

# Counting Polygon Triangulations is Hard

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# WARNING

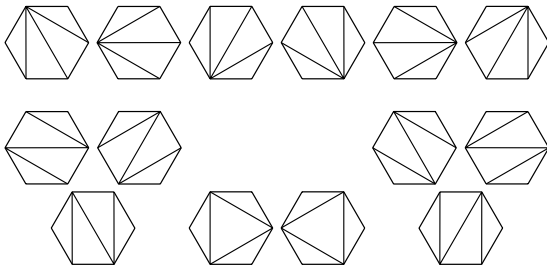
This is all just a gadget-based reduction.

There are no new ideas.



# Triangulations of $n$ convex points

How many are there?



[Dmharvey 2006]

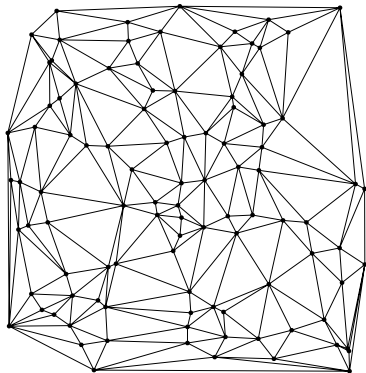
= Catalan numbers 1, 2, 5, 14, 42, ...

## When we let the point set vary...

Non-crossing graphs  
of any type are subgraphs of  
triangulations

Any  $n$ -point set has  
 $2^{O(n)}$  triangulations  
[Ajtai et al. 1982]

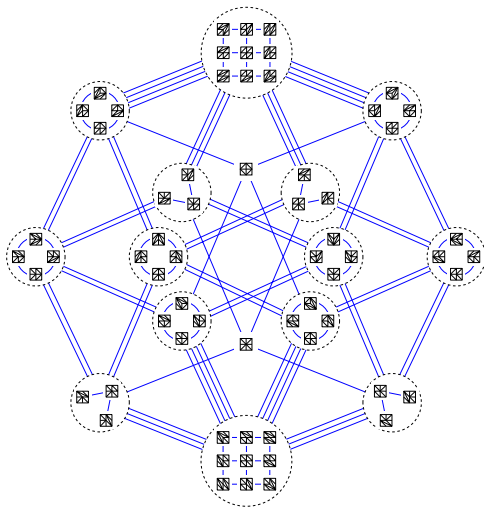
Therefore, there are  $2^{O(n)}$   
non-crossing graphs



[Inductiveload 2007]

# Overview of research on geometric counting

Counting configurations for special classes of point sets:  
[Flajolet and Noy 1999; Anclin 2003; Kaibel and Ziegler 2003]



All triangulations  
of a  $3 \times 3$  grid  
[Eppstein 2010]

# Overview of research on geometric counting

## Tightening the upper and lower bounds

[Aichholzer et al. 2007, 2008; Sharir and Welzl 2006; Sharir and Sheffer 2011; Dumitrescu et al. 2013; Sharir et al. 2013; Aichholzer et al. 2016; Santos and Seidel 2003; Aichholzer et al. 2004; Seidel 1998; García et al. 2000; Asinowski and Rote 2018]



[Farrington 2011]

# Overview of research on geometric counting

Exponential- or subexponential-time counting algorithms

[Bespamyatnikh 2002; Alvarez and Seidel 2013; Wettstein 2017; Alvarez et al. 2015a; Marx and Miltzow 2016; Brönnimann et al. 2006]



[NASA 2016]

# Overview of research on geometric counting

Faster approximation algorithms

[Alvarez et al. 2015b; Karpinski et al. 2018]



THE AUTHOR OF THE WINDOWS FILE COPY DIALOG VISITS SOME FRIENDS.

[Munroe 2009]



# Overview of research on geometric counting

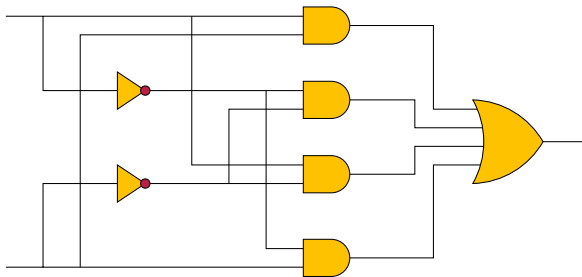
Complexity theory of 2d counting problems:  
Almost no past research



[9591353082 2006]

(but see Dittmer and Pak [2018] for # dominance orderings)

## Complexity classes from Boolean circuits



**P:** Problems that can be transformed into circuit evaluation  
(by writing a program and running it on a computer)

**NP:** Problems that can be transformed into satisfiability

**#P:** Problems that can be transformed into satisfiability counting

All transformations must be *efficient* (polynomial time overhead)

# Completeness

Complete problems for a given class are defined by two properties:

- ▶ They belong to the class
- ▶ Everything else in the class can be transformed into them

Automatically true for the defining circuit problem of the class

So, equivalently:

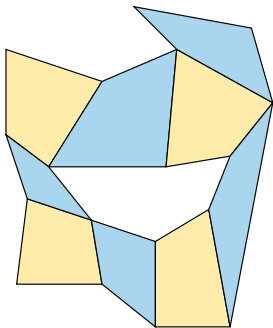
- ▶ Has transformations both from and to the circuit problem



# NP-complete problems are also hard to count

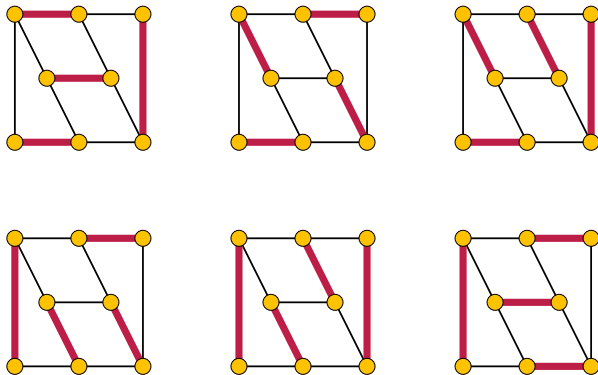
Some NP-complete problems in 2d geometry:

- ▶ Triangulations with restricted edges [Alvarez et al. 2015a]
- ▶ Min-max-degree triangulation [Kant and Bodlaender 1997]
- ▶ Partitions of a polygon into  $k$  trapezoids [Asano et al. 1986]
- ▶ Convex quadrangulation [Lubiw 1985; Pilz and Seara 2017]



## Valiant's insight

Problems for which existence is easy can still be #P-complete

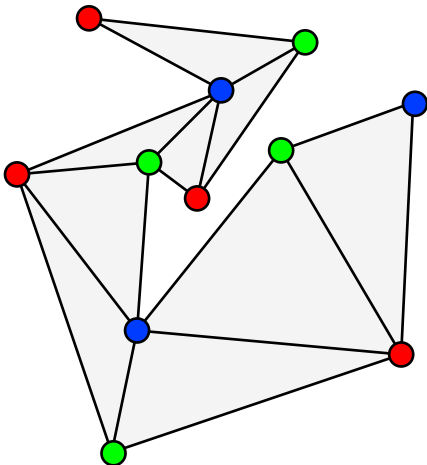


Example: Counting bipartite perfect matchings

[Valiant 1979]

# The main theorem

Counting triangulations of polygons  
(allowing holes) is #P-complete



# Easy problems for polygon triangulation

Existence: Answer is always yes

Construction: Polynomial-time greedy algorithm

Counting when there are no holes:

Polynomial-time dynamic programming

[Epstein and Sack 1994; Ray and Seidel 2004; Ding et al. 2005]

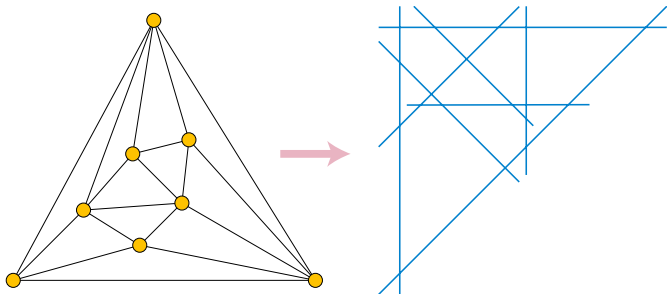


[Comet 1994]

So, holes are necessary for hardness of counting

# Intuitive/sloppy version of hardness proof

Every planar graph is a line segment intersection graph  
[Scheinerman 1984; Chalopin and Gonçalves 2009]



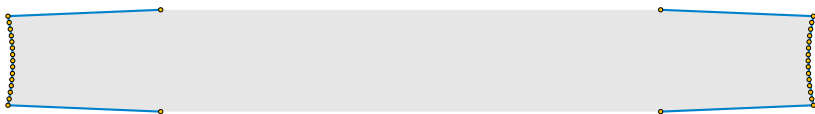
Redrawn from [Taxipom 2005a] and [Taxipom 2005b]

So hardness for planar graph problems  $\Rightarrow$  hardness for segments



# Intuitive/sloppy version of hardness proof

Gadget for replacing line segments:



Open in the middle where other line segments cross it

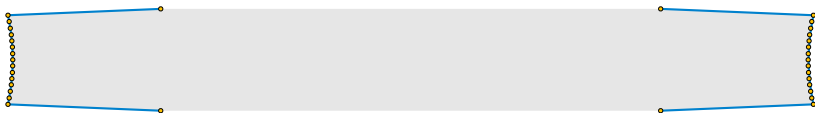
Exponentially many triangulations connect one concave chain to the other, but each chain has only polynomially many by itself

Guard edges protect unrelated chains from seeing each other

# Intuitive/sloppy version of hardness proof

Represent planar graph as a line segment intersection graph and  
replace each line segment with a gadget

Connect the dots to link gadgets into a polygon



$$\log(\# \text{ triangulations}) \approx X \cdot Y$$

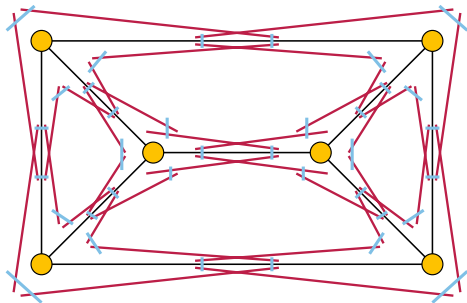
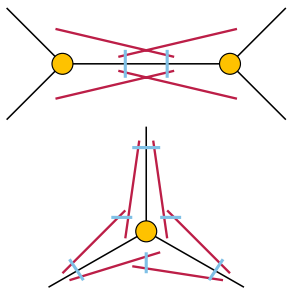
where  $X$  = size of maximum independent set  
and  $Y$  =  $\log(\#$  triangulations of a single gadget)

But this only proves NP-hardness!

Difficulty: Controlling the number of triangulations more precisely

## More careful hardness proof

Translate # planar 3-regular independent sets (not max!)  
to max non-crossing subsets of *red-blue segment arrangements*



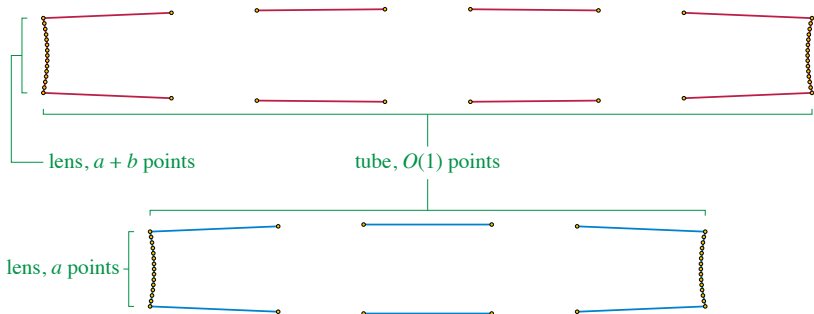
Each red segment has three crossings in the order blue, red, blue

Each blue segment has two red crossings

Max non-crossing set uses either blue or red from each graph vertex

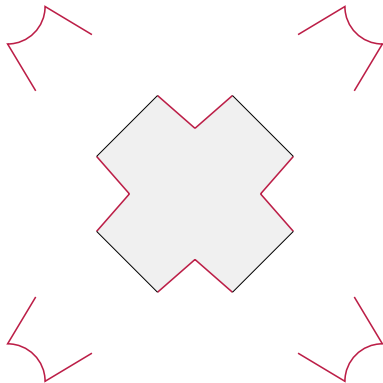
# More careful hardness proof

Two versions of line segment gadget  
with more triangulations for the red ones



## More careful hardness proof

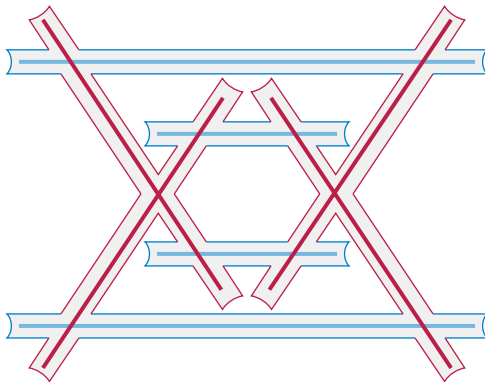
Triangulations that fill in a max non-crossing set of segments leave predictable shapes and numbers of remaining regions



So non-crossing sets with the same number of red segments always correspond to equal numbers of triangulations

## More careful hardness proof

Counting reduction showing  $\#\text{triangulations}$  is  $\#\text{P}$ -complete:



Translate 3-regular planar graph into red-blue segments, then replace segments by gadgets with shared vertices at crossings

Decode  $\#\text{triangulations}$  into sequence of numbers of non-crossing subsets for each possible number of red segments in the subset

# Conclusions

First 2d geometry problem with easy existence and hard counting

For polygons with  $h$  holes, can count in  $O(n^{3+h})$ ;  
is linear dependence on  $h$  in exponent necessary?

Possible stepping-stone to hardness of counting for point sets?



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