Rate control as a market equilibrium

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Abstract

This note extracts the definition of a rate control equilibrium from [2]. An interesting question is whether the equilibrium can be found by a polynomial time algorithm: it can in the case where every route is of unit length [1]. (Continuous time algorithms similar to TCP are known - see [3] - but insights from finite algorithms may be provocative.)

1 The equilibrium

Consider a network with a set J of *resources*, and let C_j be the finite capacity of resource j, for $j \in J$. Let a *route* r be a non-empty subset of J, and write R for the set of possible routes. Set $A_{jr} = 1$ if $j \in r$, so that resource j lies on route r, and set $A_{jr} = 0$ otherwise. This defines a $0-1$ matrix $A = (A_{jr}, j \in J, r \in R)$. Suppose that several routes through the network may substitute for one another: formally, suppose that a *source-sink* s is a subset of R and write S for the set of possible source-sinks. Set $H_{sr} = 1$ if $r \in s$, so that route r serves the source-sink s, and set $H_{sr} = 0$ otherwise. This defines a $0-1$ matrix $H = (H_{sr}, s \in S, r \in R)$. For each $r \in R$ let $s(r)$ identify a value $s \in S$ such that $H_{sr} = 1$, and suppose this value is unique; we view $s(r)$ as the source-sink served by route r.

A flow pattern $y = (y_r, r \in R)$ supports the rates $x = (x_s, s \in S)$ if $Hy = x$, so that the flows y_r over routes r serving the source-sink s sum to the rate x_s. A flow pattern $y = (y_r, r \in R)$ is feasible if $y \geq 0$ and $Ay \leq C$, where $C = (C_j, j \in J)$, so that the flows over routes through resource j

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sum to not more than the capacity C_j of resource j. Suppose that user s is prepared to pay an amount m_s per unit time, and let $m = (m_s, s \in S)$.

Say that (x, y) is an *equilibrium* if there exist multipliers (λ, μ) such that:

$$Hy = x, \quad Ay \leq C, \quad x, y \geq 0 \tag{1}$$

$$\lambda^T H \leq \mu^T A, \quad \lambda, \mu \geq 0 \tag{2}$$

$$\mu^T(C - Ay) = 0, \quad (\mu^T A - \lambda^T H)y = 0, \quad m_s = \lambda_s x_s, \ s \in S. \tag{3}$$

Interpret μ_j as the price of unit flow through resource j, and λ_s as the price of unit flow between source-sink s. The first row of conditions expresses capacity constraints; the other rows insist that no flow goes along a route unless it is the cheapest route serving the relevant source-sink, that no link has a positive price unless it is full, and (the final equation) that all money is spent.

An equilibrium exists, since it solves the following optimization problem (e.g. [2], page 13):

$$\text{maximize} \quad \sum m_s \log x_s \tag{4}$$

$$\text{subject to} \quad Hy = x, \ Ay \leq C \tag{5}$$

$$\text{over} \quad x, y \geq 0. \tag{6}$$

For this optimization problem row (1) is *primal feasibility*; row (2) is *dual feasibility*; and row (3) comprises *complementary slackness.*

Indeed an equilibrium exists if H and A are more general than $0-1$ matrices. The case where all routes are of unit length is the model of [1], where a polynomial time algorithm is presented. Is there a polynomial time algorithm for the case of longer routes?

References

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[2] F. P. Kelly (1997) Charging and rate control for elastic traffic. European Transactions on Telecommunications, volume 8, pages 33-37. www.statslab.cam.ac.uk/~frank/elastic.html

[3] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan (1998) Rate control in communication networks: shadow prices, proportional fairness and stability. Journal of the Operational Research Society 49, 237-252. www.statslab.cam.ac.uk/~frank/rate.html.