Diamond-Kite Meshes: Adaptive Quadrilateral Meshing and Orthogonal Circle Packing

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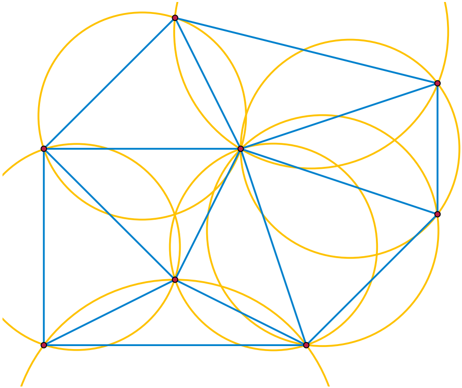
Circle packing has a long history in mesh generation

Prototypical result: triangles of a Delaunay triangulation correspond to empty circles through triples of vertices



Circle packing has a long history in mesh generation

Introduced as an explicit mesh generation technique by Bern, Mitchell, and Ruppert ("Linear-size nonobtuse triangulation of polygons", SoCG 1994 and DCG 1995)

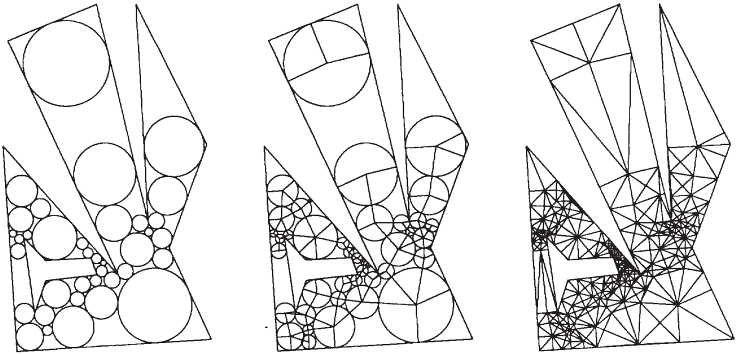
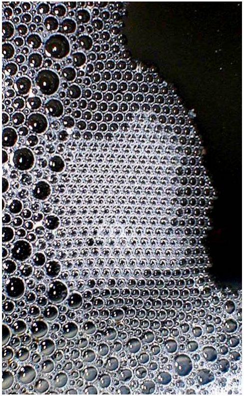


Figure 1. (a) Disk packing. (b) Induced small polygons. (c) Final triangulation.

How to construct circle packings

Unstructured circle packings have been constructed by

- Placing circles one at a time until all regions have few sides [Bern, Mitchell, and Ruppert et seq.]
- Simulating physical motion of incompressible circles [Shimada and Gossard, "Bubble mesh", SMA 1995]
- Oversampling too many circles then removing overlaps [Miller, Talmor, Teng, and Walkington, STOC 1995]



File:Soapbubbles1b.jpg by

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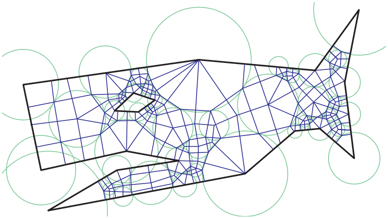
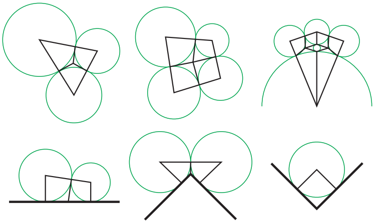
Meshing applications of circle packings

Circle packings can lead to well-behaved meshes

- Nonobtuse
- Bounded aspect ratio
- Well-graded sizes
- Well-shaped quadrilaterals

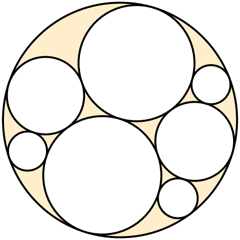




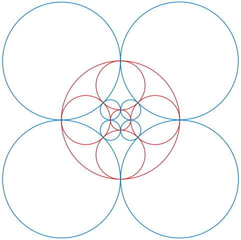
But the irregularity of the gaps between circles can lead to messy case analysis

The Koebe–Andreev–Thurston circle packing theorem

An old method for generating highly structured circle packings



The vertices of every maximal planar graph may be represented by interior-disjoint circles such that vertices are adjacent iff circles are tangent



The vertices of every 3-connected planar graph and its dual may be represented by circles that are perpendicular for incident vertex-face pairs

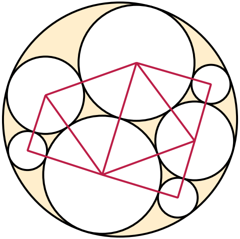
Applications of the circle packing theorem

The circle packing theorem has many applications:

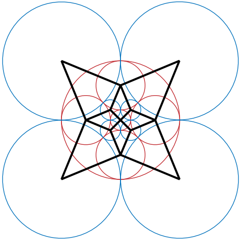
- Approximate conformal maps
  $\Rightarrow$ structured mesh generation
  [Rodin & Sullivan 1987]
- Mesh partitioning
  [Miller, Teng, Thurston, Vavasis 1997]
- Graph drawing [many]
- Brain visualization [Hurdal et al. 1999]
- Constructing soap bubbles [E., 2012]

So why hasn't it been applied directly in unstructured meshing?

Is the theorem useful for unstructured meshing?



Connecting centers of tangent circles in a maximal packing produces a triangulation



Connecting centers of crossing circles in primal-dual packing produces a quadrilateral mesh

So these types of packing give meshes without any case analysis

But, because the packing is generated from a graph, there's **no control of the shape** of the meshed region

Our results

New family of quadrilateral meshes generated by recursive subdivision operations (like quadtrees): "diamond–kite meshes"

We generate a circle packing from a mesh, rather than vice versa: the mesh vertices are centers of a valid primal-dual circle packing

Quadtree squares must be subdivided to make a mesh; instead, we generate the mesh directly without additional modifications

Mesh elements are well-shaped, well-graded, optimal in number, and adapt easily to changes in element size requirements

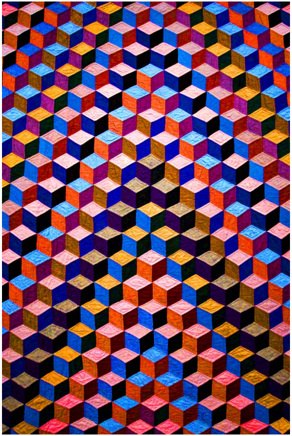
Rhombille tiling

Uniform quad mesh of 60°–120° rhombi ("diamonds")

Two kinds of vertices:

- Three rhombi meet at their obtuse angles
- Six rhombi meet at their acute angles

Short diagonals of rhombi form a hexagonal tiling of the plane

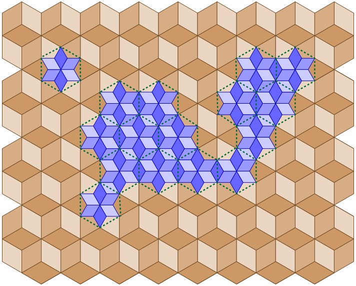


"Tumbling Blocks, Stairway to Heaven", 1935, *Amish Abstractions: Quilts from the Collection of Faith and Stephen Brown*, De Young Museum, 2009

Local replacement step

Select a subset of hexagons formed by short diagonals of diamonds

Place six smaller diamonds (reduced by $\sqrt{3}$) in selected hexes



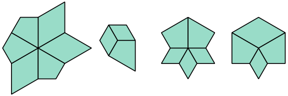
Contiguous groups of subdivided hexes form a finer rhombille tiling (rotated by 90°) within which recursive refinement is possible

Adapting a mesh to a local size function

Specify desired element size by a function that maps points in plane to the maximum allowed side length of a containing quad

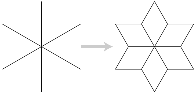
To refine at vertex p (incident to a 60° angle):

- Find all kites with right angles at p
- Recursively refine their 60° vertices
- p must have degree six; surround it by diamonds

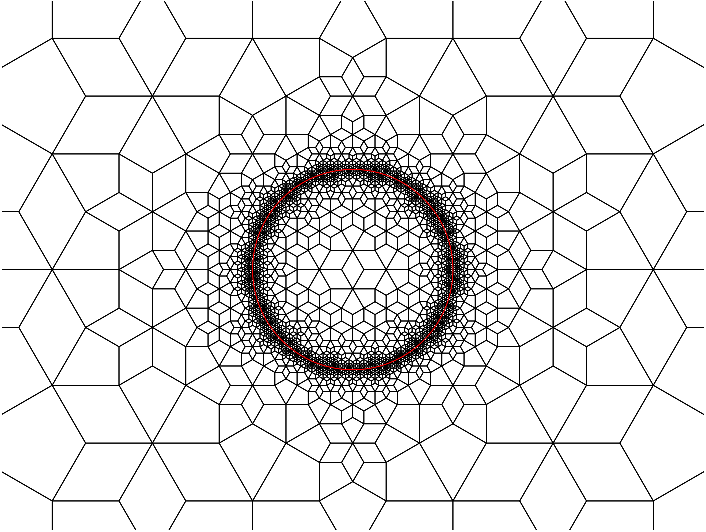


While there exists a too-large quad q:

- If q is a kite, refine its 60° vertex
- Else at least one of the two overlapping kites that cover q is too large; refine its 60° vertex



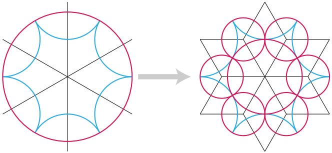
Example output

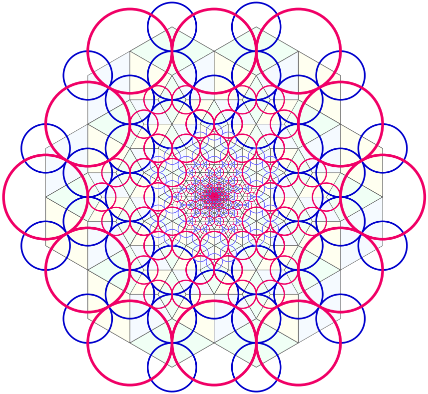


Properties of diamond-kite meshes

The vertices of the mesh are circle centers for a valid primal-dual circle packing

Proof idea: validity of the circle packing is preserved by local refinement steps





Properties of diamond-kite meshes

The prerequisite relationship between local refinement steps
(parameterized by location and size of refinement)
forms a partially ordered set

$\Rightarrow$ [Birkhoff]

Meshes formed by refining the same initial mesh
form a distributive lattice

Any two meshes have a unique coarsest common refinement
and a unique finest common coarsening

Properties of diamond-kite meshes

Define local feature size at q =
$\inf_{p \in \mathbb{R}^2}$ distance(p, q) + size requirement at p

Then every mesh with bounded-aspect-ratio elements
has element size $\mathcal{O}$(local feature size), and
the diamond-kite mesh has element size Θ(local feature size)

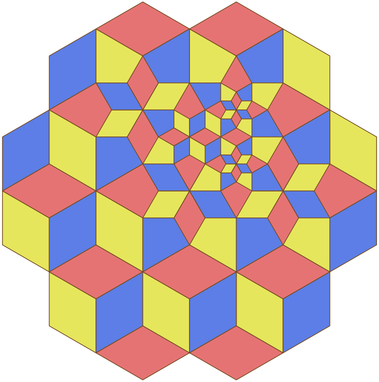
$\Rightarrow$ [Ruppert]

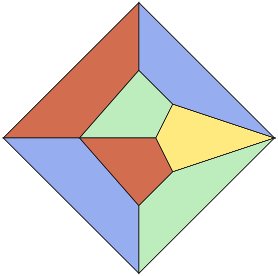
Number of elements is $\mathcal{O}$(optimal) for the size requirements

Total perimeter of elements is $\mathcal{O}$(optimal)

Properties of diamond-kite meshes

Quads can be 3-colored
by orientation of diagonals





In contrast, some other quad
meshes require four colors

Conclusions

- New type of quad mesh
- Only two element shapes, diamonds and kites
- Easy to refine and unrefine in time linear in change to mesh
- All the usual properties of guaranteed-quality meshes
- Corresponds to a primal-dual circle packing

Future work and open problems

- How to handle domain boundaries?
- Does this have any practical significance?
- The same local refinement operation makes sense in arbitrary quadrilateral meshes; does it have other applications?
- Is it possible to define a locally refinable triangle mesh that corresponds in the same way to a maximal circle packing?