Counting Polygon Triangulations is Hard

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WARNING

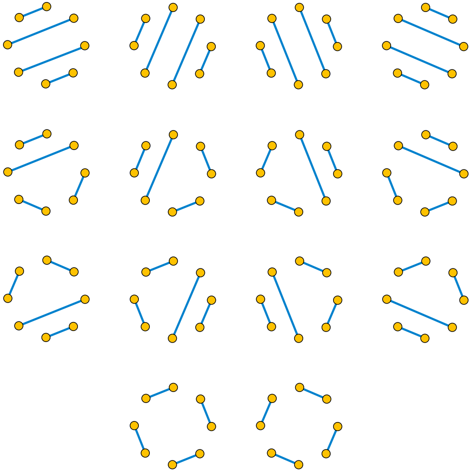
This is all just a gadget-based reduction.

There are no new ideas.

Background on geometric counting

Examples of geometric enumeration problems

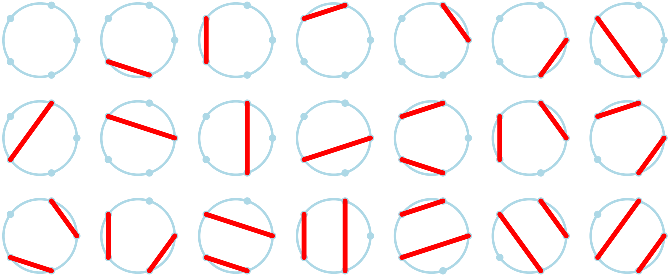
Non-crossing perfect matchings of $2n$ convex points



$=$ Catalan numbers 1, 2, 5, 14, 42, ...

Examples of geometric enumeration problems

Non-crossing (non-perfect) matchings of n convex points

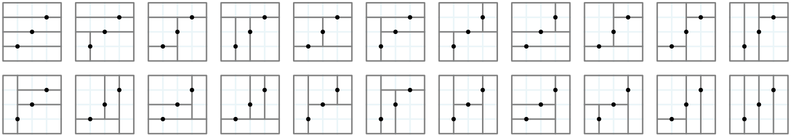


[Robertd 2008a]

$=$ Motzkin numbers 1, 2, 4, 9, 21, ...

Examples of geometric enumeration problems

Hierarchical rectangle subdivisions through points on a diagonal line

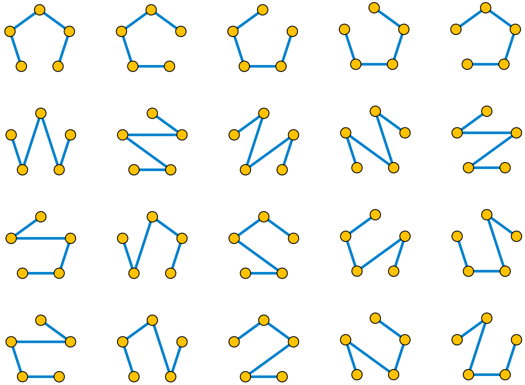


[Robertd 2008b]

= Schröder numbers 1, 2, 6, 22, 90, ...

Examples of geometric enumeration problems

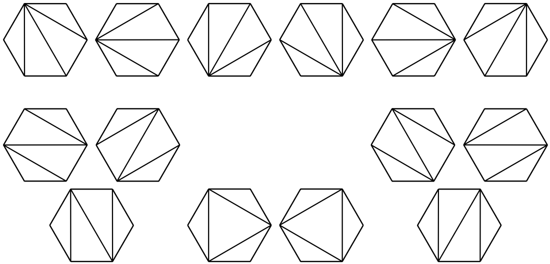
Non-crossing paths through n convex points



$$= 1, 1, 3, 8, 20, \cdots = n \cdot 2^{n-3}$$

Examples of geometric enumeration problems

Triangulations of n convex points



[Dmharvey 2006]

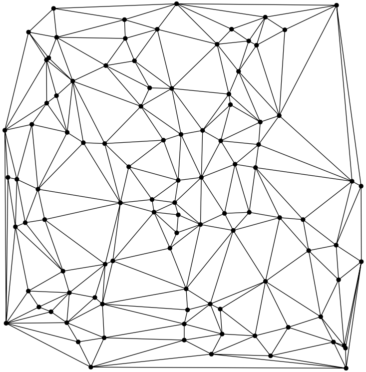
= Catalan numbers again

When we let the point set vary...

Non-crossing graphs
of any type are subgraphs of
triangulations

Any n-point set has
$2^{O(n)}$ triangulations
[Ajtai et al. 1982]

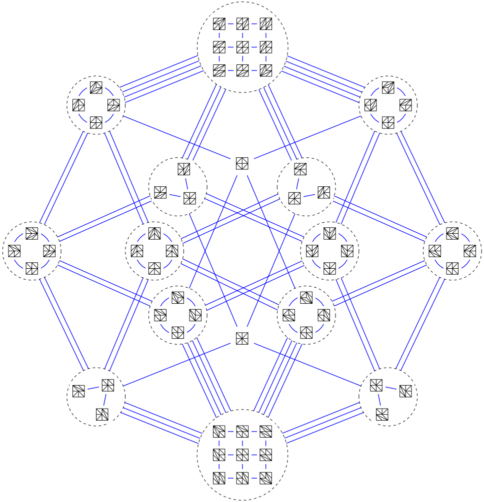
Therefore, there are $2^{O(n)}$
non-crossing graphs



[Inductiveload 2007]

Overview of research on geometric counting

Counting configurations for special classes of point sets:
[Flajolet and Noy 1999; Anclin 2003; Kaibel and Ziegler 2003]



All triangulations
of a 3 × 3 grid
[Eppstein 2010]

Overview of research on geometric counting

Tightening the upper and lower bounds
[Aichholzer et al. 2007, 2008; Sharir and Welzl 2006; Sharir and Sheffer 2011; Dumitrescu et al. 2013; Sharir et al. 2013; Aichholzer et al. 2016; Santos and Seidel 2003; Aichholzer et al. 2004; Seidel 1998; García et al. 2000; Asinowski and Rote 2018]

[Farrington 2011]

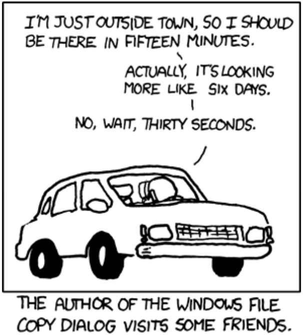
Overview of research on geometric counting

Exponential- or subexponential-time counting algorithms
[Bespamyatnikh 2002; Alvarez and Seidel 2013; Wettstein 2017; Alvarez et al. 2015a; Marx and Miltzow 2016; Brönnimann et al. 2006]

[NASA 2016]

Overview of research on geometric counting

Faster approximation algorithms
[Alvarez et al. 2015b; Karpinski et al. 2018]



[Munroe 2009]

Overview of research on geometric counting

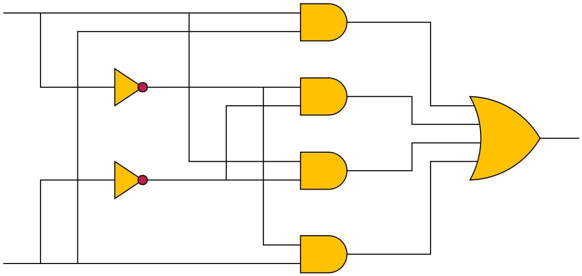
Complexity theory of 2d counting problems:
Almost no past research

[9591353082 2006]

(but see Dittmer and Pak [2018] for # dominance orderings)

Background on counting complexity

Computational problems on Boolean formulas

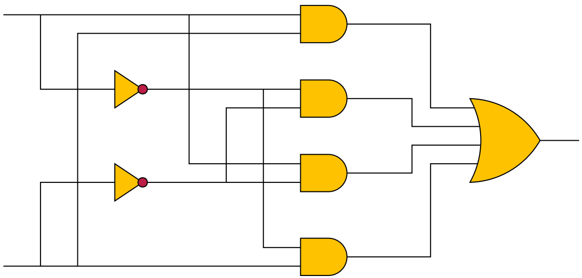


Evaluation: What is the output of this formula?

Satisfiability: Is there an input that makes the output true?

Counting: How many inputs make the output true?

Complexity classses from Boolean circuits



P: Problems that can be transformed into circuit evaluation
(by writing a program and running it on a computer)

NP: Problems that can be transformed into satisfiability

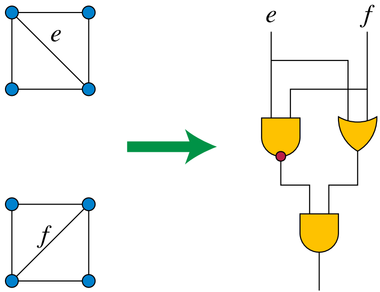
#P: Problems that can be transformed into satisfiability counting

All transformations must be *efficient* (polynomial time overhead)

Many geometric counting problems are in #P

To transform a counting problem on geometric graphs to circuit satisfiability counting:

- Make a Boolean variable for each potential edge
- Construct a circuit that checks that the edges are non-crossing and form the right kind of graph



Completeness

Complete problems for a given class are defined by two properties:

- They belong to the class
- Everything else in the class can be transformed into them

Automatically true for the defining circuit problem of the class

So, equivalently:

- Has transformations both from and to the circuit problem

Types of problem transformation for counting

Parsimonious reduction

Polynomial time count-preserving function
from inputs of problem X to inputs of problem Y

($\nexists$ when one problem can have count $= 0$ and the other cannot)

Polynomial-time counting reduction

Polynomial-time function from inputs of X to inputs of Y
Second polynomial-time function from count of Y to count of X

(My preference, corresponds to many-one reduction for NP-completeness)

Turing reduction

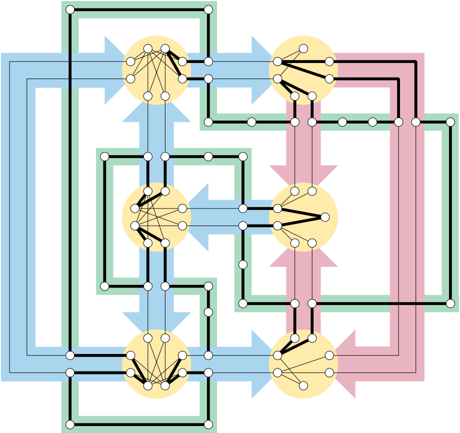
Algorithm for solving Y using a subroutine for X
in polynomially many calls and polynomial added time

(The original and most commonly used)

Standard method for reductions: Gadgets

Gadget = part of input(Y)
that can simulate an element of input(X)

Transform X to Y by replacing each element by a gadget

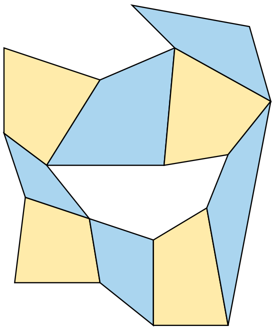


[Demaine et al. 2019]

NP-complete problems are also hard to count

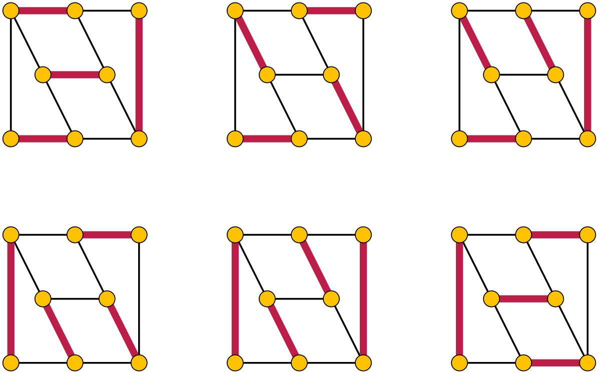
Some NP-complete problems in 2d geometry:

- Triangulations with restricted edges [Alvarez et al. 2015a]
- Min-max-degree triangulation [Kant and Bodlaender 1997]
- Partitions of a polygon into k trapezoids [Asano et al. 1986]
- Convex quadrangulation [Lubiw 1985; Pilz and Seara 2017]



Valiant's insight

Problems for which existence is easy can still be #P-complete



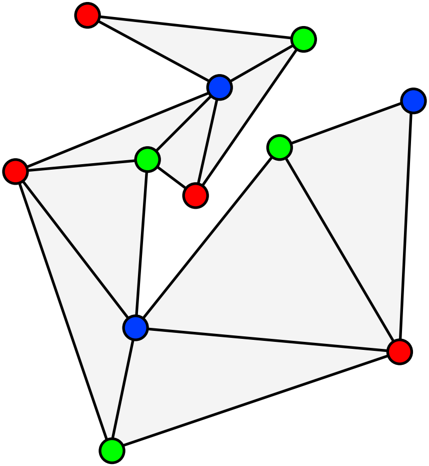
Example: Counting bipartite perfect matchings

[Valiant 1979]

Complexity of counting triangulations

The main theorem

Counting triangulations of polygons
(allowing holes) is #P-complete



Easy problems for polygon triangulation

Existence: Answer is always yes

Construction: Polynomial-time greedy algorithm

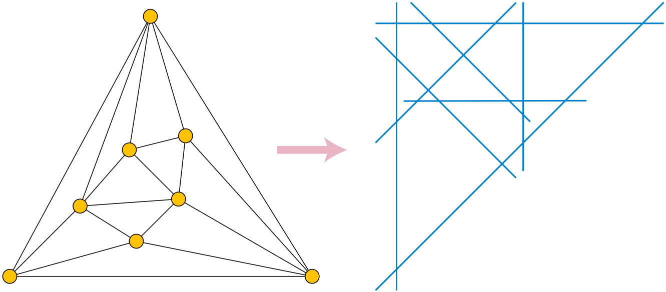
Counting when there are no holes:
Polynomial-time dynamic programming
[Epstein and Sack 1994; Ray and Seidel 2004; Ding et al. 2005]

[Comet 1994]

So, holes are necessary for hardness of counting

Intuitive/sloppy version of hardness proof

Every planar graph is a line segment intersection graph
[Scheinerman 1984; Chalopin and Gonçalves 2009]

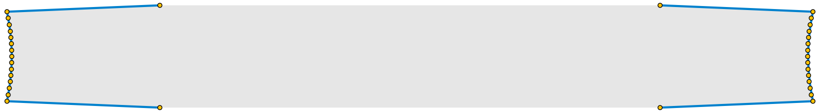


Redrawn from [Taxipom 2005a] and [Taxipom 2005b]

So hardness for planar graph problems ⇒ hardness for segments

Intuitive/sloppy version of hardness proof

Gadget for replacing line segments:



Open in the middle where other line segments cross it

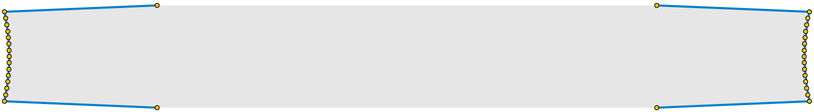
Exponentially many triangulations connect one concave chain to the other, but each chain has only polynomially many by itself

Guard edges protect unrelated chains from seeing each other

Intuitive/sloppy version of hardness proof

Represent planar graph as a line segment intersection graph and replace each line segment with a gadget

Connect the dots to link gadgets into a polygon



$$\log(\#\text{ triangulations}) \approx X \cdot Y$$

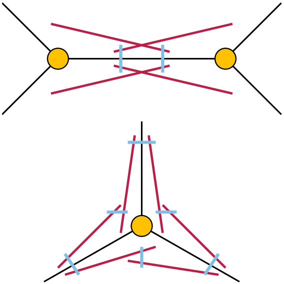
where $X =$ size of maximum independent set
and $Y = \log(\#\text{ triangulations of a single gadget})$

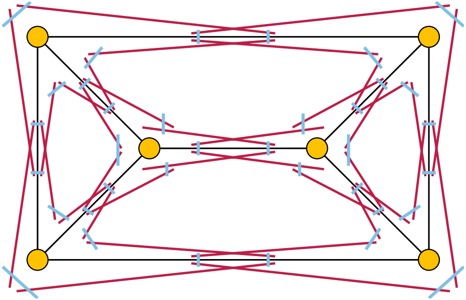
But this only proves NP-hardness!

Difficulty: Controlling the number of triangulations more precisely

More careful hardness proof

Translate # planar 3-regular independent sets (not max!)
to max non-crossing subsets of *red-blue segment arrangements*





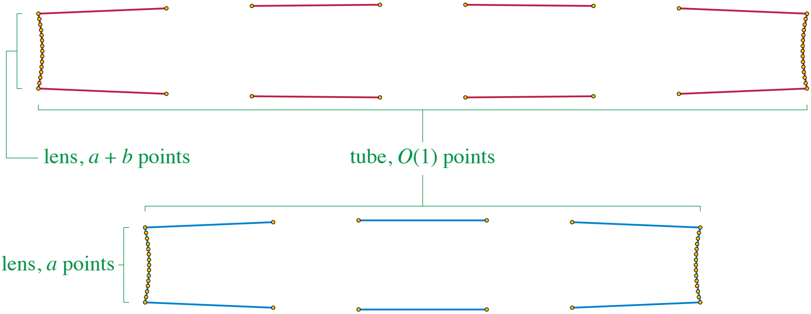
Each red segment has three crossings in the order blue, red, blue

Each blue segment has two red crossings

Max non-crossing set uses either blue or red from each graph vertex

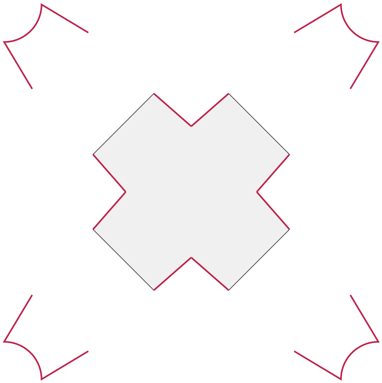
More careful hardness proof

Two versions of line segment gadget
with more triangulations for the red ones



More careful hardness proof

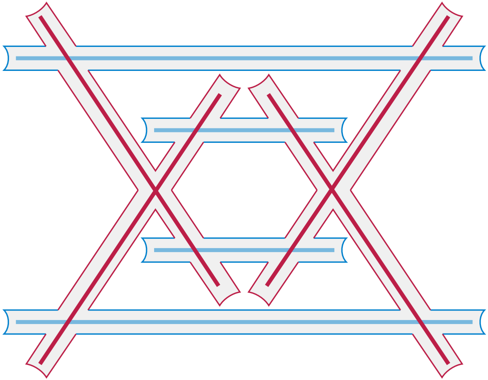
Triangulations that fill in a max non-crossing set of segments
leave predictable shapes and numbers of remaining regions



So non-crossing sets with the same number of red segments
always correspond to equal numbers of triangulations

More careful hardness proof

Counting reduction showing #triangulations is #P-complete:



Translate 3-regular planar graph into red-blue segments, then replace segments by gadgets with shared vertices at crossings

Decode #triangulations into sequence of numbers of non-crossing subsets for each possible number of red segments in the subset

Conclusions

First 2d geometry problem with easy existence and hard counting

For polygons with h holes, can count in $O(n^{3+h})$;
is linear dependence on h in exponent necessary?

Possible stepping-stone to hardness of counting for point sets?

[Ingles-Le Nobel 2010]

References and image credits, I

9591353082. Desert in Qatar. Public-domain image, 2006. URL `https://commons.wikimedia.org/wiki/File:Desert_Qatar.JPG`.

Oswin Aichholzer, Ferran Hurtado, and Marc Noy. A lower bound on the number of triangulations of planar point sets. *Comput. Geom. Theory and Applications*, 29(2):135–145, 2004. doi: 10.1016/j.comgeo.2004.02.003.

Oswin Aichholzer, Thomas Hackl, Clemens Huemer, Ferran Hurtado, Hannes Krasser, and Birgit Vogtenhuber. On the number of plane geometric graphs. *Graphs and Combinatorics*, 23(suppl. 1):67–84, 2007. doi: 10.1007/s00373-007-0704-5.

Oswin Aichholzer, David Orden, Francisco Santos, and Bettina Speckmann. On the number of pseudo-triangulations of certain point sets. *J. Combin. Theory Ser. A*, 115(2):254–278, 2008. doi: 10.1016/j.jcta.2007.06.002.

References and image credits, II

Oswin Aichholzer, Victor Alvarez, Thomas Hackl, Alexander Pilz, Bettina Speckmann, and Birgit Vogtenhuber. An improved lower bound on the minimum number of triangulations. In Sándor Fekete and Anna Lubiw, editors, *Proceedings of the 32nd International Symposium on Computational Geometry (SoCG 2016)*, volume 51 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages A7:1–A7:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi: 10.4230/LIPIcs.SoCG.2016.7.

M. Ajtai, V. Chvátal, M. M. Newborn, and E. Szemerédi. Crossing-free subgraphs. In *Theory and Practice of Combinatorics*, volume 60 of *North-Holland Math. Stud.*, pages 9–12. North-Holland, Amsterdam, 1982.

Victor Alvarez and Raimund Seidel. A simple aggregative algorithm for counting triangulations of planar point sets and related problems. In Timothy Chan and Rolf Klein, editors, *Proceedings of the 29th Annual Symposium on Computational Geometry (SoCG'13)*, pages 1–8, New York, 2013. ACM. doi: 10.1145/2462356.2462392.

References and image credits, III

Victor Alvarez, Karl Bringmann, Radu Curticapean, and Saurabh Ray. Counting triangulations and other crossing-free structures via onion layers. *Discrete Comput. Geom.*, 53(4):675–690, 2015a. doi: 10.1007/s00454-015-9672-3.

Victor Alvarez, Karl Bringmann, Saurabh Ray, and Raimund Seidel. Counting triangulations and other crossing-free structures approximately. *Comput. Geom. Theory and Applications*, 48(5): 386–397, 2015b. doi: 10.1016/j.comgeo.2014.12.006.

Emile E. Anclin. An upper bound for the number of planar lattice triangulations. *J. Combin. Theory Ser. A*, 103(2):383–386, 2003. doi: 10.1016/S0097-3165(03)00097-9.

Takao Asano, Tetsuo Asano, and Hiroshi Imai. Partitioning a polygonal region into trapezoids. *J. ACM*, 33(2):290–312, 1986. doi: 10.1145/5383.5387.

Andrei Asinowski and Günter Rote. Point sets with many non-crossing perfect matchings. *Comput. Geom. Theory and Applications*, 68:7–33, 2018. doi: 10.1016/j.comgeo.2017.05.006.

References and image credits, IV

Sergei Bespamyatnikh. An efficient algorithm for enumeration of triangulations. *Comput. Geom. Theory and Applications*, 23(3): 271–279, 2002. doi: 10.1016/S0925-7721(02)00111-6.

Hervé Brönnimann, Lutz Kettner, Michel Pocchiola, and Jack Snoeyink. Counting and enumerating pointed pseudotriangulations with the greedy flip algorithm. *SIAM J. Comput.*, 36(3):721–739, 2006. doi: 10.1137/050631008.

Jérémie Chalopin and Daniel Gonçalves. Every planar graph is the intersection graph of segments in the plane: extended abstract. In Michael Mitzenmacher, editor, *Proceedings of the 41st Annual ACM Symposium on Theory of Computing, STOC 2009, Bethesda, MD, USA, May 31 - June 2, 2009*, pages 631–638, 2009. doi: 10.1145/1536414.1536500.

Renee Comet. Cheese. Public-domain image, 1994. URL https://commons.wikimedia.org/wiki/File:Cheese.jpg.

References and image credits, V

E. Demaine, D. Eppstein, A. Hesterberg, K. Jain, A. Lubiw, R. Uehara, and Y. Uno. Undirected path reconfiguration. In *Proceedings of the 16th Algorithms and Data Structures Symposium (WADS 2019)*, 2019. To appear.

Q. Ding, J. Qian, W. Tsang, and C. Wang. Randomly generating triangulations of a simple polygon. In Lusheng Wang, editor, *Computing and Combinatorics: 11th Annual International Conference, COCOON 2005, Kunming, China, August 16–19, 2005, Proceedings*, volume 3595 of *Lecture Notes in Computer Science*, pages 471–480. Springer, Berlin, 2005. doi: 10.1007/11533719_48.

Samuel Dittmer and Igor Pak. Counting linear extensions of restricted posets. Electronic preprint arxiv:1802.06312, 2018.

Dmharvey. Illustrating the fifth Catalan number. Public-domain image, 2006. URL https://commons.wikimedia.org/wiki/File:Catalan-Hexagons-example.svg.

References and image credits, VI

Adrian Dumitrescu, André Schulz, Adam Sheffer, and Csaba D. Tóth. Bounds on the maximum multiplicity of some common geometric graphs. *SIAM J. Discrete Math.*, 27(2):802–826, 2013. doi: 10.1137/110849407.

David Eppstein. Happy endings for flip graphs. *Journal of Computational Geometry*, 1(1):3–28, 2010. doi: 10.20382/jocg.v1i1a2.

Peter Epstein and Jörg-Rüdiger Sack. Generating triangulations at random. *ACM Trans. Model. Comput. Simul.*, 4(3):267–278, 1994. doi: 10.1145/189443.189446.

Christopher Farrington. Sailors heave a fuel line aboard the guided-missile destroyer USS Halsey. Public-domain image, 2011. URL `https://commons.wikimedia.org/wiki/File:US_Navy_110918-N-ZF681-266_Sailors_heave_a_fuel_line_aboard_the_guided-missile_destroyer_USS_Halsey_(DDG_97)_during_a_replenishment_at_sea.jpg`.

Philippe Flajolet and Marc Noy. Analytic combinatorics of non-crossing configurations. *Discrete Math.*, 204(1-3):203–229, 1999. doi: 10.1016/S0012-365X(98)00372-0.

References and image credits, VII

Alfredo García, Marc Noy, and Javier Tejel. Lower bounds on the number of crossing-free subgraphs of K_N. *Comput. Geom. Theory and Applications*, 16(4):211–221, 2000. doi: 10.1016/S0925-7721(00)00010-9.

Inductiveload. Diagram of the Delaunay Triangulation of 100 random points. Public-domain image, 2007. URL https://commons.wikimedia.org/wiki/File:Delaunay_Triangulation_(100_Points).svg.

Johan J. Ingles-Le Nobel. Stepping stones over the River Mole, Box Hill, Dorking. CC-BY licensed image, 2010. URL https://commons.wikimedia.org/wiki/File:Stepping_Stones_over_the_River_Mole,_Box_Hill,_Dorking.jpg.

Volker Kaibel and Günter M. Ziegler. Counting lattice triangulations. In C. D. Wensley, editor, *Surveys in Combinatorics 2003: Papers from the 19th British Combinatorial Conference held at the University of Wales, Bangor, June 29–July 4, 2003*, volume 307 of *London Math. Soc. Lecture Note Ser.*, pages 277–307. Cambridge Univ. Press, Cambridge, UK, 2003.

References and image credits, VIII

Goos Kant and Hans L. Bodlaender. Triangulating planar graphs while minimizing the maximum degree. *Inform. and Comput.*, 135(1):1–14, 1997. doi: 10.1006/inco.1997.2635.

Marek Karpinski, Andrzej Lingas, and Dzmitry Sledneu. A QPTAS for the base of the number of crossing-free structures on a planar point set. *Theoretical Computer Science*, 711:56–65, 2018. doi: 10.1016/j.tcs.2017.11.003.

Anna Lubiw. Decomposing polygonal regions into convex quadrilaterals. In Joseph O'Rourke, editor, *Proceedings of the 1st Symposium on Computational Geometry, Baltimore, Maryland, USA, June 5-7, 1985*, pages 97–106, New York, 1985. ACM. doi: 10.1145/323233.323247.

References and image credits, IX

Dániel Marx and Tillmann Miltzow. Peeling and nibbling the cactus: subexponential-time algorithms for counting triangulations and related problems. In Sándor Fekete and Anna Lubiw, editors, *Proceedings of the 32nd International Symposium on Computational Geometry (SoCG 2016)*, volume 51 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages A52:1–A52:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. doi: 10.4230/LIPIcs.SoCG.2016.52.

Randall Munroe. Estimation. CC-BY-NC licensed image, 2009. URL https://xkcd.com/612/.

NASA. Human computers – Jet Propulsion Laboratory employees. Public-domain image, 2016. URL https://commons.wikimedia.org/wiki/File:Human_computers_-_Jet_Propulsion_Laboratory_employees.jpg.

References and image credits, X

Alexander Pilz and Carlos Seara. Convex quadrangulations of bichromatic point sets. In *Proceedings of the 33rd European Workshop on Computational Geometry (EuroCG 2017)*, 2017. URL https://mat-web.upc.edu/people/carlos.seara/data/publications/internationalConferences/EuroCG-17_paper_23.pdf.

Saurabh Ray and Raimund Seidel. A simple and less slow method for counting triangulations and for related problems. In *Proceedings of the 20th European Workshop on Computational Geometry (EuroCG 2004)*, 2004. URL https://hdl.handle.net/11441/55368.

Robertd. Chord diagrams representing Motzkin numbers. CC-BY licensed image, 2008a. URL https://commons.wikimedia.org/wiki/File:MotzkinChords5.svg.

Robertd. Diagram showing guillotine rectangulations corresponding to Schroeder numbers (4). CC-BY licensed image, 2008b. URL https://commons.wikimedia.org/wiki/File:Schroeder_rectangulation_4.svg.

References and image credits, XI

Francisco Santos and Raimund Seidel. A better upper bound on the number of triangulations of a planar point set. *J. Combin. Theory Ser. A*, 102(1):186–193, 2003. doi: 10.1016/S0097-3165(03)00002-5.

Edward R. Scheinerman. *Intersection Classes and Multiple Intersection Parameters of Graphs*. PhD thesis, Princeton University, 1984.

Raimund Seidel. On the number of triangulations of planar point sets. *Combinatorica*, 18(2):297–299, 1998. doi: 10.1007/PL00009823.

Micha Sharir and Adam Sheffer. Counting triangulations of planar point sets. *Electron. J. Combin.*, 18(1):P70:1–P70:74, 2011. URL `https://emis.ams.org/journals/EJC/Volume_18/PDF/v18i1p70.pdf`.

Micha Sharir and Emo Welzl. On the number of crossing-free matchings, cycles, and partitions. *SIAM J. Comput.*, 36(3):695–720, 2006. doi: 10.1137/050636036.

Micha Sharir, Adam Sheffer, and Emo Welzl. Counting plane graphs: perfect matchings, spanning cycles, and Kasteleyn's technique. *J. Combin. Theory Ser. A*, 120(4):777–794, 2013. doi: 10.1016/j.jcta.2013.01.002.

References and image credits, XII

Taxipom. Intersect1. CC-BY-SA licensed image, 2005a. URL https://commons.wikimedia.org/wiki/File:Intersect1.png.

Taxipom. Intersect2. CC-BY-SA licensed image, 2005b. URL https://commons.wikimedia.org/wiki/File:Intersect2.png.

L. G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8(2):189–201, 1979. doi: 10.1016/0304-3975(79)90044-6.

Manuel Wettstein. Counting and enumerating crossing-free geometric graphs. *J. Comput. Geom.*, 8(1):47–77, 2017. URL https://jocg.org/index.php/jocg/article/view/280.