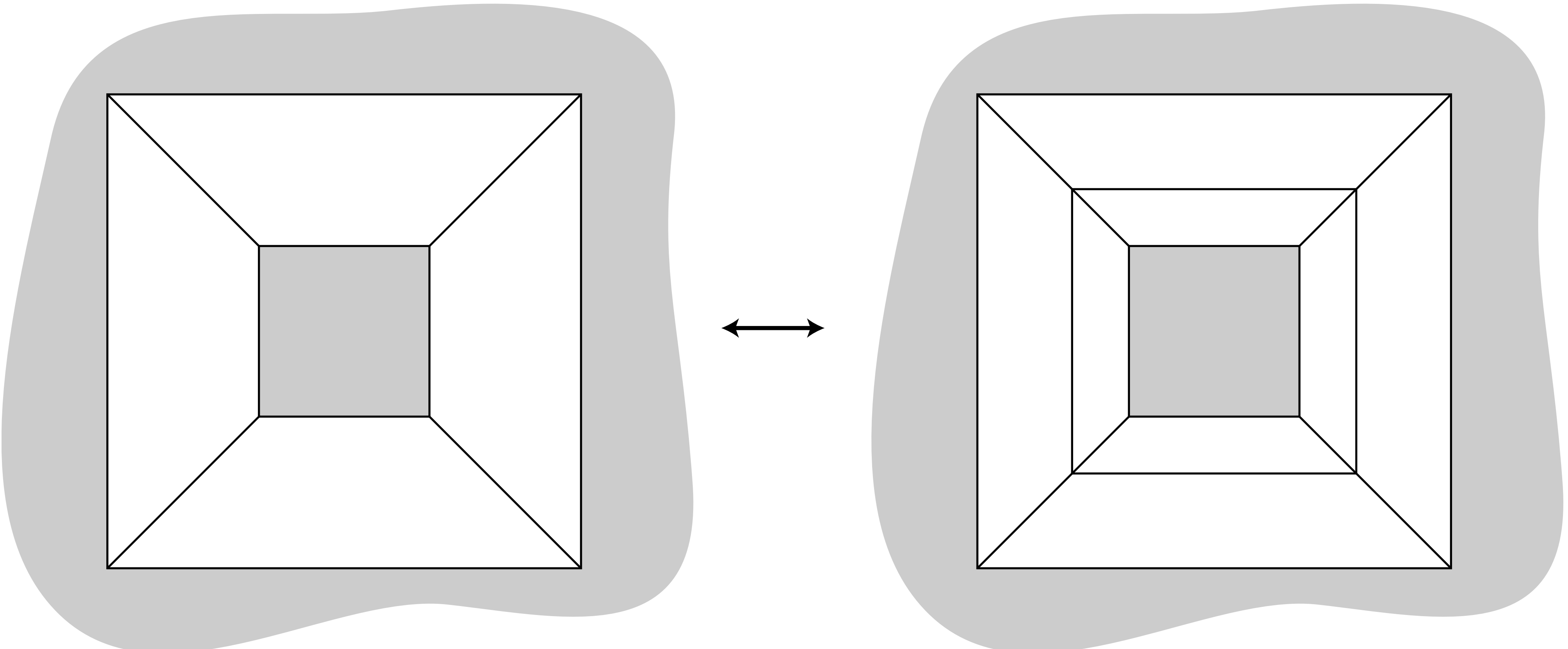
Mesh 3d domain [e.g. via Mitchell & Thurston]
Form dual surface arrangement

Add additional concentric spheres to arrangement
(forming concentric layers of cuboids in mesh)

Drill to center by removing one cuboid per layer
Then remove one layer at a time inside-out
Use drilling + layer removal as shelling/flipping order

Shellability of planar maps allows correct removal of each layer

Flips are not enough when the domain is not a disk



Two even-parity meshes of an annular domain

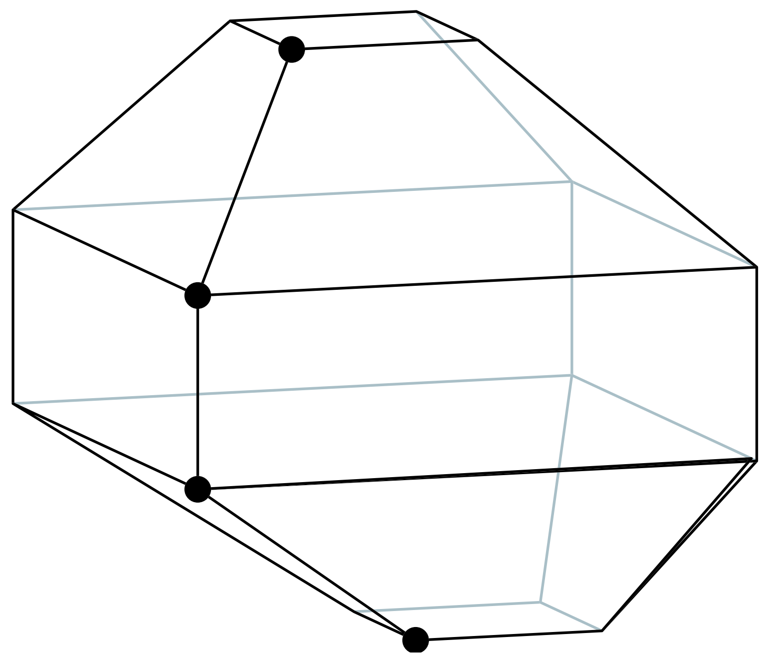
If they could be connected by flips,
flip sequence would give hexahedral mesh of 3 by 4 torus
impossible due to interior triangle

Bicuboid revisited

1-7, 2-6, 6-2, 7-1 flips preserve flatness of facets

Meshes reachable from warped 2-cuboid mesh of bicuboid must also be warped
Rules out many but not all meshes for the bicuboid

However, not all flips preserve flatness:



This polytope has a flat 3-hex mesh but not the flipped 5-hex mesh.

Conclusions

Defined natural set of flips

Based on exchanging sets of faces of cube or hypercube

For quad flips, sufficient (together with parity change) to simulate any other local connectivity change

Flip sequence closely related to 3d hex mesh

Good bounds on number of flips needed

For hex flips, progress on bicuboid mesh existence

Many potential geometric meshes do not work
due to flatness-preserving flips

More open questions:

Non-simply-connected 2d domains?

Classify connected components of quad-mesh flip graph

Since all local changes can be simulated by flips,
some non-local changes are needed – what is a good set?

3d flip graph connectivity?

Can use same idea of lifting dimensions and using mesh to guide flips

Need to understand which 4d domains have hypercube meshes