The Traveling Salesman Problem for Cubic Graphs

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Why study worst case time bounds for exponential algorithms?

Worst case time analysis is most important
for slower compute-bound processes

Better time bounds improve solvable instance size
by constant factor; Moore's law much less effective

Less well-studied, so more interesting problems

Interesting gap between theory (exponential times)
and practice (much smaller exponentials)

Problems studied in this paper

Hamiltonian cycle

Input: undirected, unweighted graph
Output: simple cycle containing all vertices, if one exists
Decision version is NP-complete

Traveling salesman problem (TSP)

Input: undirected graph with edge weights
Output: shortest Hamiltonian cycle, if one exists
Decision version is NP-complete

Cycle counting

Input: undirected, unweighted graph
Output: number of simple cycles containing all vertices
#P-complete

Weighted cycle counting

Input: undirected graph with edge weights
Output: Sum, over all Hamiltonian cycles,
of product of weights of edges in cycle

Best previously-known TSP algorithm: dynamic programming

Choose arbitrary starting vertex v

For each pair (S,w), where S is any set of vertices
that contains both v and w,
let A(S,w) be min weight of a path from v to w via S

Compute $A(S,w) = \min \{ A(S\setminus\{w\},x) + d(x,w) \mid x \text{ in } S\setminus\{v\} \}$

Global TSP length = A(V(G),v)

Total time: $O(2^n n^2)$
Total space: $O(2^n n^{1/2})$
[order subproblems by set size, only store problems with similar size]

Same approach works also for counting
and weighted counting

New results

TSP and related problems for graphs
with maximum degree at most three

Hamiltonian cycle and TSP

time: $O(2^{n/3})$
space: linear

Cycle counting and weighted cycle counting

time: $O(2^{3n/8}\, n^{O(1)})$
space: polynomial

Main ideas of the new algorithms

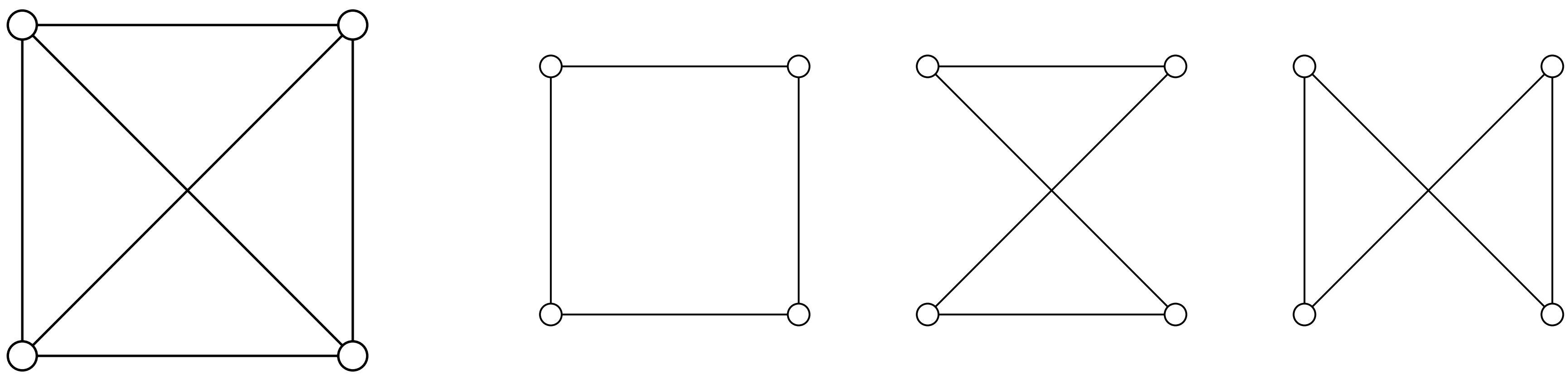
Generalize the problem
(allow graphs with forced edges)

Branch and simplify
(complicated case analysis)

Improve analysis via polynomial-time special case
(reduction from TSP to minimum spanning tree)

Forced edges

Input = graph + set of "forced edges" (shown thick)
Output cycle(s) must include all forced edges



Example: K_4 with no forced edges has three Hamiltonian cycles

Forced edges

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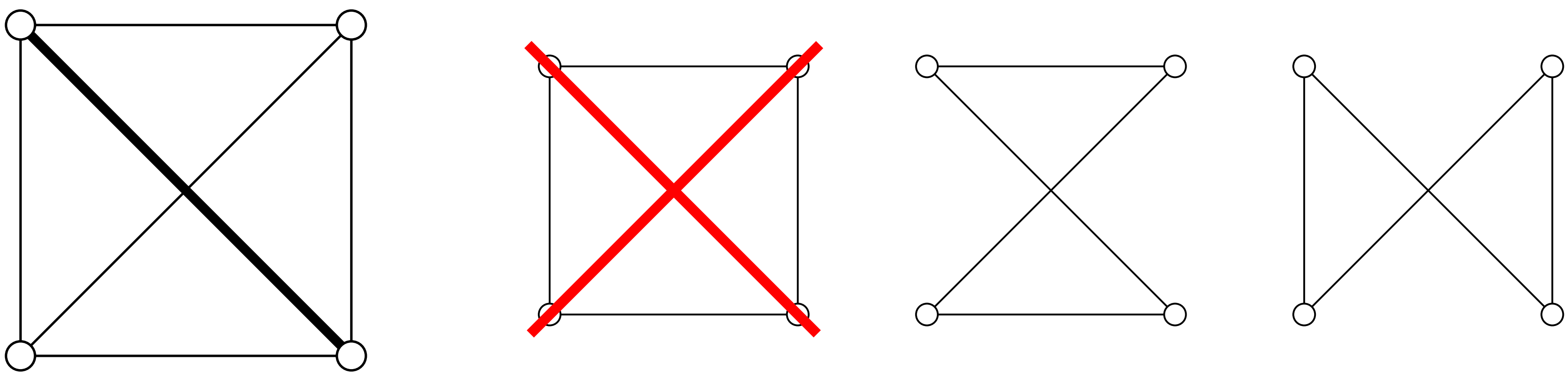
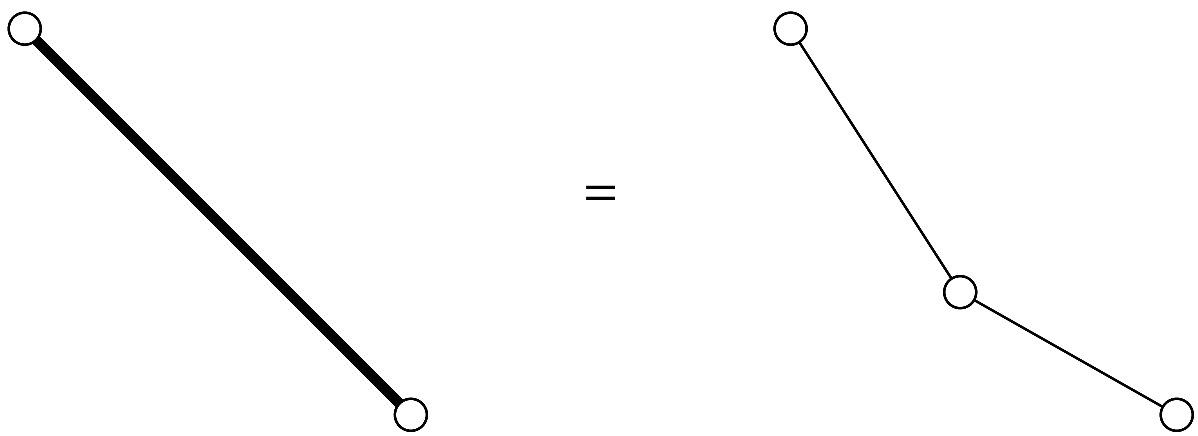
Output cycle(s) must include all forced edges



Example: K_4 with one forced edge has only two Hamiltonian cycles

Forced edges

Equivalent in effect to inserting a vertex inside the edge



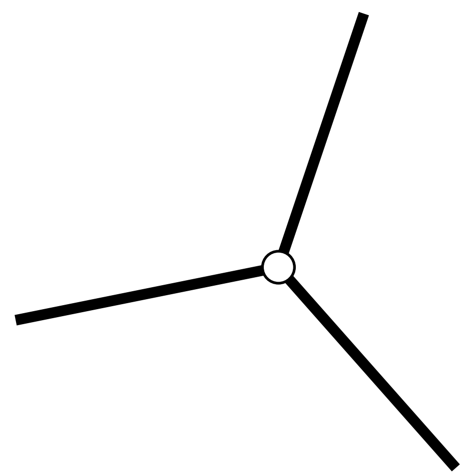
But, unlike new vertex, doesn't count against total size of graph

More forced edges = fewer possible cycles = easier problem instance

Problem simplifications

Certain configurations force no Hamiltonian cycle to exist

Vertex with degree one

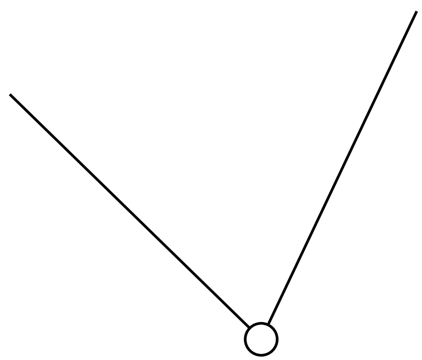


Three mutually incident forced edges

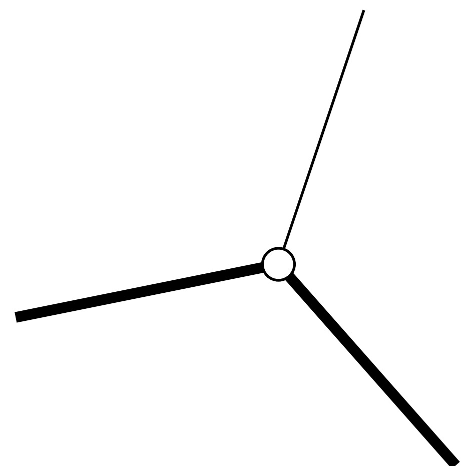
If encountered, immediately stop this branch of the search algorithm

Problem simplifications

Certain configurations allow additional edges to be forced or removed



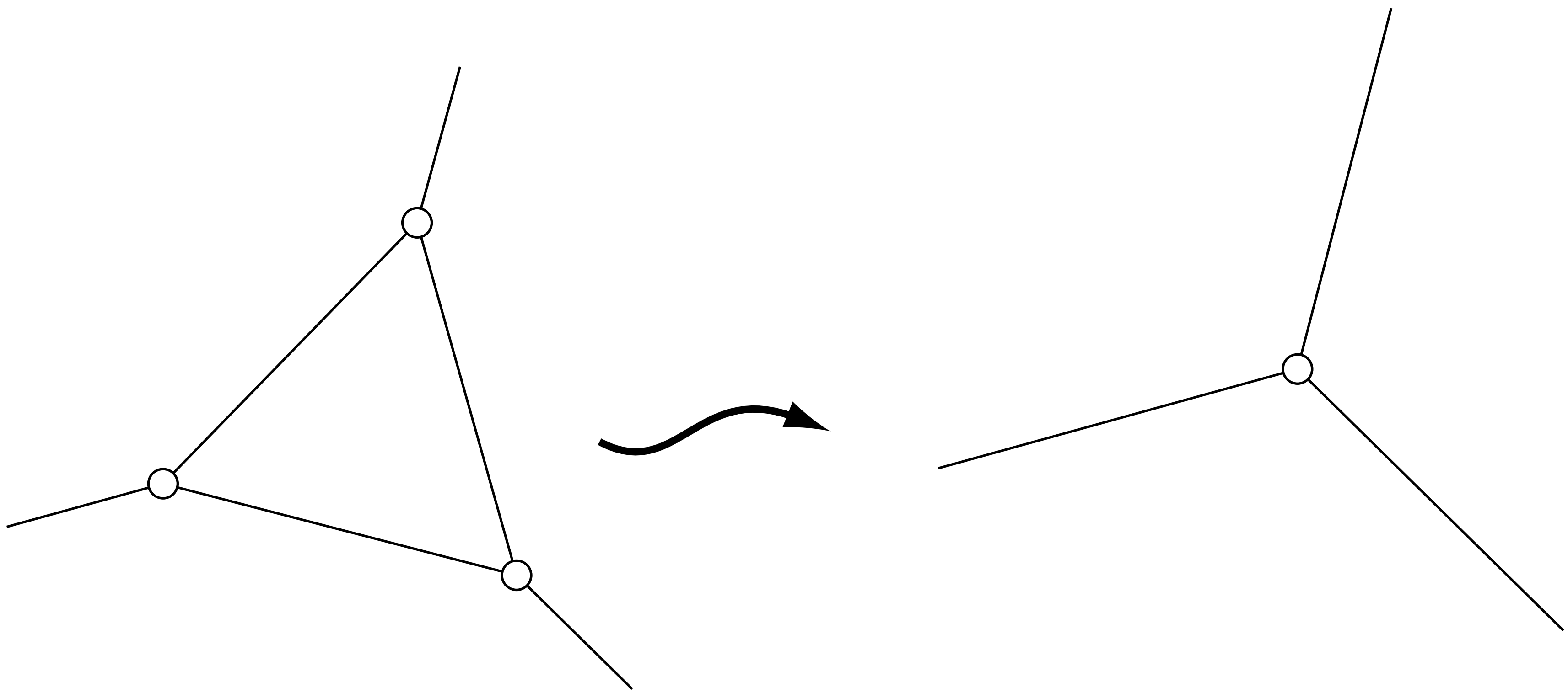
Vertex with degree two
both edges can be forced



Two mutually incident forced edges
remove third edge (if it exists)
and contract to a single forced edge

Problem simplifications

Triangles in the input graph can be contracted (Delta-Y transformation)



Can be made to preserve weights of each Hamiltonian cycle
and respect forced edges within the triangle

Branch and simplify

Standard general technique for exponential algorithms

Choose a variable (unforced edge)

Force the edge
Perform all possible simplifications
Recurse on smaller instance
finds all Hamiltonian cycles containing the edge

Restore original graph
Try removing the edge
Perform all possible simplifications
Recurse on smaller instance
finds all Hamiltonian cycles not containing the edge

Choose variable to maximize simplification
(complex case analysis)

Analysis

Let U = # unforced edges in instance, T(U) = # subproblems in algorithm

Worst case:
Cycle of four unforced edges with all neighbors forced
Each subproblem removes or forces all edges in cycle

$$T(U) = 2T(U - 4)$$

Not-quite-as-bad case:
Cycle of four unforced edges with two neighbors forced
Removing side between forced neighbors forces two adjacent sides
Forcing side removes adjacent sides, forces three other edges

$$T(U) = T(U - 3) + T(U - 6)$$

Overall $T(U) = O(2^{U/4}) = O(2^{3n/8})$

Polynomial special case

If all components of unforced edges form 4-cycles,
TSP can be transformed to minimum spanning tree

In general, perform branch and simplify
until reaching polynomial special case

Removes worst case from previous algorithm

New analysis (using different variable for subproblem size):
Total time = total # subproblems = $O(2^{n/3})$

Conclusions

Significantly more efficient algorithms
for interesting special case of TSP, related problems

Open problems

Extend TSP-MST reduction to cycle counting?

Any hope of $o(2^n)$ for general case of TSP?

Generalize low degree to low total number of edges?

Other important NP-hard special cases not yet considered?