Limitations on Realistic Hyperbolic Graph Drawing

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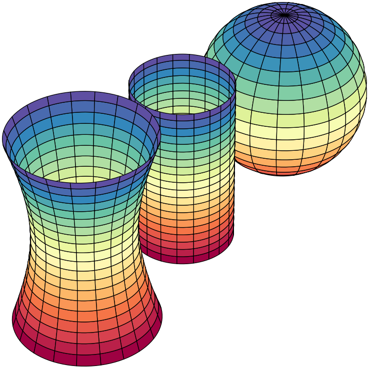
Gaussian curvature



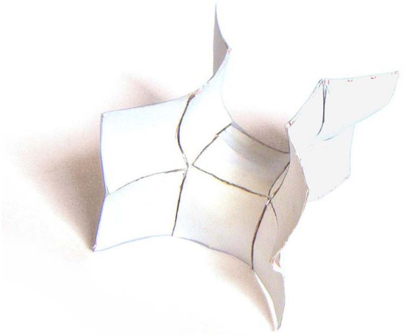
We usually draw graphs in the Euclidean plane (curvature $= 0$) but other surfaces like the cylinder also have curvature 0

Sphere has uniformly positive curvature

Other surfaces can have negative curvature everywhere

Hyperbolic geometry

Topologically equivalent to the usual Euclidean plane
negative Gaussian curvature, equal at all points



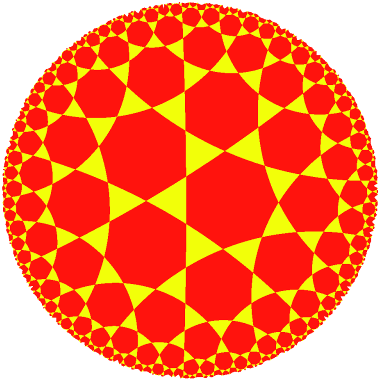
Small patches embed into $\mathbb{R}^3$

Exponential area vs radius
makes bigger more difficult

Euclidean models of the hyperbolic plane

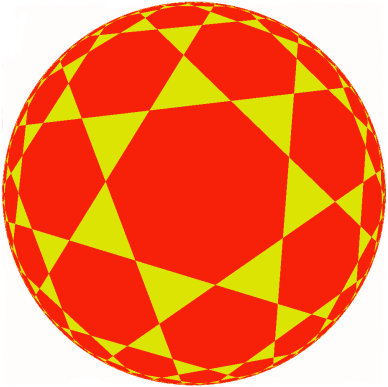
View hyperbolic plane by 1-to-1 map to (part of) Euclidean plane

Many possibilities, including. . .



Poincaré disk model

Plane ↦ disk, lines ↦ circular arcs perpendicular to disk



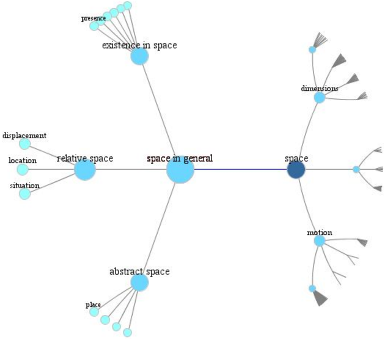
Klein model

Plane ↦ disk, lines ↦ straight line segments

Hyperbolic graph drawing

Idea: Draw graph in hyperbolic plane, view in Euclidean model

[Lamping and Rao 1996]



Initial motivation: Poincaré "fisheye view" allows both focus at a movable point of interest + context of whole drawing in single view

Drawings are not embeddings

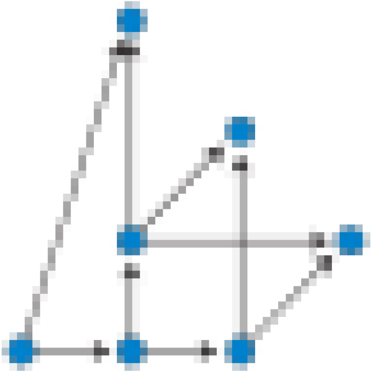
All features have nonzero thickness

- Vertex-vertex separation $\Omega(1)$
- Vertex-edge separation $\Omega(1)$

All features should be visible
(not hidden by other thick features)

[Barequet et al. 2004; Duncan et al. 2006; van Kreveld 2011; Pach 2015]

“Bold” or “realistic” graph drawing

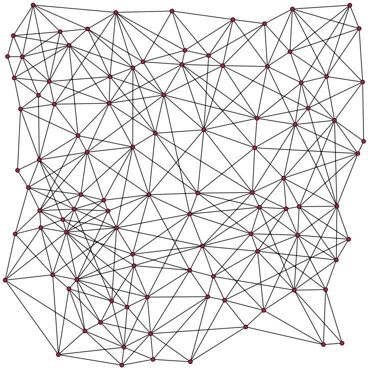


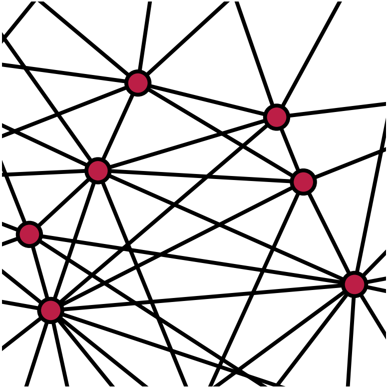
This work: Explore what this means for hyperbolic drawing

Embed – Zoom – Draw

For drawings in the Euclidean plane, scale is arbitrary

You can zoom an embedding to focus on an arbitrary detail, then render a drawing of the embedded graph at the zoomed scale



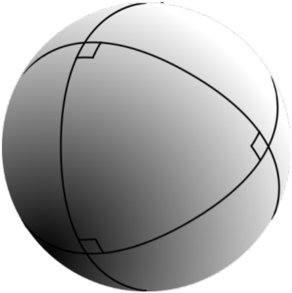


Hyperbolic scale is not arbitrary

Hyperbolic geometry has a natural unit of distance

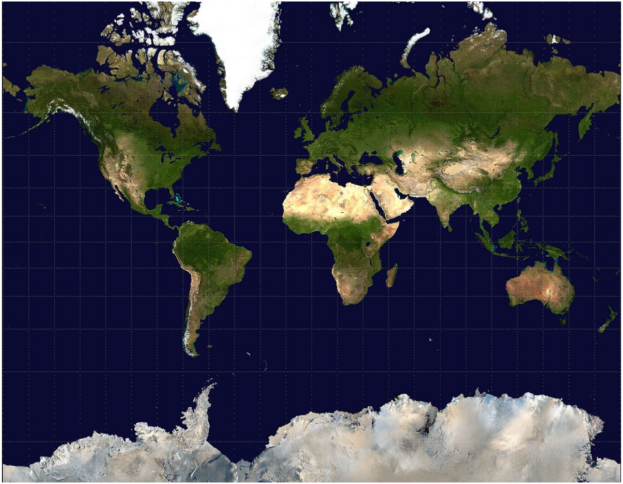
Objects of unit diameter, at the center of a Poincaré view, fill a constant fraction of the view

Analogy: natural unit in spherical geometry is the radian, measuring the angle between two points as viewed from the center of the sphere



Focus+context Poincaré view cannot zoom

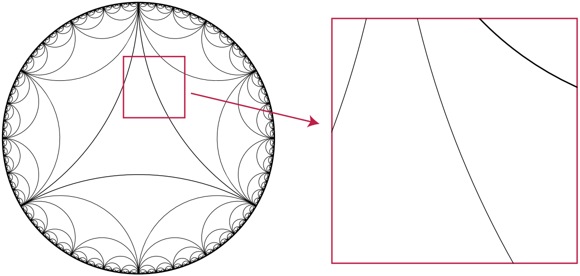
It's a fixed map whose only choice is what point to put at its center



Analogy: Mercator world map, fitted to your screen, cannot zoom

Even if you zoomed...

Suppose you crop a small rectangle from the Poincaré view

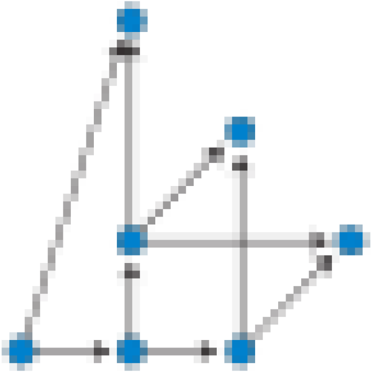


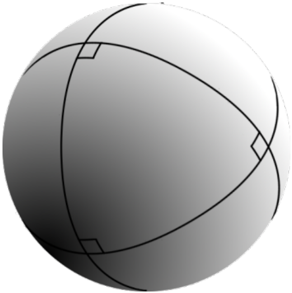
You lose the context from the focus+context

Within the crop, geometry is approximately Euclidean
(so no added flexibility from hyperbolic geometry)

What has realistic drawings in constant curvature?

For sphere drawings (uniform positive curvature),
vertex separation $\Omega(1) \Rightarrow O(1)$ vertices, trivial!





Hyperbolic plane is nontrivial: all trees have good drawings

What more can we say?

New results

Several types of hyperbolic drawing are forced to have sub-constant feature sizes, in natural hyperbolic length units:

- Every straight-line maximal planar graph has vertex-edge resolution $O(1/\sqrt{n})$; nested triangles has $O(1/n)$
- Constant vertex-vertex resolution is possible but leads to exponentially small angular resolution
- Non-planar graphs with constant vertex-vertex resolution may have angular resolution $O(1/n^2)$ (versus $O(1/n)$ Euclidean)
- Some graphs require edge width $O(1/n)$ to make all features visible in drawings with constant vertex-vertex resolution

Planar vertex-edge resolution intuition

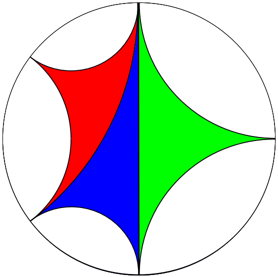
Hyperbolic triangle area $\leq \pi$

This is true of the outer face of any maximal planar drawing

$\Rightarrow$ smallest interior face has area $O(1/n)$

area of a hyperbolic triangle is $\Omega\big(\min(1, \text{height}^2)\big)$

$\Rightarrow$ vertex is close to an edge



The largest hyperbolic triangles have area exactly π, with all three vertices at infinity

Nonplanar edge thickness intuition

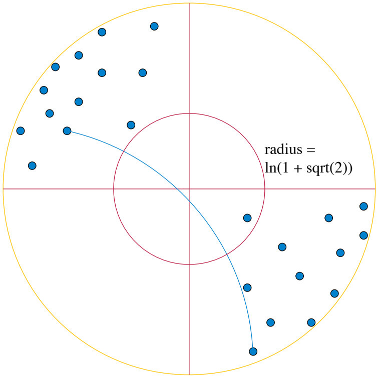
Start with any drawing of K_n

Some two perpendicular lines bound opposite quadrants with $n/4$ points (same as Euclidean)

Edges from one quadrant to the other all pass within distance $O(1)$ of line crossing

Pigeonhole $\Rightarrow$ spacing of edges with same endpoint near crossing is $O(1/n)$

If lines are thick, some edge completely obscured by other edges with same endpoints

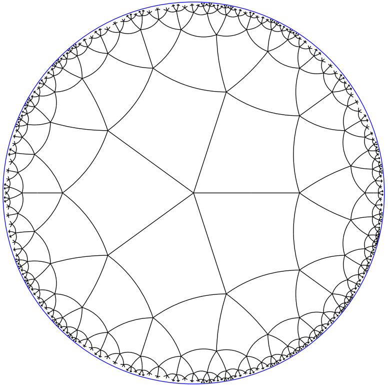


Conclusions

Some specific types of hyperbolic drawings don't work well

. . . but hyperbolic embeddings are still useful!

greedy routing, construction of Lombardi drawings, etc



Some graphs (e.g. trees or the tiling above) have nice hyperbolic drawings; can we characterize them?

References and image credits I

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