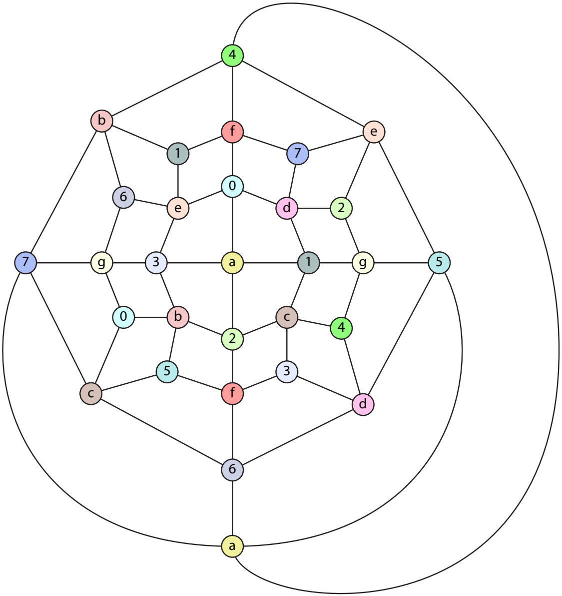
# On the Planar Split Thickness of Graphs

**David Eppstein**, Philipp Kindermann, Stephen Kobourov,
Giuseppe Liotta, Anna Lubiw, Aude Maignan,
Debajyoti Mondal, Hamideh Vosoughpour,
Sue Whitesides, and Stephen Wismath

12th Latin American Theoretical Informatics Symposium
(LATIN 2016)
Ensenada, Mexico, April 2016

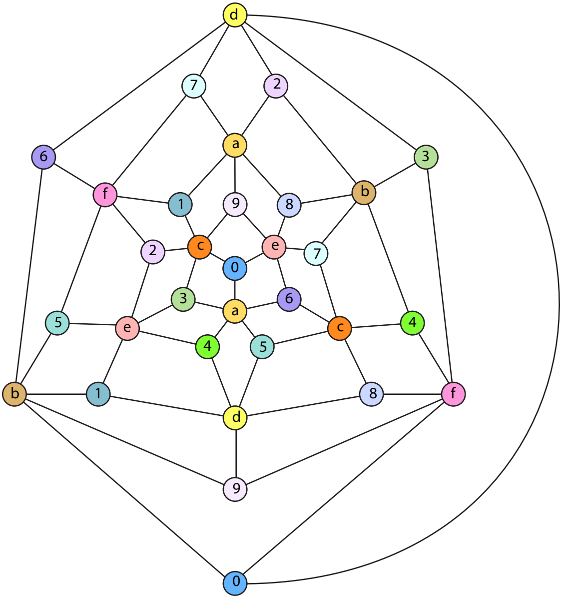
# Definition by example, I




Draw a graph (here, $K_{7,8}$) with:

- Each vertex $\Rightarrow$ O(1) points (here, 2 points/vertex)
- Each edge $\Rightarrow$ curve between representatives of its endpoints
- No crossings

# Definition by example, II

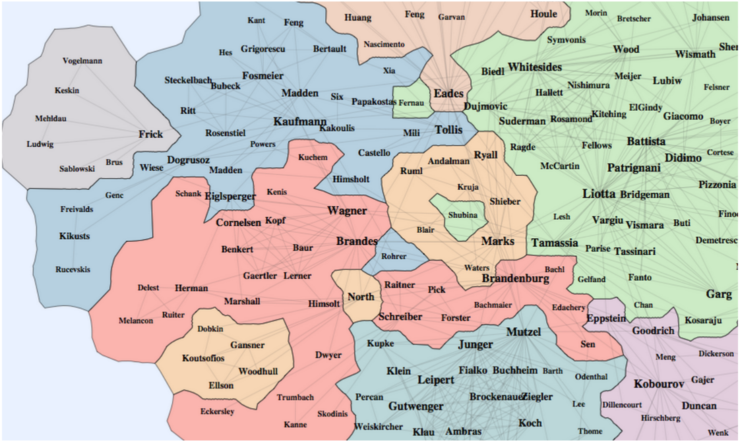



Split thickness:
max points/vertex
(here, 2)

$G$ is $k$-splittable:
it has a drawing with
split thickness $\leq k$

E.g. this drawing
shows that $K_{6,10}$ is
2-splittable

# Motivation: Maps of clustered social networks




Network itself drawn conventionally (no split vertices)

Clusters drawn as regions with $\leq k$ connected components

To construct drawing, need to show cluster graph is $k$-splittable

# Related research

Rephrased into our terminology:

Heawood 1890:
$K_{12}$ is 2-splittable

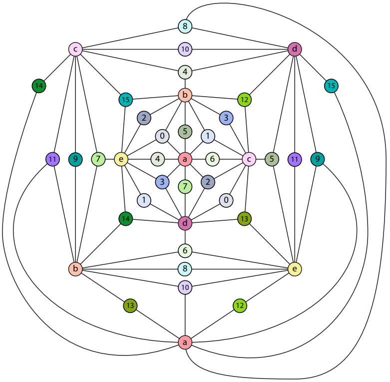
Ringel and Jackson 1984:
Optimal $k$-splittability for $K_n$ ($n > 6$) is $k = \lceil n/6 \rceil$

Hartsfield et al 1985 and later researchers:
Split to planar minimizing total # splits rather than splits/vertex

Knauer and Ueckerdt 2012:
Split vertices to transform graph into several types of trees

# Complete bipartite graphs

Theorem: $K_{a,b}$ is 2-splittable if and only if $ab \leq 4(a+b) - 4$




Proof:

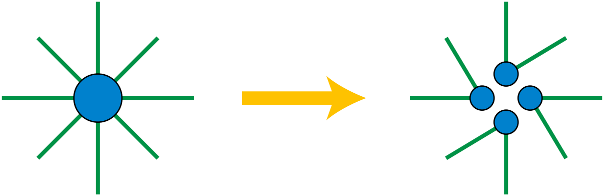
$ab \leq 4(a+b) - 4 \Rightarrow G \subset K_{4,b}$, $K_{5,16}$ (above), $K_{6,10}$, or $K_{7,8}$

$ab > 4(a+b) - 4 \Rightarrow$ too many edges for bipartite planar drawing

# Splittability by maximum degree

Let max degree $= \Delta(G)$

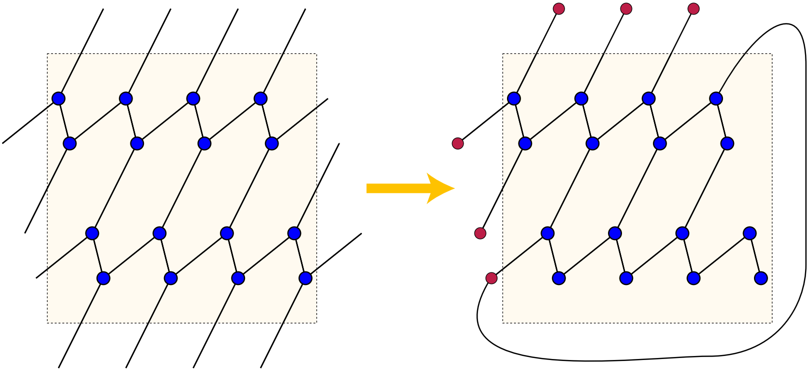
Then every graph $G$ is $\lceil \Delta(G)/2 \rceil$-splittable



Regular graphs with odd $\Delta$, high girth are not $\lfloor \Delta/2 \rfloor$-splittable:
high-girth planar graphs have edges/vertices $\leq 1 + o(1)$
but any $\lfloor \Delta/2 \rfloor$-split would have edges/vertices $= 1 + \frac{1}{\Delta - 1}$.

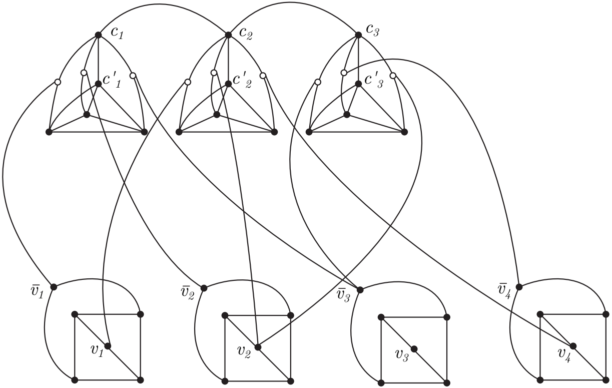
# Splittability by genus

Theorem: Toroidal and projective-planar graphs are 2-splittable



# Computational complexity

Theorem: 2-splittability is NP-complete




Reduction from planar 3-SAT with a cycle through clause vertices

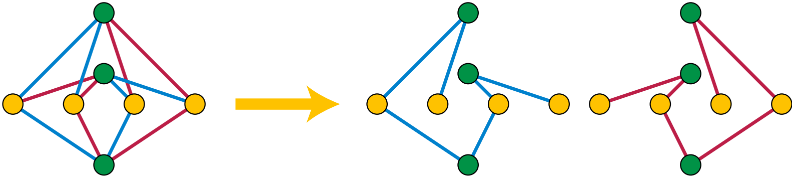
(shown NPC by Kratochvíl, Lubiw, & Nešetřil 1991)

# Approximation

Part of a family of graph parameters (arboricity, thickness, degeneracy, etc) all within constant factors of each other

Arboricity $a(G)$: minimum # trees whose union is the given graph

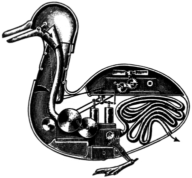
Every graph is $a(G)$-splittable: draw the trees disjointly



Every $n$-vertex $k$-splittable graph has $\leq (3k+1)(n-1)$ edges $\Rightarrow$ (Nash-Williams 1964) $a(G) \leq 3k+1$

So arboricity is a $(3+\frac{1}{k})$-approximation to splittability (can improve to 3-approximation using pseudoarboricity)

# Fixed-parameter tractability



Theorem: can test $k$-splittability of graphs of treewidth $\leq w$ in time $O(f(k, w) \cdot n)$

Main ideas:

- Use monadic second-order logic (MSO) to represent graph properties as quantified formulae over vertex and edge sets

$$\forall S \subset E(G) : \exists T \subset G(V) : \ldots$$

  - A standard DFS-tree trick distinguishes endpoints of each edge
  - Use edge-set variables to partition the edges according to the vertex-copies that each endpoint connects to
  - Simulate any MSO formula on the split graph by a more complex formula on the original graph
  - Planarity = absence of $K_5$ and $K_{3,3}$ minors
- Use Courcelle's theorem to construct an automaton that tests whether tree-decompositions obey the formula

# Conclusions

Defined a new concept of $k$-splittability, used it to draw nonplanar graphs in a planar way

Tight bounds for complete graphs, complete bipartite graphs, and graphs of bounded maximum degree

NP-complete but $O(1)$-approximable, FPT for bounded treewidth

Future work: splitting vertices to produce near-planar graphs (e.g. low genus or bounded local crossing number)