The Graphs of Stably Matchable Pairs

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Stable matching

Pair up two sets of agents, each with preferences

Goal: No two agents prefer each other over their assigned pairing

[Heyboer 2021]

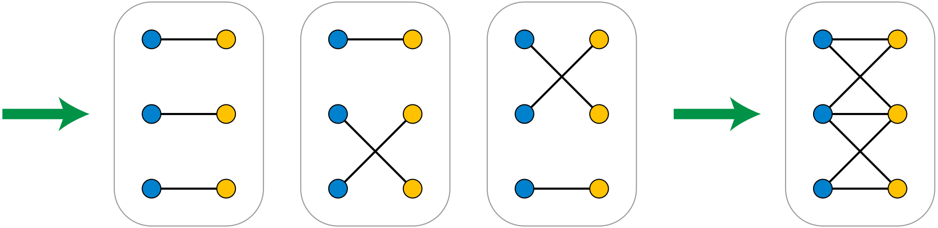
Real-world uses include pairing US medical students with residencies and French high school students with universities [Mathieu 2018]

The graphs of stably matchable pairs

Preferences ⇒ all stable matchings ⇒ union of edges

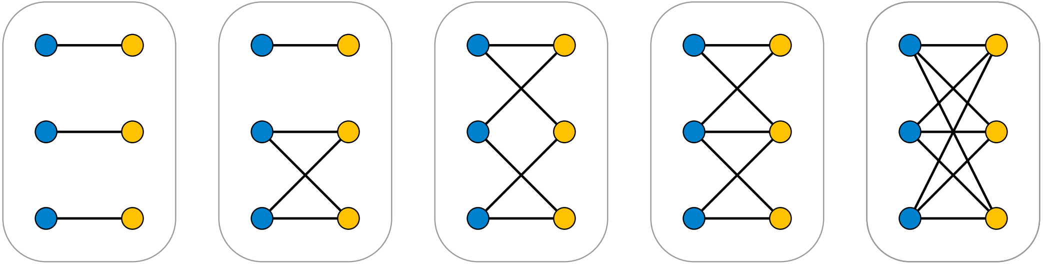
A: Y,X,Z
B: X,Y,Z
C: Z,Y,X

X: B,A,C
Y: A,B,C
Z: C,B,A



Our questions

Which graphs are possible?



What structural and algorithmic properties do these graphs have?

First, the bad news

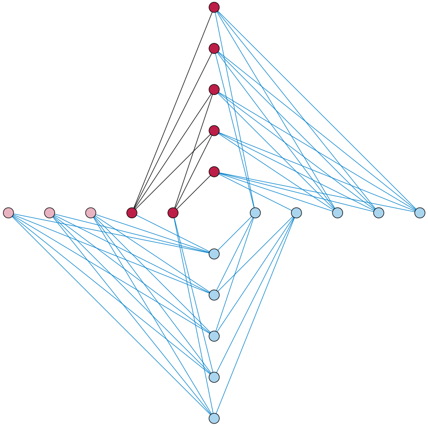
There is no hereditary structure

Recognizing these graphs is hard

Constructing them by listing
all stable matchings is slow

[Hartmann 1892]

There are no forbidden induced subgraphs



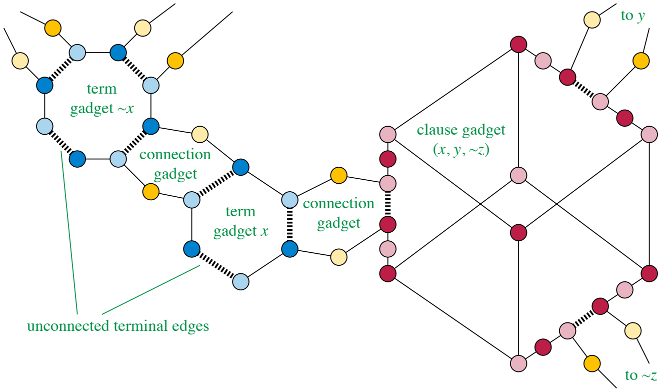
Every bipartite graph is induced subgraph of a regular bipartite graph (see figure)

Every regular bipartite graph comes from a stable matching instance

- Decompose into a sequence of perfect matchings [Kőnig 1916]
- Set preferences for one side of the matching by sequence order
- Set preferences for the other side by reverse sequence order

Recognition is NP-complete

Reduction from not-all-equal 3-satisfiability:



More complicated reduction from monotone planar 2-in-4-SAT

$\Rightarrow$ remains NP-complete even for subcubic planar graphs

Naive construction from all matchings is too slow

Example with 2^{n-1} matchings [Irving and Leather 1986]:

- Divide applicants and positions into subsets of size $n/2$
- Applicants in each subset prefer positions in corresponding subset
- Positions in each subset prefer applicants in opposite subset
- Recurse!

Even though there are $n!$ matchings, only a single-exponential number of them can be stable [Karlin et al. 2018]

Good news!

Fast construction without having to list all stable matchings

Exponential and FPT recognition

Relations between graph structure and lattice structure

[Snowman]

The lattice of stable matchings

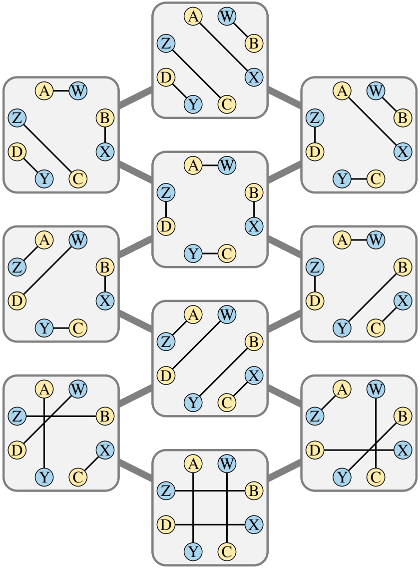
Join: Combine two matchings by giving each applicant their preferred match of the two

Meet: Combine using positions' preferred match

These operations form a *distributive lattice* [Knuth 1976]

Neighboring matchings differ by alternating cycles

Same cycles may be repeated



A: X, W, Z, Y
B: W, X, Y, Z
C: Z, Y, X, W
D: Y, Z, W, X

W: B, A, D, C
X: A, B, C, D
Y: D, C, B, A
Z: C, D, A, B

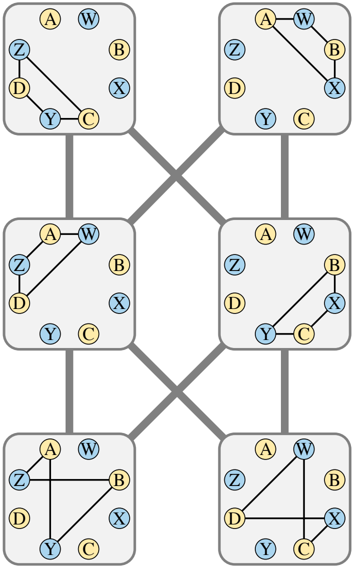
Concise representation of all stable matchings

Partially ordered set of alternating cycles

Each matched edge is in ≤ 2 cycles
(swapped in once and swapped out once)
$\Rightarrow O(n^2)$ cycles

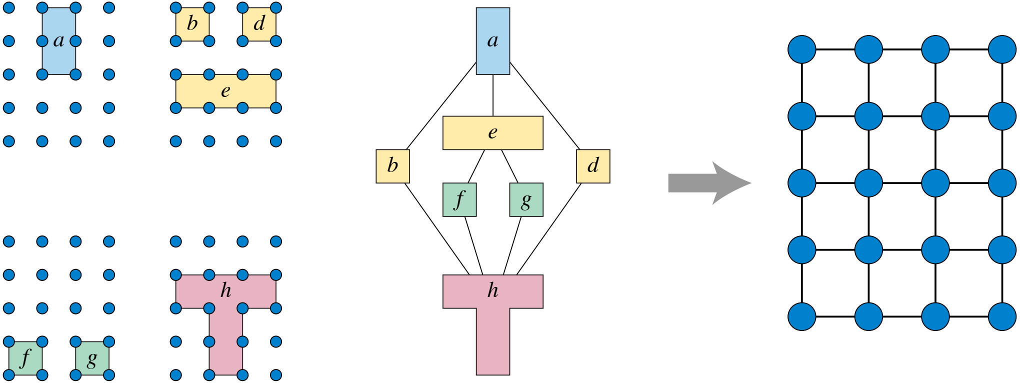
Stable matchings correspond to
downward-closed subsets of cycles
(Birkhoff's representation theorem)

Can construct in time linear in input size
[Gusfield 1987]



Efficient construction of graph

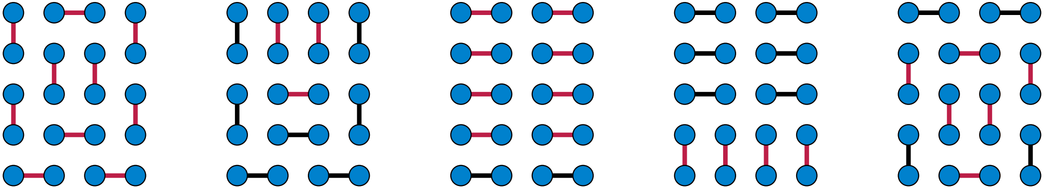
Union of cycles of concise representation



Covering sequence of matchings

Main idea of exponential-time recognition algorithm:

Find sequence of perfect matchings that introduces each edge exactly once



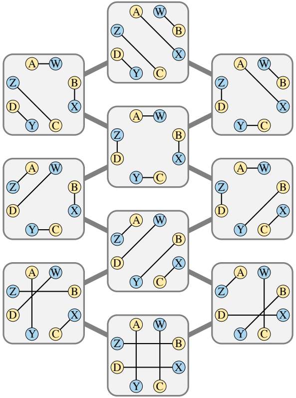
Example: 4×5 grid, with red showing the introduction (leftmost copy) of each edge

Stable matching instance ⇒ covering sequence

Use any top-to-bottom path through the lattice of stable matchings

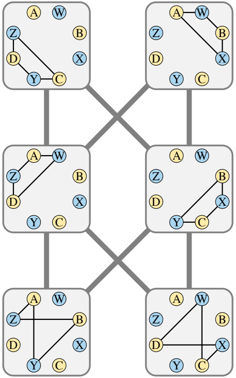
Equivalently, any topological order of the partial order of cycles

These sequences have an extra property – consecutive matchings differ by a single alternating cycle – that we don't use



A: X, W, Z, Y
B: W, X, Y, Z
C: Z, Y, X, W
D: Y, Z, W, X

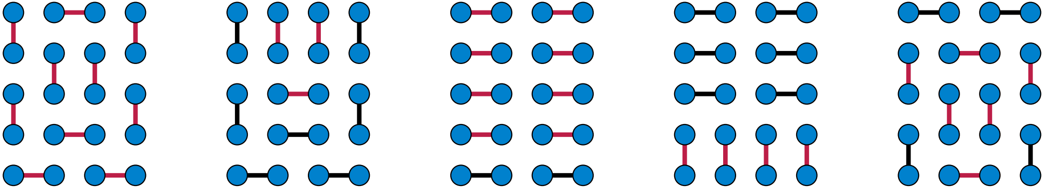
W: B, A, D, C
X: A, B, C, D
Y: D, C, B, A
Z: C, D, A, B



Graph with a covering sequence ⇒ stable matching instance

Applicant preferences: left-to-right order of sequence (with all non-matched pairs last)

Position preferences: right-to-left order (with all non-matched pairs last)



Each matching in the sequence is stable

There might be other stable matchings but they must use the same edges

Exponential-time recognition

Basic state-space search:

State: (M, S)

- M is a perfect matching
- S must be disjoint from M (set of already-introduced edges)

Transition: $(M, S) \Rightarrow (M', S')$

- M' must be disjoint from S
- $S' = S \cup (M - M')$

Use breadth-first or depth-first search to find path empty set $\Rightarrow$ all edges

Space-efficient version

Number the states and store for each number its matching, edge set, and predecessor

Build a table addressed by edge sets, listing disjoint matchings and their state numbers

Use the table to find neighbors of states when we need them without keeping all state space transitions in memory at once

Exponential-time analysis

$\#$ perfect matchings $= O(1.22028^m)$

- Comes from bound of Alon and Friedland [2008] on matchings from degrees

Space $= O(\#\text{ states}) = O(2.21435^m)$

- Apply binomial theorem to bound on matchings

Time $= O(\#\text{ transitions}) = O(2.48475^m)$

- Analyze a backtracking algorithm to generate a matching $+$ a disjoint set of edges, and use state bound to count transitions for each edge set

FPT parameterized by carving width

Bounded carving width = bounded treewidth + bounded degree

Dynamic programming on tree decomposition

Local state = what does concise representation look like at a bag of the tree decomposition?

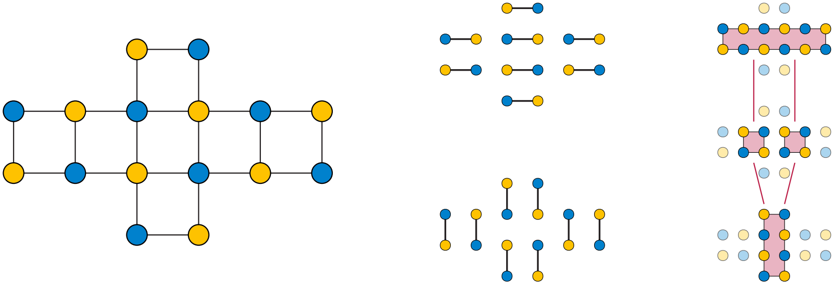
Bounded degree $\Rightarrow$ bounded
\# elements in local view of partial order of concise representation

An easy special case

Bipartite planar graph is a stable matching graph $\iff$ no articulation vertex

At an articulation vertex, not all edges can be covered by perfect matchings

Can use faces as concise representation (but other realizations may also exist)

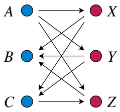


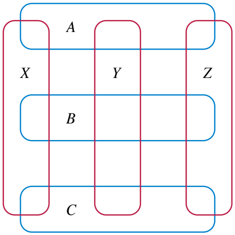
Open: Stable matching graph recognition for unbounded-degree series-parallel graphs

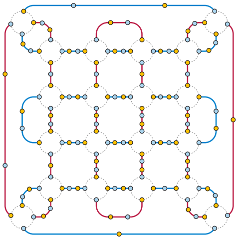
Graph structure $\iff$ lattice structure

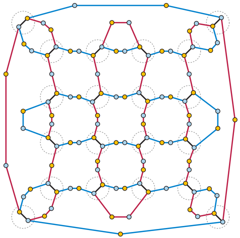
For each of the following, graphs of the given class can only represent stable matching instances with lattices of the given class, and all lattices of that class are representable:

- All graphs $\iff$ all distributive lattices [Gusfield et al. 1987]
- Subcubic graphs $\iff$ all distributive lattices
- Subcubic + can perfectly match equal-degree vertices
  $\iff$ Birkhoff representation has height two
- Subcubic + outerplanar/series-parallel $\iff$ lattice of closures of an oriented forest
- Planar $\iff$ lattice of closures of an oriented string graph









Conclusions

New class of graphs coming from important real-world applications

All bipartite graphs are induced subgraphs of one of these graphs

Recognition is NP-complete but has exponential time and FPT algorithms

Connections between graph structure and lattice structure

Unsolved: FPT for bounded treewidth but unbounded degree, even for treewidth $= 2$

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