Continuity Properties of Equilibrium Prices and Allocations in Linear Fisher Markets

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Abstract. Continuity of the mapping from initial endowments and utilities to equilibria is an essential property for a desirable model of an economy – without continuity, small errors in the observation of parameters of the economy may lead to entirely different predicted equilibria.
We show that for the linear case of Fisher's market model, the (unique) vector of equilibrium prices, $\mathbf{p} = \mathbf{p}(\mathbf{m}, \mathbf{U})$ is a continuous function of the initial amounts of money held by the agents, $\mathbf{m}$, and their utility functions, $\mathbf{U}$. Furthermore, the correspondence $X(\mathbf{m}, \mathbf{U})$, giving the set of equilibrium allocations for any specified $\mathbf{m}$ and $\mathbf{U}$, is upper hemicontinuous, but not lower hemicontinuous. However, for a fixed $\mathbf{U}$, this correspondence is lower hemicontinuous in $\mathbf{m}$.

1 Introduction

Mathematical economists have studied extensively three basic properties that a desirable model of an economy should possess: existence, uniqueness, and continuity of equilibria.[3] An equilibrium operating point ensures parity between demand and supply, uniqueness of the equilibrium ensures stability, and continuity is essential for this theory to have predictive value – without continuity, small errors in the observation of parameters of an economy may lead to entirely different predicted equilibria.

The questions of existence and uniqueness (or its relaxation to local uniqueness) were studied for several concrete and realistic models. However, to the best of our knowledge, the question of continuity was studied only in an abstract setting; for example, demand functions of agents were assumed to be continuously differentiable and, using differential topology, the set of "bad" economies was shown to be "negligible" (of Lebesgue measure zero if the set of economies is finite-dimensional).[4]

In this paper, we study continuity of equilibrium prices and allocations for perhaps the simplest market model – the linear case of Fisher's model. It is

[3] See [3], Chapter 15, "Smooth preferences".
[4] See [3], Chapter 19, "The application to economies of differential topology and global analysis: regular differentiable economies".

well known that equilibrium prices are unique for this case [5]. An instance of this market is specified by $\mathbf{m}$ and $\mathbf{U}$, the initial amounts of money held by the agents and their utility functions, respectively. We denote by $\mathbf{p} = \mathbf{p}(\mathbf{m}, \mathbf{U})$ be the corresponding (unique) vector of equilibrium prices. In Section 3 we prove that the equilibrium utility values are continuous functions of the unit utility values and the initial amounts of money. In Section 4 we prove that $\mathbf{p}(\mathbf{m}, \mathbf{U})$ is a continuous mapping.

Such linear markets can, however, have more than one equilibrium allocation of goods; let $X(\mathbf{m}, \mathbf{U})$ denote the correspondence giving the set of equilibrium allocations. In Section 5 we prove that this correspondence is upper hemicontinuous, but not lower hemicontinuous. For a fixed $\mathbf{U}$, however, this correspondence turns out to be lower hemicontinuous in $\mathbf{m}$ as well.

2 Fisher's linear case and some basic polyhedra

Fisher's linear market model (see [2]) consists of N *buyers* and n divisible *goods*; without loss of generality, the amount of each good may be assumed to be unity. Let u_{ij} denote the utility derived by i on obtaining a unit amount of good j. Thus, the utility of buyer i from receiving x_{ij} units of good j, $j = 1, \ldots, n$, is equal to $\sum_{j=1}^{n} u_{ij}x_{ij}$. Let m_i, $i = 1, \ldots, N$, denote the initial amount of money of buyer i. Unit prices, $p_1, \ldots, p_n$, of the goods are said to be *equilibrium prices* if there exists an allocation $\mathbf{x} = (x_{ij})$ of all the goods to the buyers so that each buyer receives a bundle of maximum utility value among all bundles that the buyer can afford, given these prices; in this case $\mathbf{x}$ is called an *equilibrium allocation.*

Denote by P_X the polytope of feasible allocations, i.e.,

$$P_X \;\equiv\; \{\mathbf{x} = (x_{ij}) \in \mathbb{R}^{Nn} \,:\, \sum_{i=1}^{N} x_{ij} \leq 1 \ (j = 1, \ldots, n),\ \mathbf{x} \geq \mathbf{0}\}\ .$$

Obviously, $\mathbf{x}$ is a vertex of P_X if and only if for all i and j, $x_{ij} \in \{0, 1\}$, and for each j, there is at most one i such that $x_{ij} = 1$. In other words, an allocation $\mathbf{x}$ is a vertex of P_X if and only if in $\mathbf{x}$ each good is given in its entirety to one agent. Denote by $\mathbf{U}$ the $(N \times (Nn))$-matrix that maps a vector $\mathbf{x}$ to the associated vector $\mathbf{y} = (y_1, \ldots, y_N)$ of utilities, where $y_i = \sum_{j=1}^{n} u_{ij}\, x_{ij}$, i.e., $\mathbf{y} = \mathbf{U}\mathbf{x}$. Uniqueness of equilibrium prices implies uniqueness of $\mathbf{y}$ at equilibrium.

Denote by $P_Y = P_Y(\mathbf{U})$ the polytope of feasible N-tuples of utility values, i.e., $P_Y = \mathbf{U}P_X$. Obviously, $\mathbf{y} \geq \mathbf{0}$ for every $\mathbf{y} \in P_Y$. It follows that for every vertex $\mathbf{y}$ of P_Y, there exists a vertex $\mathbf{x}$ of P_X such that $\mathbf{y} = \mathbf{U}\mathbf{x}$. Denote by S_i the set of goods that i receives under vertex allocation $\mathbf{x}$. Then, $y_i = \sum_{j \in S_i} u_{ij}$, $i = 1, \ldots, N$.[5]

[5] The converse is not true in general. In fact, in the case of $N = n = 2$, if $u_{ij} = 1$ for all i and j, then the allocation $(1, 0, 0, 1)$, where good 1 is allocated to agent 1 and good 2 is allocated to agent 2, is a vertex of P_X but the associated vector of utilities $(1, 1)$ is not a vertex because it is a convex combination of the feasible vectors of utilities $(2, 0)$ and $(0, 2)$.

3 Continuity of equilibrium utility values

Denote $G(\mathbf{y}, \mathbf{U}, \mathbf{x}) \equiv \|\mathbf{y} - \mathbf{U}\mathbf{x}\|^2$. Obviously,

(i) G is continuous,
(ii) $G(\mathbf{y}, \mathbf{U}, \mathbf{x}) \geq 0$ for all $\mathbf{y}$, $\mathbf{U}$, and $\mathbf{x}$,
(iii) $G(\mathbf{y}, \mathbf{U}, \mathbf{x}) = 0$ if and only if $\mathbf{y} = \mathbf{U}\mathbf{x}$, and
(iv) for every $\mathbf{y}$ and $\mathbf{U}$, the function $g(\mathbf{x}) \equiv G(\mathbf{y}, \mathbf{U}, \mathbf{x})$ has a minimum over P_X.

Denote by $F(\mathbf{y}, \mathbf{U})$ the minimum of $G(\mathbf{y}, \mathbf{U}, \mathbf{x})$ over $\mathbf{x} \in P_X$. It is easy to verify the following:

(i) F is continuous, because G is continuous and P_X is compact,
(ii) $F(\mathbf{y}, \mathbf{U}) \geq 0$ for all $\mathbf{y}$ and $\mathbf{U}$, and
(iii) $F(\mathbf{y}, \mathbf{U}) = 0$ if and only if $\mathbf{y} \in P_Y(\mathbf{U})$.

For $\mathbf{y} \geq \mathbf{0}$, $\mathbf{m} > \mathbf{0}$ and $\mathbf{U} \geq \mathbf{0}$, denote

$$f(\mathbf{y}; \mathbf{m}, \mathbf{U}) \equiv \sum_{i=1}^{n} m_i \cdot \log y_i - M \cdot F(\mathbf{y}, \mathbf{U}) \ , \tag{1}$$

where M is a sufficiently large scalar. By definition, $P_Y(\mathbf{U}) \neq \varnothing$ for every $\mathbf{U} \geq \mathbf{0}$. For $\mathbf{m} > \mathbf{0}$, f is strictly concave in $\mathbf{y}$ over $P_Y(\mathbf{U})$, and hence has a unique maximizer over $P_Y(\mathbf{U})$. For M sufficiently large, this is also a maximizer over all $\mathbf{y} \geq \mathbf{0}$. Thus, for $\mathbf{m} > \mathbf{0}$ and $\mathbf{U} \geq \mathbf{0}$, denote by $\mathbf{y}^* = \mathbf{y}^*(\mathbf{m}, \mathbf{U})$ that unique maximizer.

Theorem 1. *The mapping* $\mathbf{y}^*(\mathbf{m}, \mathbf{U})$ *is continuous.*

Proof. Suppose $\{(\mathbf{m}^k, \mathbf{U}^k)\}_{k=1}^{\infty}$ is a sequence that converges to $(\mathbf{m}^0, \mathbf{U}^0)$, where for every $k \geq 0$, $\mathbf{m}^k > \mathbf{0}$ and $\mathbf{U}^k \geq \mathbf{0}$. Denote $\mathbf{y}^k = \mathbf{y}^*(\mathbf{m}^k, \mathbf{U}^k)$, $k = 0, 1, \ldots$ By continuity of f as a function of $(\mathbf{y}; \mathbf{m}, \mathbf{U})$, $\{f(\mathbf{y}^0; \mathbf{m}^k, \mathbf{U}^k)\}$ converges to $f(\mathbf{y}^0; \mathbf{m}^0, \mathbf{U}^0)$. Since $\mathbf{y}^k \in P_Y(\mathbf{U}^k)$ and $\{\mathbf{U}^k\}$ converges, there exists a bound u such that $\|\mathbf{y}^k\| \leq u$ for every k. Thus, we may assume without loss of generality that $\mathbf{y}$ is restricted to a compact set. Let $\{\mathbf{y}^{k_j}\}_{j=1}^{\infty}$ be any convergent subsequence, and denote its limit by $\overline{\mathbf{y}}$. By continuity of f, $\{f(\mathbf{y}^{k_j}; \mathbf{m}^{k_j}, \mathbf{U}^{k_j})\}$ converges to $f(\overline{\mathbf{y}}; \mathbf{m}^0, \mathbf{U}^0)$. Since $f(y^{k_j}; \mathbf{m}^{k_j}, \mathbf{U}^{k_j}) \geq f(\mathbf{y}^0; \mathbf{m}^{k_j}, \mathbf{U}^{k_j})$, it follows that $f(\overline{\mathbf{y}}; \mathbf{m}^0, \mathbf{U}^0) \geq f(\mathbf{y}^0; \mathbf{m}^0, \mathbf{U}^0)$. Since $\mathbf{y}^0$ maximizes $f(\mathbf{y}; \mathbf{m}^0, \mathbf{U}^0)$ and the maximum is unique, it follows that $\overline{\mathbf{y}} = \mathbf{y}^0$. This implies that $\{\mathbf{y}^k\}$ converges to $\mathbf{y}^0$.

4 Continuity of equilibrium prices

Denote by $\mathbf{p} = \mathbf{p}(\mathbf{m}, \mathbf{U}) = (p_1(\mathbf{m}, \mathbf{U}), \ldots, p_n(\mathbf{m}, \mathbf{U}))$ the prices that are generated as dual variables in the Eisenberg-Gale convex program, whose optimal solutions give equilibrium allocations and dual variables give equilibrium prices [4]:[6]

$$\begin{aligned} &\text{Maximize } \sum_{i=1}^{n} m_i \cdot \log\left(\textstyle\sum_{j=1}^{n} u_{ij}\, x_{ij}\right) \\ &\text{subject to } \mathbf{x} = (x_{ij}) \in P_X \ , \end{aligned} \tag{2}$$

i.e., given an optimal solution $\mathbf{x} = (x_{ij})$ of (2),

$$p_j(\mathbf{m}, \mathbf{U}) = \max\left\{ \frac{m_i \cdot u_{ij}}{\sum_{k=1}^{n} u_{ik}\, x_{ik}} \ : \ i = 1, \ldots, n \right\} \ . \tag{3}$$

The vector $\mathbf{y} = \mathbf{U}\mathbf{x}$ of utilities is the same for all optimal solutions $\mathbf{x}$, and hence $\mathbf{p}$ is unique. The problem can alternately be formulated in terms of the vector of utilities:

$$\begin{aligned} &\text{Maximize } \sum_{i=1}^{n} m_i \cdot \log y_i \\ &\text{subject to } \mathbf{y} = (y_1, \ldots, y_n) \in P_Y \end{aligned} \tag{4}$$

and the prices can be represented as

$$p_j(\mathbf{m}, \mathbf{U}) = \max\left\{ \frac{m_i \cdot u_{ij}}{y_i} \ : \ i = 1, \ldots, n \right\} \ . \tag{5}$$

The latter, together with Theorem 1 gives:

Theorem 2. *The mapping* $\mathbf{p}(\mathbf{m}, \mathbf{U})$ *is continuous.*

5 Hemicontinuity of equilibrium allocations

5.1 Upper hemicontinuity

For every $\mathbf{m} > \mathbf{0}$ and $\mathbf{U} \geq \mathbf{0}$, denote

$$g(\mathbf{x}) = g(\mathbf{x}; \mathbf{m}, \mathbf{U}) \equiv \sum_{i=1}^{n} m_i \cdot \log\left(\textstyle\sum_{j=1}^{n} u_{ij}\, x_{ij}\right) \ .$$

Denote by $X(\mathbf{m}, \mathbf{U})$ the set of optimal solutions of (2). Obviously, $X(\mathbf{m}, \mathbf{U})$ is compact and nonempty for every $\mathbf{m}$ and $\mathbf{U}$. Denote by $v(\mathbf{m}, \mathbf{U})$ the maximum of $g(\mathbf{x})$ over P_X.

Theorem 3. *The correspondence* $X(\mathbf{m}, \mathbf{U})$ *is upper hemicontinuous.*

Proof. To prove that X is upper hemicontinuous at $(\mathbf{m}^0, \mathbf{U}^0)$, one has to show the following: *for every sequence* $\{\mathbf{m}^k, \mathbf{U}^k\}_{k=1}^{\infty}$ *that converges to* $(\mathbf{m}^0, \mathbf{U}^0)$*, and every sequence* $\{\mathbf{x}^k\}_{k=1}^{\infty}$ *such that* $\mathbf{x}^k \in X(\mathbf{m}^k, \mathbf{U}^k)$*, there exists a convergent subsequence* $\{\mathbf{x}^{k_j}\}_{j=1}^{\infty}$*, whose limit* $\mathbf{x}^0$ *belongs to* $X(\mathbf{m}^0, \mathbf{U}^0)$.

[6] We use the convention that $\log 0 = -\infty$.

Suppose $\{\mathbf{m}^k, \mathbf{U}^k\}_{k=1}^{\infty}$ converges to $(\mathbf{m}^0, \mathbf{U}^0)$, and $\{\mathbf{x}^k\}_{k=1}^{\infty}$ is a sequence such that $\mathbf{x}^k \in X(\mathbf{m}^k, \mathbf{U}^k)$. Since $\mathbf{x}^k \in P_X$ for every k, there exists a subsequence $\{\mathbf{x}^{k_j}\}_{j=1}^{\infty}$ that converges to a point $\mathbf{x}^0$. Since g is a continuous function of $(\mathbf{x}; \mathbf{m}, \mathbf{U})$, it follows that the sequence $\{g(\mathbf{x}^{k_j}; \mathbf{m}^{k_j}, \mathbf{U}^{k_j})\}$ converges to $g(\mathbf{x}^0; \mathbf{m}^0, \mathbf{U}^0)$. On the other hand, $g(\mathbf{x}^k; \mathbf{m}^k, \mathbf{U}^k) = v(\mathbf{m}^k, \mathbf{U}^k)$. By Theorem 1, $\{\mathbf{y}^k \equiv \mathbf{U}^k \mathbf{x}^k\}$ converges to an optimal $\mathbf{y}$ with respect to $(\mathbf{m}^0, \mathbf{U}^0)$, so that $\{v(\mathbf{m}^k, \mathbf{U}^k)\}$ converges to $v(\mathbf{m}^0, \mathbf{U}^0)$. Thus, $g(\mathbf{x}^0; \mathbf{m}^0, \mathbf{U}^0) = v(\mathbf{m}^0, \mathbf{U}^0)$, which means $\mathbf{x}^0 \in X(\mathbf{m}^0, \mathbf{U}^0)$.

5.2 Lower hemicontinuity

Proposition 1. *There exist* $\mathbf{m}$ *and* $\mathbf{U}^0$ *such that the correspondence* $\Xi(\mathbf{U}) \equiv X(\mathbf{m}, \mathbf{U})$ *is not lower hemicontinuous at* $\mathbf{U}^0$*.*

Proof. To prove that $\Xi(\mathbf{U})$ is lower hemicontinuous at $\mathbf{U}^0$, one has to show the following: *for every sequence* $\{\mathbf{U}^k\}_{k=1}^{\infty}$ *that converges to* $\mathbf{U}^0$*, and every* $\mathbf{x}^0 \in X(\mathbf{U}^0)$*, there exists a sequence* $\{\mathbf{x}^k \in X(\mathbf{U}^k)\}$ *that converges to* $\mathbf{x}^0$*.*

Consider a linear Fisher market with two goods and two buyers, each having one unit of money ($\mathbf{m} = (1, 1)$), and the utilities per unit $\mathbf{U}$ are: $u_{11} = u_{12} = u_{21} = 1$ and $u_{22} = u$, where $0 < u \leq 1$. Under these circumstances, the equilibrium prices are $(1, 1)$ for every u. If $u < 1$, there is only one equilibrium allocation: Buyer 1 gets Good 2 and Buyer 2 gets Good 1. However, if $u = 1$, there are infinitely many equilibrium allocations: Buyer 1 gets x units of Good 1 and $1-x$ units of Good 2, and Buyer 2 gets $1-x$ units of Good 1 and x units of Good 2, for $0 \leq x \leq 1$. This implies that the correspondence $\Xi(\mathbf{U})$ is not lower hemicontinuous at the point $\mathbf{U}^0$ where $u = 1$.

To prove that X is lower hemicontinuous in $\mathbf{m}$ we need the following lemmas:

Lemma 1. *Let* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and let* $\mathbf{x}^0 \in \mathbb{R}^n$*. For every* $\mathbf{y}$ *in the column space of* $\mathbf{A}$*, denote by* $\mathbf{x}^*(\mathbf{y})$ *the closest*[7] *point to* $\mathbf{x}^0$ *among all points* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} = \mathbf{y}$*. Under these conditions, the mapping* $\mathbf{x}^*(\mathbf{y})$ *is affine.*

Proof. Since we consider only vectors $\mathbf{y}$ in the column space of $\mathbf{A}$, we may assume, without loss of generality, that the rows of $\mathbf{A}$ are linearly independent; otherwise, we may drop dependent rows. Thus, $\mathbf{A}\mathbf{A}^T$ is nonsingular. Let $\mathbf{y}^0 = \mathbf{A}\mathbf{x}^0$. Obviously, $\mathbf{x}^0 = \mathbf{x}^*(\mathbf{y}^0)$. Let $\mathbf{y}$ in the column space of $\mathbf{A}$ be fixed, and consider the problem of minimizing $\frac{1}{2}\|\mathbf{x} - \mathbf{x}^0\|^2$ subject to $\mathbf{A}\mathbf{x} = \mathbf{y}$. It follows that there exists a vector of Lagrange multipliers $\mathbf{z} \in \mathbb{R}^m$ such that $\mathbf{x}^*(\mathbf{y}) - \mathbf{x}^0 = \mathbf{A}^T\mathbf{z}$. Thus, $\mathbf{A}\mathbf{x}^*(\mathbf{y}) - \mathbf{A}\mathbf{x}^0 = \mathbf{A}\mathbf{A}^T\mathbf{z}$, and hence $\mathbf{z} = (\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{y} - \mathbf{y}^0)$. It follows that $\mathbf{x}^*(\mathbf{y}) = \mathbf{x}^0 + \mathbf{A}^T\mathbf{z} = \mathbf{x}^0 + \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{y} - \mathbf{y}^0)$.

Lemma 2. *Let* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *be a matrix whose columns are linearly independent. Let* $\mathbf{x}^0 \in \mathbb{R}^n$ *and* $\mathbf{y}^0 \in \mathbb{R}^m$ *be such that* $\mathbf{A}\mathbf{x}^0 \leq \mathbf{y}^0$*. For every* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} \leq \mathbf{y}\} \neq \emptyset$*, denote by* $\mathbf{x}^*(\mathbf{y})$ *the closest point to* $\mathbf{x}^0$ *among all points* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} \leq \mathbf{y}$*. Under these conditions, the mapping* $\mathbf{x}^*(\mathbf{y})$ *is continuous at* $\mathbf{y}^0$*.*

[7] We use the Euclidean norm throughout; thus the, closest point is unique.

Proof. For every $S \subseteq M \equiv \{1,\ldots,m\}$, denote $\overline{S} \equiv M \setminus S$. Denote by $\mathbf{A}_S$ the matrix consisting of the rows of $\mathbf{A}$ whose indices i belong to S. Similarly, let $\mathbf{y}_S$ denote the projection of $\mathbf{y}$ on the coordinates in S. Denote $F_S(\mathbf{y}) = F_S(\mathbf{y}_S) \equiv \{\mathbf{x} : \mathbf{A}_S\,\mathbf{x} = \mathbf{y}_S\}$. Let $\mathbf{x}^*_S(\mathbf{y})$ be the point in $F_S(\mathbf{y})$ that is closest to $\mathbf{x}^0$. By Lemma 1, $\mathbf{x}^*_S(\mathbf{y})$ is an affine transformation of $\mathbf{y}_S$. It follows that there exists an $\alpha > 0$ such that for every $\mathbf{y}$, $\|\mathbf{x}^*_S(\mathbf{y}) - \mathbf{x}^*_S(\mathbf{y}^0)\| \le \alpha \cdot \|\mathbf{y} - \mathbf{y}^0\|$. Let $\epsilon > 0$ be any number. Fix $S \equiv \{i : (\mathbf{A}\mathbf{x}^0)_i = y^0_i\}$. Obviously, $\mathbf{x}^0 = \mathbf{x}^*_S(\mathbf{y}^0)$ and $\mathbf{A}_{\overline{S}}\,\mathbf{x}^0 < \mathbf{y}^0_{\overline{S}}$. Let $0 < \delta < \epsilon/\alpha$ be sufficiently small so that $\|\mathbf{y} - \mathbf{y}^0\| < \delta$ implies $\mathbf{A}_{\overline{S}}\,\mathbf{x}^*_S(\mathbf{y}) < \mathbf{y}_{\overline{S}}$. It follows that $\|\mathbf{y} - \mathbf{y}^0\| < \delta$ implies $\|\mathbf{x}^*(\mathbf{y}) - \mathbf{x}^0\| \le \|\mathbf{x}^*_S(\mathbf{y}) - \mathbf{x}^0\| < \alpha\delta < \epsilon$.

Theorem 4. *For every fixed* $\mathbf{U}$*, the correspondence* $\Xi(\mathbf{m}) = X(\mathbf{m},\mathbf{U})$ *is lower hemicontinuous at every* $\mathbf{m}^0 > \mathbf{0}$.

Proof. To prove that $\Xi(\mathbf{m})$ is lower hemicontinuous at $\mathbf{m}^0 > \mathbf{0}$, one has to show the following: *for every sequence* $\{\mathbf{m}^k\}_{k=1}^{\infty}$ *that converges to* $\mathbf{m}^0$*, and every* $\mathbf{x}^0 \in \Xi(\mathbf{m}^0)$*, there exists a sequence* $\{\mathbf{x}^k \in \Xi(\mathbf{m}^k)\}$ *that converges to* $\mathbf{x}^0$.

Suppose $\{\mathbf{m}^k\}_{k=1}^{\infty}$ converges to $\mathbf{m}^0$, and let $\mathbf{x}^0 \in \Xi(\mathbf{m}^0)$ be any point. Let $y^k = y^*(\mathbf{m}^k)$, $k = 0,1,\ldots$, i.e., $\mathbf{y}^k$ is the unique maximizer of $f(\mathbf{y};\mathbf{m}^k,\mathbf{U})$ (see (1)) or, equivalently, the optimal solution of (4). By Theorem 1, $\{\mathbf{y}^k\}$ converges to $\mathbf{y}^0$. Thus, $\Xi(\mathbf{m}^k)$ is the set of all vectors $\mathbf{x} \in P_X$ such that $\mathbf{U}\mathbf{x} = \mathbf{y}^k$. In particular, $\mathbf{x}^0 \in P_X$ and $\mathbf{U}\mathbf{x}^0 = \mathbf{y}^0$. Let $\mathbf{x}^k$ denote the minimizer of $\|\mathbf{x} - \mathbf{x}^0\|$ subject to $\mathbf{x} \in P_X$ and $\mathbf{U}\mathbf{x} = \mathbf{y}^k$. Denote by $\mathbf{x}^* = \mathbf{x}^*(\mathbf{y})$ the optimal solution of the following optimization problem:

$$\begin{array}{rl} \text{Minimize}_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}^0\| \\ \text{subject to} & \mathbf{U}\mathbf{x} = \mathbf{y} \\ & \displaystyle\sum_i x_{ij} \le 1 \quad (\forall j) \\ & x_{ij} \ge 0 \quad (\forall i)(\forall j)\ . \end{array}$$

Thus, $\mathbf{x}^k = \mathbf{x}^*(\mathbf{y}^k)$. By Lemma 2, the mapping $\mathbf{x}^*(\mathbf{y})$ is continuous. Since $\{\mathbf{y}^k\}$ converges to $\mathbf{y}^0$, $\{\mathbf{x}^k\}$ converges to $\mathbf{x}^0$.

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References

1. K. J. Arrow and G. Debreu, "Existence of an equilibrium for a competitive economy," *Econometrica*, **22** (1954) 265–290.
2. W. C. Brainard and H. E. Scarf, "How to compute equilibrium prices in 1891," Cowles Foundation Discussion Paper 1270, 2000.
3. G. Debreu, *Mathematical Economics: Twenty papers of Gerard Debreu*, Cambridge University Press, 1986.
4. E. Eisenberg and D. Gale, "Consensus of subjective probabilities: the Pari-Mutuel method," *The Annals of Mathematical Statistics*, **30** (1959) 165–168.
5. D. Gale, *Theory of Linear Economic Models*, McGraw Hill, 1960.