New Characterizations of Core Imputations of Matching and b-Matching Games (Extended Abstract)

Vijay V. Vazirani ✉
University of California, Irvine, Irvine, CA, USA

Abstract

We give new characterizations of core imputations for the following games:

1. The assignment game.
2. Concurrent games, i.e., general graph matching games having non-empty core.
3. The unconstrained bipartite b-matching game (edges can be matched multiple times).
4. The constrained bipartite b-matching game (edges can be matched at most once).

The classic paper of Shapley and Shubik [11] showed that core imputations of the assignment game are precisely optimal solutions to the dual of the LP-relaxation of the game. Building on this, Deng et al. [5] gave a general framework which yields analogous characterizations for several fundamental combinatorial games. Interestingly enough, their framework does not apply to the last two games stated above. In turn, we show that some of the core imputations of these games correspond to optimal dual solutions and others do not. This leads to the tantalizing question of understanding the origins of the latter.

We also present new characterizations of the profits accrued by agents and teams in core imputations of the first two games. Our characterization for the first game is stronger than that for the second; the underlying reason is that the characterization of vertices of the Birkhoff polytope is stronger than that of the Balinski polytope.

2012 ACM Subject Classification Theory of computation → Algorithmic game theory

Keywords and phrases LP-duality theory, cooperative game theory, core of a game, assignment game, general graph matching game, bipartite b-matching game

Digital Object Identifier 10.4230/LIPIcs...13

Funding *Vijay V. Vazirani*: Supported in part by NSF grants CCF-1815901 and CCF-2230414.

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

The matching game forms one of the cornerstones of cooperative game theory and the *core* is a quintessential solution concept in this theory; the latter captures all possible ways of distributing the total worth of a game among individual agents in such a way that the grand coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself and therefore has no incentive to secede from the grand coalition. The matching game can also be viewed as a matching market in which utilities of the agents are stated in monetary terms and side payments are allowed, i.e., it is a *transferable utility (TU) market.* For an extensive coverage of these notions, see the book by Moulin [9]. Due to space restrictions, we have not presented proofs in this version of the paper; for that, see the full version [12].

The classic paper of Shapley and Shubik [11] showed that the set of core imputations of the assignment game as the set of optimal solutions to the dual of the LP-relaxation of the maximum weight matching problem in the underlying graph. Among their other insights was a characterization of the two "antipodal" points — imputations which maximally favor one side of the bipartition[1] — in the core of this game. This in-depth understanding makes the assignment game a paradigmatic setting for studying the core; in turn, insights gained provide valuable guidance on profit-sharing in real-life situations.

Deng et al. [5] distilled the ideas underlying the Shapley-Shubik Theorem to obtain a general framework (see Section 4.1.1) which helps characterize the core of several games that are based on fundamental combinatorial optimization problems, including maximum flow in unit capacity networks both directed and undirected, maximum number of edge-disjoint s-t paths, maximum number of vertex-disjoint s-t paths, maximum number of disjoint arborescences rooted at a vertex r, and concurrent games (defined below).

In this paper, we study the core of the assignment game and some of its generalizations, including two versions of the bipartite b-matching game (Section 4); in the first version (Section 4.2) edges can be matched multiple number of times and in the second, edges can be matched at most once (Section 5). The intriguing aspect of the latter two games is that they don't fall in framework of Deng et al.; see Section 4.1.1 for the reason. In turn, we show that some of the core imputations of these games correspond to optimal dual solutions and some not. This leads to a tantalizing question: is there a "mathematical structure" that produces the latter?

For the assignment game (Section 3), we start with the realization is that despite the in-depth work of Shapely and Shubik, and the passage of half a century, there are still basic questions about the core which have remained unexplored:

1. Do core imputations spread the profit more-or-less evenly or do they restrict them to certain well-chosen agents? If the latter, what characterizes these "chosen" agents?
2. By definition, under any core imputation, the sum of profits of two agents i and j is at least the profit they make by being matched, say w_{ij}. What characterizes pairs (i, j) for which this sum strictly exceed w_{ij}?
3. How do core imputations behave in the presence of degeneracy?

An assignment game is said to be *degenerate* if the optimal assignment is not unique. Although Shapley and Shubik had mentioned this phenomenon, they brushed it away, claiming that "in the most common case" the optimal assignment will be unique, and if not, their suggestion was to perturb the edge weights to make the optimal assignment

[1] Much like the top and bottom elements in a lattice of stable matchings.

unique. However, this is far from satisfactory, since perturbing the weights destroys crucial information contained in the original instance and the outcome becomes a function of the vagaries of the randomness imposed on the instance.

The following broad idea helps answer all three questions. A well-known theorem in matching theory says that the LP-relaxation of the optimal assignment problem always has an integral optimal solution [8]. Therefore, the worth of the assignment game is given by the optimal objective function value of this LP. Next, the Shapley-Shubik Theorem says that the set of core imputations of this game are precisely the optimal solutions to the dual of this LP. These two facts naturally raise the question of viewing core imputations through the lens of complementarity; in turn, it leads to a resolution of all three questions.

The following setting, taken from [6] and [2], vividly captures the issues underlying profit-sharing in an assignment game. Suppose a coed tennis club has sets U and V of women and men players, respectively, who can participate in an upcoming mixed doubles tournament. Assume $|U| = m$ and $|V| = n$, where m, n are arbitrary. Let $G = (U, V, E)$ be a bipartite graph whose vertices are the women and men players and an edge (i, j) represents the fact that agents $i \in U$ and $j \in V$ are eligible to participate as a mixed doubles team in the tournament. Let w be an edge-weight function for G, where $w_{ij} > 0$ represents the expected earnings if i and j do participate as a team in the tournament. The total worth of the game is the weight of a maximum weight matching in G.

Assume that the club picks such a matching for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club. We will use this setting to discuss the issues involved in the questions raised above.

Under core imputations, the profit allocated to an agent is a function of the value he/she brings to the various sub-coalitions he/she belongs to, i.e., it is consistent with his/her negotiating power. Indeed, it is well known that core imputations provide profound insights into the negotiating power of individuals and sub-coalitions, see [9]. The first question provides further insights into this issue. Our answer to this question is that the core rewards only *essential* agents, namely those who are matched by *every* maximum weight matching, see Theorem 10.

Our answer to the second question is quite counter-intuitive: we show that a pair of players (i, j) get overpaid by core allocations if and only if they are so incompetent, as a team, that they don't participate in any maximum weight matching! Since i and j are incompetent as a team, w_{ij} is small. On the other hand, a least one of i and j does team up with other agents in maximum weight matchings – if not, (i, j) would have been matched in a maximum weight matching. Therefore, the sum of the profits of i and j exceeds w_{ij} in at least one core imputation; this is shown in Theorem 15.

Our insight into degeneracy is that it treats teams and agents in totally different ways, see Section 3.4. Section 2 discusses past approaches to degeneracy.

Whereas the core of the assignment game is always non-empty, that of the general graph matching game can be empty. Deng et al. [5] showed that the core of this game is non-empty if and only if the weights of maximum weight integral and fractional matchings concur. For this reason, we have named such games as *concurrent games*. As stated above, their core imputations are precisely the set of optimal solutions to the dual LP.

In the full paper [12], we study the three questions, raised above, for concurrent games as well.

2 Related Works

An imputation in the core has to ensure that *each* of the exponentially many sub-coalitions is "happy" — clearly, that is a lot of constraints. As a result, the core is non-empty only for a handful of games, some of which are mentioned in the Introduction. A different kind of game, in which preferences are cardinal, is based on the stable matching problem defined by Gale and Shapley [7]. The only coalitions that matter in this game are ones formed by one agent from each side of the bipartition. A stable matching ensures that no such coalition has the incentive to secede and the set of such matchings constitute the core of this game.

Over the years, researchers have approached the phenomenon of degeneracy in the assignment game from directions that are different from ours. Nunez and Rafels [10], studied relationships between degeneracy and the dimension of the core. They defined an agent to be *active* if her profit is not constant across the various imputations in the core, and non-active otherwise. Clearly, this notion has much to do with the dimension of the core, e.g., it is easy to see that if all agents are non-active, the core must be zero-dimensional. They prove that if all agents are active, then the core is full dimensional if and only if the game is non-degenerate. Furthermore, if there are exactly two optimal matchings, then the core can have any dimension between 1 and $m - 1$, where m is the smaller of $|U|$ and $|V|$; clearly, m is an upper bound on the dimension.

In another work, Chambers and Echenique [3] study the following question: Given the entire set of optimal matchings of a game on $m = |U|$, $n = |V|$ agents, is there an $m \times n$ surplus matrix which has this set of optimal matchings. They give necessary and sufficient conditions for the existence of such a matrix.

3 The Core of the Assignment Game

In this section, we provide answers to the three questions, for assignment games, which were raised in the Introduction.

3.1 Definitions and Preliminary Facts

The *assignment game*, $G = (U, V, E),\ w : E \to \mathcal{R}_+$, has been defined in the Introduction. We start by giving definitions needed to state the Shapley-Shubik Theorem.

▶ **Definition 1.** *The set of all players, $U \cup V$, is called the* grand coalition. *A subset of the players, $(S_u \cup S_v)$, with $S_u \subseteq U$ and $S_v \subseteq V$, is called a* coalition *or a* sub-coalition.

▶ **Definition 2.** *The* worth *of a coalition $(S_u \cup S_v)$ is defined to be the maximum profit that can be generated by teams within $(S_u \cup S_v)$ and is denoted by $p(S_u \cup S_v)$. Formally, $p(S_u \cup S_v)$ is the weight of a maximum weight matching in the graph G restricted to vertices in $(S_u \cup S_v)$ only. $p(U \cup V)$ is called the* worth of the game. *The* characteristic function *of the game is defined to be $p : 2^{U \cup V} \to \mathcal{R}_+$.*

▶ **Definition 3.** *An* imputation[2] *gives a way of dividing the worth of the game, $p(U \cup V)$, among the agents. It consists of two functions $u : U \to \mathcal{R}_+$ and $v : V \to \mathcal{R}_+$ such that $\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)$.*

[2] Some authors prefer to call this a pre-imputation, while using the term imputation when individual rationality is also satisfied.

▶ **Definition 4.** *An imputation* (u, v) *is said to be in the* core of the assignment game *if for any coalition* $(S_u \cup S_v)$*, the total worth allocated to agents in the coalition is at least as large as the worth that they can generate by themselves, i.e.,* $\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)$.

We next describe the characterization of the core of the assignment game given by Shapley and Shubik [11][3].

As stated in Definition 2, the worth of the game, $G = (U, V, E),\ w : E \rightarrow \mathcal{R}_+$, is the weight of a maximum weight matching in G. Linear program (1) gives the LP-relaxation of the problem of finding such a matching. In this program, variable x_{ij} indicates the extent to which edge (i, j) is picked in the solution. Matching theory tells us that this LP always has an integral optimal solution [8]; the latter is a maximum weight matching in G.

$$
\begin{aligned}
\max \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall i \in U, \\
& \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall j \in V, \\
& x_{ij} \geq 0 \quad \forall (i,j) \in E
\end{aligned}
\tag{1}
$$

Taking u_i and v_j to be the dual variables for the first and second constraints of (1), we obtain the dual LP:

$$
\begin{aligned}
\min \quad & \sum_{i \in U} u_i + \sum_{j \in V} v_j \\
\text{s.t.} \quad & u_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\
& u_i \geq 0 \quad \forall i \in U, \\
& v_j \geq 0 \quad \forall j \in V
\end{aligned}
\tag{2}
$$

▶ **Theorem 5.** *(Shapley and Shubik [11]) The imputation* (u, v) *is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).*

By Theorem 5, the core of the assignment game is a convex polyhedron. Shapley and Shubik shed further light on the structure of the core by showing that it has two special imputations which are furtherest apart and so can be thought of as antipodal imputations. In the tennis club setup, one of these imputations maximizes the earnings of women players and the other maximizes the earnings of men players.

Finally, we state a fundamental fact about LP (1); its corollary will be used in a crucial way in Theorems 10 and 15.

▶ **Theorem 6.** *(Birkhoff [1]) The vertices of the polytope defined by the constraints of LP (1) are* $0/1$ *vectors, i.e., they are matchings in* G.

[3] Shapley and Shubik had described this game in the context of the housing market in which agents are of two types, buyers and sellers. They had shown that each imputation in the core of this game gives rise to unique prices for all the houses. In this paper we will present the assignment game in a variant of the tennis setting given in the Introduction; this will obviate the need to define "prices", hence leading to simplicity.

▶ **Corollary 7.** *Any fractional matching in a bipartite graph is a convex combination of integral matchings.*

3.2 The first question: Allocations made to agents by core imputations

▶ **Definition 8.** *A generic player in $U \cup V$ will be denoted by q. We will say that q is:*
1. essential *if q is matched in every maximum weight matching in G.*
2. viable *if there is a maximum weight matching M such that q is matched in M and another, M' such that q is not matched in M'.*
3. subpar *if for every maximum weight matching M in G, q is not matched in M.*

▶ **Definition 9.** *Let y be an imputation in the core. We will say that q* gets paid in *y if $y_q > 0$ and* does not get paid *otherwise. Furthermore, q is* paid sometimes *if there is at least one imputation in the core under which q gets paid, and it is* never paid *if it is not paid under every imputation.*

▶ **Theorem 10.** *For every player $q \in (U \cup V)$:*

$$q \text{ is paid sometimes} \iff q \text{ is essential}$$

Theorem 10 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus core imputations pay only essential players and each of them is paid in some core imputation. Since we have assumed that the weight of each edge is positive, so is the worth of the game, and all of it goes to essential players. Hence we get:

▶ **Corollary 11.** *In the assignment game, the set of essential players is non-empty and in every core imputation, the entire worth of the game is distributed among essential players; moreover, each of them is paid in some core imputation.*

▶ Remark 12. Theorem 5 and Corollary 11 are of much consequence.
1. Corollary 11 reveals the following surprising fact: the set of players who are allocated profits in a core imputation is *independent* of the set of teams that play.
2. The identification of these players, and the exact manner in which the total profit is divided among them, follows the negotiating process. In turn, this process identifies agents who play in *all* possible maximum weight matchings.
3. Perhaps the most remarkable aspect of Theorem 5 is that each possible outcome of this very real process is captured by an inanimate object, namely an optimal solution to the dual, LP (2).

By Corollary 11, core imputations reward only essential players. This raises the following question: Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is "No", because the dual (2) has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

3.3 The second question: Allocations made to teams by core imputations

▶ **Definition 13.** *By a* mixed doubles team *we mean an edge in G; a generic one will be denoted as $e = (u, v)$. We will say that e is:*

1. essential *if e is matched in every maximum weight matching in G.*
2. viable *if there is a maximum weight matching M such that $e \in M$, and another, M' such that $e \notin M'$.*
3. subpar *if for every maximum weight matching M in G, $e \notin M$.*

▶ **Definition 14.** *Let y be an imputation in the core of the game. We will say that e is* fairly paid *in y if $y_u + y_v = w_e$ and it is* overpaid *if $y_u + y_v > w_e$*[4]*. Finally, we will say that e is* always paid fairly *if it is fairly paid in every imputation in the core.*

▶ **Theorem 15.** *For every team $e \in E$:*

$$e \text{ is always paid fairly} \iff e \text{ is viable or essential}$$

Negating both sides of the implication proved in Theorem 15, we get the following implication. For every team $e \in E$:

$$e \text{ is subpar} \iff e \text{ is sometimes overpaid}$$

Clearly, this statement is equivalent to the statement proved Theorem 15 and hence contains no new information. However, it provides a new viewpoint. These two equivalent statements yield the following assertion, which at first sight seems incongruous with what we desire from the notion of the core and the just manner in which it allocates profits:

Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.

How can the core favor subpar teams over viable and essential teams? An explanation is provided in the Introduction, namely a subpar team (i, j) gets overpaid because i and j create worth by playing in competent teams with other players.

Finally, we observe that contrary to Corollary 11, which says that the set of essential players is non-empty, the set of essential teams may be empty.

3.4 The third question: Degeneracy

Next we use Theorems 10 and 15 to get insights into degeneracy. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in G, i.e., there are no viable teams or players. Since viable teams and players arise due to degeneracy, in order to understand the phenomenon of degeneracy, we need to understand how viable teams and players behave with respect to core imputations; this is done in the next corollary.

▶ **Corollary 16.** *In the presence of degeneracy, imputations in the core of an assignment game treat:*

- *viable players in the same way as subpar players, namely they are never paid.*
- *viable teams in the same way as essential teams, namely they are always fairly paid.*

4 The Core of Bipartite b-Matching Games

In this section, we will define two versions of the *bipartite b-matching game* and we will study their core imputations; both versions generalize the assignment game.

[4] Observe that by the first constraint of the dual LP (2), these are the only possibilities.

4.1 Definitions and Preliminary Facts

As in the assignment game, let $G = (U, V, E)$, $w : E \to \mathcal{R}_+$ be the underlying bipartite graph and edge-weight function. Let function $b : U \cup V \to \boldsymbol{Z}_+$ give a bound on the number of times a vertex can be matched. Under the *unconstrained bipartite b-matching game*, each edge can be matched multiple number of times and under the *constrained bipartite b-matching game*, each edge can be matched at most once. Observe that even in the first version, limits imposed by b on vertices will impose limits on edges — thus edge (i, j) can be matched at most $\min\{b_i, b_j\}$ times.

The *worth* of a coalition $(S_u \cup S_v)$, with $S_u \subseteq U, S_v \subseteq V$, is the weight of a maximum weight b-matching in the graph G restricted to vertices in $(S_u \cup S_v)$ only; we will denote this by $p(S_u \cup S_v)$. Whether an edge can be matched at most once or more than once depends on the version of the problem we are dealing with. $p(U \cup V)$ is called the *worth of the game*. The *characteristic function* of the game is defined to be $p : 2^{U \cup V} \to \mathcal{R}_+$. Definitions 3 and 4, defining an imputation and the core, carry over unchanged from the assignment game.

The tennis setting, given in the Introduction, provides a vivid description of these two variants of the b-matching game as well. Let K denote the maximum b-value of a vertex and assume that the tennis club needs to enter mixed doubles teams into K tennis tournaments. In the first variant, a team can play in multiple tournaments and in the second version, a team can play in at most one tournament. In both cases, a player i can play in at most b_i tournaments. The goal of the tennis club is to maximize its profit over all the tournaments and hence picks a maximum weight b-matching in G. An imputation in the core gives a way of distributing the profit in such a way that no sub-coalition has an incentive to secede.

Linear program (3) gives the LP-relaxation of the problem of finding a maximum weight b-matching for the unconstrained version. In this program, variable x_{ij} indicates the extent to which edge (i, j) is picked in the solution; observe that there is no upper bound on the variables x_{ij} since an edge can be matched any number of times.

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \leq b_i \quad \forall i \in U, \\
& \sum_{(i,j)\in E} x_{ij} \leq b_j \quad \forall j \in V, \\
& \qquad\quad x_{ij} \geq 0 \quad \forall (i,j) \in E
\end{aligned}
\tag{3}
$$

Taking u_i and v_j to be the dual variables for the first and second constraints of (3), we obtain the dual LP:

$$
\begin{aligned}
\min \quad & \sum_{i\in U} b_i u_i + \sum_{j \in V} b_j v_j \\
\text{s.t.} \quad & u_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\
& \quad u_i \geq 0 \qquad \forall i \in U, \\
& \quad v_j \geq 0 \qquad \forall j \in V
\end{aligned}
\tag{4}
$$

Linear program (5) gives the LP-relaxation of the problem of finding a maximum weight

b-matching for the constrained version. Observe that in this program, variables x_{ij} are upper bounded by 1, since an edge can be matched at most once.

$$\begin{aligned} \max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \le b_i \quad \forall i \in U, \\ & \sum_{(i,j)\in E} x_{ij} \le b_j \quad \forall j \in V, \\ & x_{ij} \le 1 \quad \forall (i,j) \in E, \\ & x_{ij} \ge 0 \quad \forall (i,j) \in E \end{aligned} \tag{5}$$

▶ Remark 17. In both in LPs (3) and (5), the matrices of coefficients of the constraints are totally unimodular [8], and therefore both LPs always have integral optimal solutions.

Taking u_i, v_j and z_{ij} to be the dual variables for the first, second and third constraints of (5), we obtain the dual LP:

$$\begin{aligned} \min \quad & \sum_{i\in U} b_i u_i + \sum_{j\in V} b_j v_j + \sum_{(i,j)\in E} z_{ij} \\ \text{s.t.} \quad & u_i + v_j + z_{ij} \ge w_{ij} \quad \forall (i,j) \in E, \\ & u_i \ge 0 \quad \forall i \in U, \\ & v_j \ge 0 \quad \forall j \in V, \\ & z_{ij} \ge 0 \quad \forall (i,j) \in E \end{aligned} \tag{6}$$

4.1.1 The Framework of Deng et al. [5]

In this section, we present the framework of Deng et al. [5], which was mentioned in the Introduction, and point out why it does not apply to the two versions of the b-matching game. Let $T = \{1, \cdots, n\}$ be the set of n agents of the game. Let $w \in \mathbb{R}_+^m$ be an m-dimensional non-negative real vector specifying the weights of certain objects; in the assignment game, the objects are edges of the underlying graph. Let A be an $n \times m$ matrix with 0/1 entries whose i^{th} row corresponds to agent $i \in T$. Let x be an m-dimensional vector of variables and $\mathbb{1}$ be the n-dimensional vector of all 1s. Assume that the worth of the game is given by the objective function value of following integer program.

$$\begin{aligned} \max \quad & w \cdot x \\ \text{s.t.} \quad & Ax \le \mathbb{1}, \\ & x \in \{0,1\} \end{aligned} \tag{7}$$

Moreover, for a sub-coalition, $T' \subseteq T$ assume that its worth is given by the integer program obtained by replacing A by A' in (7), where A' picks the set of rows corresponding to agents in T'. The LP-relaxation of (7) is:

$$\begin{aligned} \max \quad & w \cdot x \\ \text{s.t.} \quad & Ax \le \mathbb{1}, \\ & x \ge 0 \end{aligned} \tag{8}$$

Deng et al. proved that if LP (8) always has an integral optimal solution, then the set of core imputations of this game is exactly the set of optimal solutions to the dual of LP (8).

As stated in Remark 17, the matrices of coefficients of both LPs (3) and (5) are totally unimodular and therefore these LPs always have integral optimal solutions. However, they still don't fall in the above-stated framework because their right-hand-sides are b values of the vertices and not $\mathbb{1}$.

4.2 The Core of the Unconstrained Bipartite b-Matching Game

Let I denote an instance of this game and let $C(I)$ denote its set of core imputations. We will show in Theorem 18 that corresponding to every optimal solution to the dual LP (4), there is an imputation in $C(I)$. Let $D(I)$ denote the set of all such core imputations. Since $D(I) \neq \emptyset$, we get Corollary 19 stating that the core of this game is non-empty. Next, we will give an instance I such that $D(I) \subset C(I)$, i.e., unlike the assignment game, I has core imputations that don't correspond to optimal solutions to the dual LP.

The correspondence between optimal solutions to the dual LP (4) and core imputations in $D(I)$ is as follows. Given an optimal solution (u, v), define the profit allocation to $i \in U$ to be $\alpha_i = b_i \cdot u_i$ and that to $j \in V$ to be $\beta_j = b_j \cdot v_j$.

▶ **Theorem 18.** *The profit-sharing method (α, β), which corresponds to an optimal solution (u, v) to the dual LP (4), is an imputation in the core of the unconstrained bipartite b-matching game.*

▶ **Corollary 19.** *The core of the unconstrained bipartite b-matching game is always non-empty.*

▶ Remark 20. Observe that the mapping given from optimal solutions to the dual LP (4) to core imputations in $D(I)$ is a bijection.

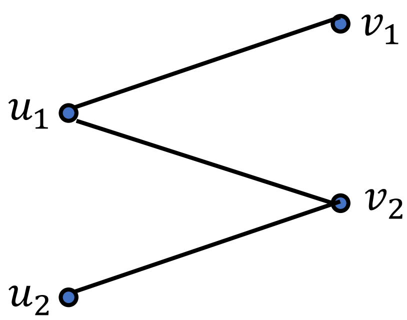

Figure 1 The graph for Example 21.

▶ **Example 21.** For the bipartite b-matching game defined by the graph of Figure 1, let the b values be $2, 1, 2, 1$ for u_1, u_2, v_1, v_2, and let the edge weights be $1, 3, 1$ for $(u_1, v_1), (u_1, v_2), (u_2, v_2)$.

In this section, we will view the game defined in Example 21 as an unconstrained bipartite b-matching game and will show that it has a set of core imputations which do not correspond to optimal dual solutions, i.e., they lie in $C(I) - D(I)$. The optimal matching picks edges

$(u_1, v_1), (u_1, v_2)$ once each, for a total profit of 4. The unique optimal dual solution is $1, 0, 0, 2$ for u_1, u_2, v_1, v_2, and the corresponding core imputation is $2, 0, 0, 2$.

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the profits allocated to u_1, u_2, v_1, v_2. The solutions of the system of linear inequalities (9), for non-negative values of the variables, capture all possible core imputations, i.e., the set $C(I)$.

$$\begin{aligned} \alpha_1 + \beta_1 &\geq 2 \\ \alpha_1 + \beta_2 &\geq 3 \\ \alpha_1 + \beta_1 + \beta_2 &\geq 4 \\ \alpha_2 + \beta_2 &\geq 1 \\ \alpha_1 + \alpha_2 + \beta_2 &\geq 3 \\ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 &= 4 \end{aligned} \tag{9}$$

On solving this system, we find that $\alpha_1, \alpha_2, \beta_1, \beta_2$ should be $1 + a,\ 0\ b,\ 1 + c$, where a, b, c are non-negative and satisfy the system (10).

$$\begin{aligned} a + b &\geq 1 \\ a + c &\geq 1 \\ a + b + c &= 2 \end{aligned} \tag{10}$$

A fourth constraint, $b \leq 1$ follows from the last two in this system. The solution $a = 1,\ b = 0,\ c = 1$ gives the core imputation corresponding to the unique optimal dual solution; the rest give the remaining core imputations, e.g., the imputation $3, 0, 0, 1$.

For an arbitrary instance I, one can clearly capture all possible core imputations via an exponential sized system of inequalities of the type $\geq$, one corresponding to each coalition $(S_u \cup S_v)$; its r.h.s. will be $p(S_u \cup S_v)$ and its l.h.s. will be the sum of all variables denoting profits accrued to vertices in this coalition. Note that all the variables of this system will be constrained to be non-negative and it will have one equality corresponding to the worth of the grand coalition; the latter is the last equality in system (9).

The following question arises: is there a smaller system which accomplishes this task? We observe that it suffices to include in the system only those coalitions whose induced subgraph is connected. This is so because if the induced subgraph for coalition $(S_u \cup S_v)$ has two or more connected components, then the sum of the inequalities for the connected components yields the inequality for coalition $(S_u \cup S_v)$. In particular, if the underlying graph of instance I is sparse, this may lead to a much smaller system. Observe that the system (9), for Example 21, follows from this idea.

▶ Remark 22. Since for the unconstrained bipartite b-matching game, the optimal dual solutions don't capture all core imputations, the characterizations established in Theorems 10 and 15 for the assignment game, don't carry over. However, if one restricts to core imputations in the set $D(I)$ only, one can see that suitable modifications of these statements do hold.

5 The Core of the Constrained Bipartite b-Matching Game

Our results for this game are related to, though not identical with, those for the unconstrained version. In Theorem 23, we will show that corresponding to every optimal solution to the dual LP (6), there is a set of core imputations. This theorem yields Corollary 24 stating that the core of this game is also non-empty. Finally, we will give an instance which has core imputations that don't correspond to optimal solutions to the dual LP.

The corresponding to an optimal solution to the dual LP (4), (u, v, z), we define a set of imputations as follows. For each edge (i, j) define two new variables c_{ij} and d_{ij}; both are constrained to be non-negative. Furthermore, consider all possible ways of splitting z_{ij} into c_{ij} and d_{ij}, i.e., $z_{ij} = c_{ij} + d_{ij}$. Observe that is $x_{ij} = 0$ then $z_{ij} = 0$ and therefore $c_{ij} = d_{ij} = 0$. Define the profit allocation to $i \in U$ to be

$$\alpha_i = b_i \cdot u_i + \sum_{(i,j)\in E} c_{ij}$$

and that to $j \in V$ to be

$$\beta_j = b_j \cdot v_j + d_{ij} + \sum_{(i,j)\in E} d_{ij}.$$

Taken over all possible ways of splitting all z_{ij}s, this gives a set of imputations.

▶ **Theorem 23.** *All profit-sharing methods (α, β), which correspond to the optimal solution (u, v, z) to the dual LP (6), are imputations in the core of the constrained bipartite b-matching game.*

▶ **Corollary 24.** *The core of the constrained bipartite b-matching game is always non-empty.*

In this section, we will view the game defined in Example 21 as a constrained bipartite b-matching game and will again show that it has a set of core imputations which do not correspond to optimal dual solutions. The optimal matching picks edges $(u_1, v_1), (u_1, v_2)$ once each, for a total profit of 4. Unlike the unconstrained case, this time, the optimal dual is not unique. The optimal dual solutions are given by $1, 0, 0, 2 - a$, for vertices u_1, u_2, v_1, v_2, and $0, a, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$, where $a \in [0, 1]$. The corresponding core imputations are $3 - b, 0, 0, 1 + b$, for the four vertices u_1, u_2, v_1, v_2, where $b \in [0, 1]$.

As in the unconstrained case, let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the profits allocated to u_1, u_2, v_1, v_2. This time, the system of linear inequalities whose solutions capture all possible core imputations is given by system (9) after replacing the first inequality by

$$\alpha_1 + \beta_1 \geq 1.$$

This is so because edge (u_1, v_1) can be matched twice under the the unconstrained bipartite b-matching game, but only once under the constrained version. As before, non-negativity is imposed on all these variables. On solving this system, we find that $\alpha_1, \alpha_2, \beta_1, \beta_2$ should be $1,\ 0\ b,\ 1 + c$, where a, b, c are non-negative and satisfy the system (11).

$$\begin{aligned} a + b &\geq 1 \\ a + c &\geq 2 \\ a + b + c &= 3 \end{aligned} \tag{11}$$

Solutions of this system which do not correspond to dual solutions include $1, 0, 0, 3$ and $0, 0, 1, 3$. Observe that neither of these is a core imputation for the unconstrained bipartite b-matching game. The method given in Section 4.2, for finding a smaller system, holds for this case as well and so does Remark 22.

▶ Remark 25. In the assignment game, core imputations were precisely optimal dual solutions. On the other hand, in both versions of the bipartite b-matching game, core imputations are obtained from optimal dual solutions via specific operations. As stated in Remark 20, for the unconstrained version, there is a bijection between optimal dual solutions and core imputations in $D(I)$. In contrast, for the constrained version, the set of imputations corresponding to optimal dual solutions may not be disjoint.

Let us illustrate the last point of Remark 25 via Example 21. Consider the two optimal dual solutions obtained by setting $a = 0$ and $a = 1$, namely $1, 0, 0, 2$, for vertices u_1, u_2, v_1, v_2, and $0, 0, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$; and $1, 0, 0, 1$, for vertices u_1, u_2, v_1, v_2, and $0, 1, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$. Both these optimal duals yield the core imputation assigning profits of $2, 0, 0, 2$ for u_1, u_2, v_1, v_2.

6 Discussion

Our most important open question is to shed light on the origins of core imputations, for the two bipartite b-matching games, which do not correspond to optimal dual solutions. Is there a “mathematical structure” that produces them? A related question is to determine the complexity of the following question for these two games: Given an imputation for a game, decide if it belongs to the core. We believe this question should be co-NP-complete. On the other hand, the following question is clearly in P: Given an imputation for a game I, decide if it lies in $D(I)$.

As stated in Section 3.1, for the assignment game, Shapley and Shubik were able to characterize “antipodal” points in the core. An analogous understanding of the core of the general graph matching games having non-empty core will be desirable.

For the assignment game, Demange, Gale and Sotomayor [4] give an auction-based procedure to obtain a core imputation; it turns out to be optimal for the side that proposes, as was the case for the deferred acceptance algorithm of Gale and Shapley [7] for stable matching. Is there an analogous procedure for obtaining an imputation in the core of the general graph matching games having non-empty core?

7 Acknowledgements

I wish to thank Hervé Moulin for asking the interesting question of extending results obtained for the assignment game to general graph matching games having a non-empty core. I also wish to thank Federico Echenique, Hervé Moulin and Thorben Trobst for several valuable discussions.

References

1 Garrett Birkhoff. Three observations on linear algebra. *Univ. Nac. Tacuman, Rev. Ser. A*, 5:147–151, 1946.

2 Péter Biró, Walter Kern, and Daniël Paulusma. Computing solutions for matching games. *International journal of game theory*, 41(1):75–90, 2012.

3 Christopher P Chambers and Federico Echenique. The core matchings of markets with transfers. *American Economic Journal: Microeconomics*, 7(1):144–64, 2015.

4 Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *Journal of political economy*, 94(4):863–872, 1986.

5 Xiaotie Deng, Toshihide Ibaraki, and Hiroshi Nagamochi. Algorithmic aspects of the core of combinatorial optimization games. *Mathematics of Operations Research*, 24(3):751–766, 1999.

6 Kimmo Eriksson and Johan Karlander. Stable outcomes of the roommate game with transferable utility. *International Journal of Game Theory*, 29(4):555–569, 2001.
7 David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.
8 L. Lovász and M.D. Plummer. *Matching Theory*. North-Holland, Amsterdam–New York, 1986.
9 Hervé Moulin. *Cooperative microeconomics: a game-theoretic introduction*, volume 313. Princeton University Press, 2014.
10 Marina Núñez and Carles Rafels. On the dimension of the core of the assignment game. *Games and Economic Behavior*, 64(1):290–302, 2008.
11 Lloyd S Shapley and Martin Shubik. The assignment game I: The core. *International Journal of Game Theory*, 1(1):111–130, 1971.
12 Vijay V Vazirani. New characterizations of core imputations of matching and b-matching games. *arXiv preprint arXiv:2202.00619*, 2022.